Strong density in Sobolev spaces to manifolds

Antoine Detaille

Université Claude Bernard Lyon 1 – Institut Camille Jordan

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Sobolev spaces with values into manifolds

Let $\mathscr{N}$ be a smooth compact Riemannian manifold, isometrically embedded into $\mathbb{R}^\nu$.

Let $\Omega \subset \mathbb{R}^m$ be a bounded open set, $1 \leq p < +\infty$, and $0 < s < +\infty$.

Definition

$$W^{s,p}(\Omega; \mathscr{N}) = \{u \in W^{s,p}(\Omega; \mathbb{R}^\nu) \colon u(x) \in \mathscr{N} \text{ for almost every } x \in \Omega\}$$

Reminder: classical Sobolev spaces

Let $s = k + \sigma$ with $k \in \mathbb{N}$ and $\sigma \in [0, 1)$.

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) \colon D^j u \in L^p(\Omega) \text{ for every } j \in \{1, \dots, k\}\}$$

If $\sigma \in (0, 1)$,

$$W^{\sigma,p}(\Omega) = \left\{u \in L^p(\Omega) \colon \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{m+\sigma p}} \, dx \, dy < +\infty\right\}.$$

If $k \geq 1$,

$$W^{s,p}(\Omega) = \{u \in W^{k,p}(\Omega) \colon D^k u \in W^{\sigma,p}(\Omega)\}.$$

A few applications

Applications in problems from physics: liquid crystals ($\mathbb{S}^2$, $\mathbb{RP}^2$), supraconductivity (Ginzburg-Landau, $\mathbb{S}^1$), biaxial liquid crystals, superfluid helium. . .

Applications in problems from numerical methods: meshing domains.

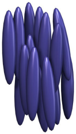

Figure: A field of liquid crystals

Figure: Meshing the earth (see the *Hextreme* project: `www.hextreme.eu`)

The strong density problem

Theorem

$C^\infty(\overline{\Omega})$ is dense in $W^{s,p}(\Omega)$

Question

Is $C^\infty(\overline{\Omega}; \mathcal{N})$ dense in $W^{s,p}(\Omega; \mathcal{N})$?

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$C^\infty(\overline{\Omega})$ *is dense in* $W^{s,p}(\Omega)$

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Is $C^\infty(\overline{\Omega}; \mathscr{N})$ dense in $W^{s,p}(\Omega; \mathscr{N})$?

A topological obstruction

For $1 \leq p < 2$, the map $u \in W^{1,p}(\mathbb{B}^2; \mathbb{S}^1)$ defined by

$$u(x) = \frac{x}{|x|}$$

cannot be approached by maps in $C^\infty(\overline{\mathbb{B}^2}; \mathbb{S}^1)$.

Theorem (Schoen, Uhlenbeck (1983); Bethuel, Zheng (1988); Escobedo (1988))

Assume that $sp < m$. If $C^\infty(\overline{\Omega}; \mathscr{N})$ is dense in $W^{s,p}(\Omega; \mathscr{N})$, then $\pi_{\lfloor sp \rfloor}(\mathscr{N}) = \{0\}$.

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Two natural questions

1. Can we characterize those $\mathscr{N}$ such that $C^{\infty}(\overline{\Omega}; \mathscr{N})$ is dense in $W^{s,p}(\Omega; \mathscr{N})$?
2. Can we find a class of *almost smooth* maps that is always dense in $W^{s,p}(\Omega; \mathscr{N})$?

Definition

The class $\mathcal{R}_i(\Omega; \mathscr{N})$ is the set of all maps u such that there exists a finite union of i-submanifolds $\mathcal{S} = \mathcal{S}_u \subset \mathbb{R}^m$ such that $u \in C^{\infty}(\overline{\Omega} \setminus \mathcal{S}; \mathscr{N})$ and

$$|D^j u(x)| \leq C \frac{1}{\operatorname{dist}(x, \mathcal{S})^j} \quad \text{for every } x \in \Omega \text{ and } j \in \mathbb{N}_*,$$

where $C > 0$ is a constant depending on u and j.

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An answer to both questions

Theorem

The class $\mathcal{R}_{m-\lfloor sp \rfloor -1}(Q^m; \mathcal{N})$ *is always dense in* $W^{s,p}(Q^m; \mathcal{N})$.
The class $C^\infty(\overline{Q^m}; \mathcal{N})$ *is dense in* $W^{s,p}(Q^m; \mathcal{N})$ *if and only if* $\pi_{\lfloor sp \rfloor}(\mathcal{N}) = \{0\}$.

- Case $s = 1$: Bethuel (1991), method of good and bad cubes;
- Case $0 < s < 1$: Brezis and Mironescu (2015), method of homogeneous extension;
- Case $s = 2, 3, \ldots$: Bousquet, Ponce, and Van Schaftingen (2015), method of good and bad cubes *plus* new tools for higher order spaces;
- Case $s > 1$ non-integer: new, method of good and bad cubes *plus* new tools for higher order spaces *plus* new ideas for fractional estimates.

Can we improve the class $\mathcal{R}_i$?

Fact

If $\pi_{\lfloor sp \rfloor}(\mathscr{N}) \neq \{0\}$, then the class $\mathcal{R}_i(Q^m; \mathscr{N})$ is dense in $W^{s,p}(Q^m; \mathscr{N})$ if and only if $i = m - \lfloor sp \rfloor - 1$.

Definition

The class $\mathscr{R}_i(\Omega; \mathscr{N})$ is the set of all $u \in \mathcal{R}_i(\Omega; \mathscr{N})$ such that the singular set $\mathscr{S}$ is a closedly embedded i-dimensional submanifold of $\mathbb{R}^m$.

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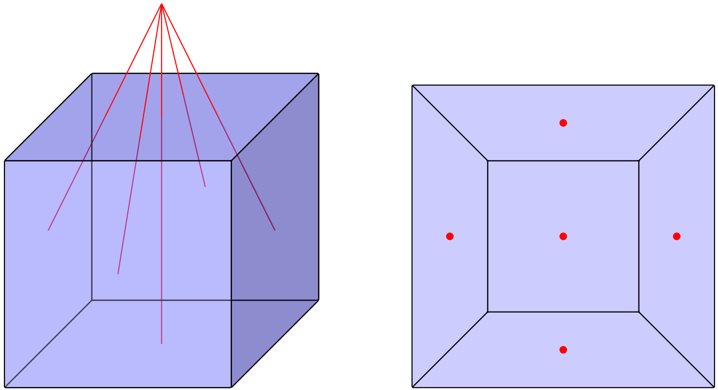
Density of the class $\mathscr{R}$

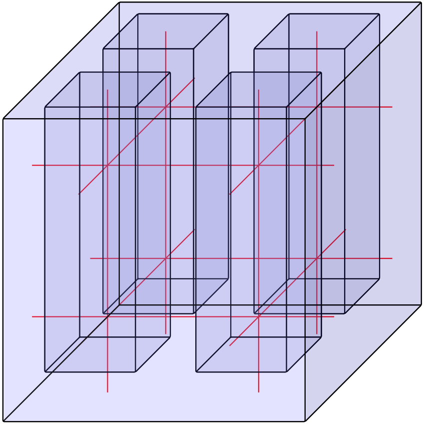
Theorem

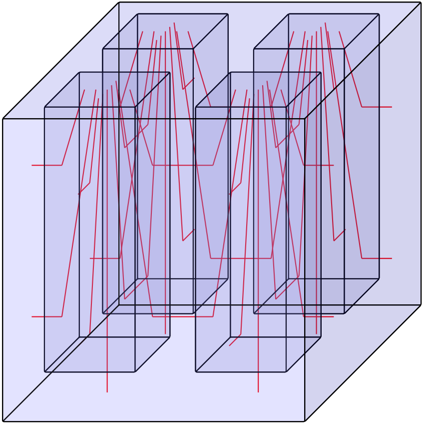
Assume that $sp < m$. The class $\mathscr{R}_{m-\lfloor sp \rfloor -1}(Q^m; \mathscr{N})$ is dense in $W^{s,p}(Q^m; \mathscr{N})$.

The crossings removal procedure: basic idea

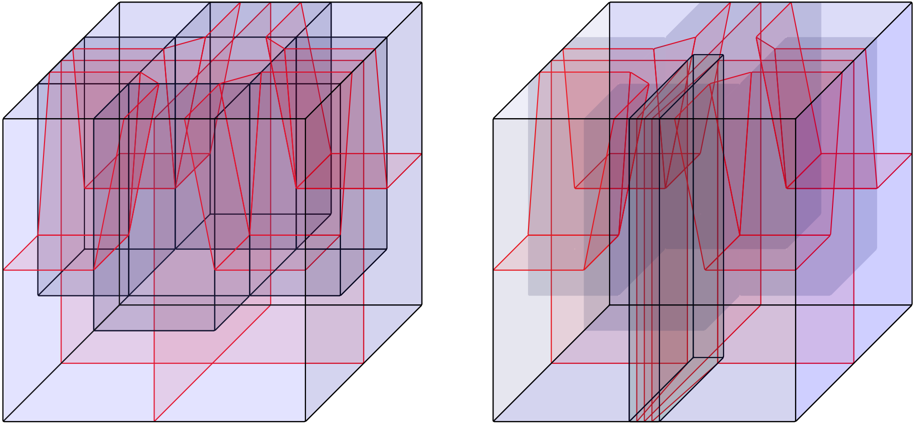


The crossings removal procedure: uncrossing lines





The crossings removal procedure: uncrossing planes



Thank you for your attention!