# A complete answer to the strong density problem for Sobolev spaces with values into manifolds

Antoine Detaille

Université Claude Bernard Lyon 1 – Institut Camille Jordan

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# Sobolev spaces with values into manifolds

Let $\mathcal{N}$ be a smooth Riemannian manifold, compact, connected, without boundary, isometrically embedded in $\mathbb{R}^\nu$.
Let $\Omega \subset \mathbb{R}^m$ be a smooth bounded open set, $1 \leq p < +\infty$, $0 < s < +\infty$.

**Definition**

$$W^{s,p}(\Omega;\mathcal{N}) = \{u \in W^{s,p}(\Omega;\mathbb{R}^\nu) : u(x) \in \mathcal{N} \text{ for almost every } x \in \Omega\}$$

# Reminder: classical Sobolev spaces

Let $s = k + \sigma$ with $k \in \mathbb{N}$ and $\sigma \in [0, 1)$.

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) \colon D^j u \in L^p(\Omega) \text{ for every } j \in \{1, \dots, k\}\}$$

If $\sigma \in (0, 1)$,

$$W^{\sigma,p}(\Omega) = \left\{u \in L^p(\Omega) \colon \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{m+\sigma p}} \, \mathrm{d}x\mathrm{d}y < +\infty\right\}.$$

If $k \geq 1$,

$$W^{s,p}(\Omega) = \{u \in W^{k,p}(\Omega) \colon D^k u \in W^{\sigma,p}(\Omega)\}.$$

# A few applications

Applications in problems from physics: liquid crystals ($\mathbb{S}^2$, $\mathbb{RP}^2$), supraconductivity (Ginzburg-Landau, $\mathbb{S}^1$), biaxial liquid crystals, superfluid helium...

Applications in problems from numerical methods: meshing domains.

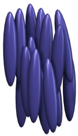

Figure: A field of liquid crystals

Figure: Meshing the earth (see the *Hextreme* project: `www.hextreme.eu`)

# The strong density problem

**Theorem**

$C^\infty(\overline{\Omega})$ is dense in $W^{s,p}(\Omega)$

**Question**

Is $C^\infty(\overline{\Omega}; \mathcal{N})$ dense in $W^{s,p}(\Omega; \mathcal{N})$?

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# A topological obstruction

For $1 \leq p < 2$, the map $u \in W^{1,p}(\mathbb{B}^2; \mathbb{S}^1)$ defined by

$$u(x) = \frac{x}{|x|}$$

cannot be approached by maps in $C^\infty(\overline{\mathbb{B}}^2; \mathbb{S}^1)$.

**Theorem (Schoen-Uhlenbeck (1983), Bethuel-Zheng (1988), Escobedo (1988))**

Assume that $sp < m$. If $C^\infty(\overline{\Omega}; \mathcal{N})$ is dense in $W^{s,p}(\Omega; \mathcal{N})$, then $\pi_{[sp]}(\mathcal{N}) = \{0\}$.

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# The easy case $sp \geq m$

**Proposition**

If $sp \geq m$, then $C^\infty(\overline{\Omega}; \mathcal{N})$ is dense in $W^{s,p}(\Omega; \mathcal{N})$.

There exists $\iota > 0$ such that the *nearest point projection* $\Pi\colon \mathcal{N} + B^{\nu}_{\iota} \longrightarrow \mathcal{N}$ is well-defined and smooth.

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# The strong density problem: $sp < m$

**Question**

Is $\mathcal{C}^{\infty}(\overline{\Omega}; \mathcal{N})$ dense in $W^{s,p}(\Omega; \mathcal{N})$ when $\pi_{[sp]}(\mathcal{N}) = \{0\}$?

Yes (when $\Omega = Q^m$) if

- $s = 1$ (Bethuel (1991));
- $s \in \mathbb{N}_*$ (Bousquet-Ponce-Van Schaftingen (2015));
- $0 < s < 1$ (Brezis-Mironescu (2015)).

Remaining case: $s \geq 1$, $s \notin \mathbb{N}_*$.

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# A complete answer to the density problem

**Theorem**

For every $1 \leq p < +\infty$ and $0 < s < +\infty$ such that $sp < m$, $C^\infty(\overline{Q}^m; \mathcal{N})$ is dense in $W^{s,p}(Q^m; \mathcal{N})$ if and only if $\pi_{[sp]}(\mathcal{N}) = \{0\}$.

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# Opening

Introduced by Brezis and Li (2001).

Let $v \in L^p(\mathbb{B}^m)$.
Can we modify $v$ to make it constant on $B^m_\eta$?

Idea: take $\Phi \colon \mathbb{B}^m \longrightarrow \mathbb{B}^m$ smooth, $\Phi = 0$ on $B^m_\eta$, $\Phi(x) = x$ on $\mathbb{B}^m \setminus B^m_{2\eta}$.
Consider $v^{\mathrm{op}}_\eta = v \circ \Phi$.
Then $v^{\mathrm{op}}_\eta$ is constant on $B^m_\eta$ and $v^{\mathrm{op}}_\eta = v$ on $\mathbb{B}^m \setminus B^m_{2\eta}$.

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Consider $v^{\mathrm{op}}_\eta = v \circ \Phi_a$, where $\Phi_a(x) = \Phi(x - a) + a$ for $a \in B^m_\eta$.
Then $v^{\mathrm{op}}_\eta$ is constant on $B^m_\eta$ and $v^{\mathrm{op}}_\eta = v$ on $\mathbb{B}^m \setminus B^m_{4\eta}$.

# Adaptative smoothing

Popularized by Schoen and Uhlenbeck (1982).

Let $\varphi \in C_c^\infty(\mathbb{B}^m)$ be a mollifier:

$$\varphi \geq 0 \quad \text{on } \mathbb{B}^m \quad \text{and} \quad \int_{\mathbb{B}^m} \varphi = 1.$$

Let $\psi \in C^\infty(\Omega)$ with $\psi > 0$.
For $v \in W^{s,p}(\Omega)$, define

$$\varphi_\psi * v(x) = \int_{\mathbb{B}^m} \varphi(z) v(x + \psi(x) z) \,\mathrm{d}z$$

$$= \frac{1}{\psi(x)^m} \int_{B^m_{\psi(x)}(x)} \varphi\left(\frac{y - x}{\psi(x)}\right) v(y) \,\mathrm{d}y.$$

# Thickening

Let $v \in C^\infty(\partial\mathbb{B}^m)$.
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# Shrinking

Let $v \in W^{1,p}(\mathbb{B}^m; \mathcal{N})$ and let $f \in \mathcal{C}^\infty(\mathbb{B}^m; \mathcal{N})$ be such that $v = f$ on $\mathbb{B}^m \setminus B^m_\mu$.
Can we obtain a better extension with control on the energy?

Idea: define

$$v_\tau(x) = \begin{cases} f(x) = v(x) & \text{if } |x| \geq 2\mu, \\ f\left(\frac{x}{\tau}\right) & \text{if } |x| \leq \tau\mu, \\ f\left(\frac{x}{|x|}\left(\frac{1}{2-\tau}(|x| - \tau\mu) + \mu\right)\right) & \text{otherwise.} \end{cases}$$

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Thank you for your attention!