A new decomposition for Borel measures dominated by the Hausdorff measure $\mathcal{H}^s$

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The Hausdorff measure $\mathcal{H}^s$

Let $0 \leq s < +\infty$ and $E \subset \mathbb{R}^N$.

Given $0 < \delta \leq +\infty$, we define the *Hausdorff capacity* $\mathcal{H}^s_\delta(E)$ by

$$\mathcal{H}^s_\delta(E) = \inf\left\{\sum_{n\in\mathbb{N}} \omega_s r_n^s \colon E \subset \bigcup_{n\in\mathbb{N}} B_{r_n}(x_n), 0 \leq r_n \leq \delta\right\}.$$

Here, $\omega_s = \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2}+1)}$.

The *Hausdorff content* is $\mathcal{H}^s_\infty(E)$.

We define the *Hausdorff measure* $\mathcal{H}^s(E)$ by

$$\mathcal{H}^s(E) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(E) = \sup_{\delta>0} \mathcal{H}^s_\delta(E).$$

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Why bother with restricting the size of the balls ?

Figure: Covering an arc of circle by balls
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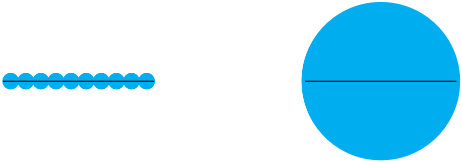

Figure: Covering a segment by balls
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The notion of s-straight sets

Definition (Foran (1995))

A Borel set $E \subset \mathbb{R}^N$ is said to be s-*straight* whenever

$$\mathcal{H}^s_\infty(E) = \mathcal{H}^s(E) < +\infty.$$

The decomposition theorem for Borel sets

Theorem (Delaware (2002))

If $E \subset \mathbb{R}^N$ is a Borel set of finite $\mathcal{H}^s$ measure, then there exists a sequence of disjoint Borel sets $(E_n)_{n\in\mathbb{N}}$ such that $E = \bigcup_{n\in\mathbb{N}} E_n$ and E_n is s-straight for each $n \in \mathbb{N}$.

From straight sets to measures

Proposition (Foran (1995))

If $E \subset \mathbb{R}^N$ is an s-straight Borel set, then every Borel set $A \subset E$ is also s-straight.

Proof.

$$\mathcal{H}^s(E) = \mathcal{H}^s(A) + \mathcal{H}^s(E \setminus A) \geq \mathcal{H}^s_\infty(A) + \mathcal{H}^s_\infty(E \setminus A) \geq \mathcal{H}^s_\infty(E) \geq \mathcal{H}^s(E)$$

Hence, all inequalities are actually equalities. □

Therefore, a Borel set $E \subset \mathbb{R}^N$ is s-straight if and only if $\mathcal{H}^s\lfloor_E \leq \mathcal{H}^s_\infty$.

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The condition $\mu \leq \mathcal{H}^s$ vs $\mu \leq \mathcal{H}^s_\infty$

If $\mu = \mathcal{H}^s\lfloor_E$, then $\mu \leq \mathcal{H}^s$.

On the other hand, $\mu\lfloor_{E_n} \leq \mathcal{H}^s_\infty$.

More generally, the condition $\mu \leq \mathcal{H}^s$ is satisfied by $\mu = f\mathcal{H}^s$ with $0 \leq f \leq 1$, or by $\mu = g\mathcal{H}^t$ with $t > s$ and $g \geq 0$.

On the other hand, $\mu \leq \mathcal{H}^s_\infty$ is equivalent to *an explicit density bound on* μ.

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From a measure inequality to a density estimate

Proposition

If $0 < \delta \leq +\infty$ and if μ is a Borel measure on $\mathbb{R}^N$, then $\mu \leq \mathcal{H}^s_\delta$ if and only if

$$\mu(B_r(x)) \leq \omega_s r^s \quad \textit{for every ball } B_r(x) \subset \mathbb{R}^N \textit{ with } 0 \leq r \leq \delta.$$

Proof.

If $\mu \leq \mathcal{H}^s_\delta$, then

$$\mu(B_r(x)) \leq \mathcal{H}^s_\delta(B_r(x)) \leq \omega_s r^s \quad \text{for every } 0 \leq r \leq \delta.$$

For the converse, let $E \subset \bigcup_{n\in\mathbb{N}} B_{r_n}(x_n)$ with $0 \leq r_n \leq \delta$. Then,

$$\mu(E) \leq \sum_{n\in\mathbb{N}} \mu(B_{r_n}(x_n)) \leq \sum_{n\in\mathbb{N}} \omega_s r_n^s. \qquad \square$$

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The decomposition theorem for Borel measures

Theorem (D. and Ponce (2023))

If μ is a finite Borel measure on $\mathbb{R}^N$ such that $\mu \leq \mathcal{H}^s$, then there exists a sequence of disjoint Borel sets $(E_n)_{n\in\mathbb{N}}$ such that $\mathbb{R}^N = \bigcup_{n\in\mathbb{N}} E_n$ and, for every $n \in \mathbb{N}$,

$$\mu\lfloor_{E_n} \leq \mathcal{H}^s_\infty.$$

An application: an existence result for a Dirichlet problem

Consider the problem

$$\begin{cases} -\Delta u + (e^u - 1) = \nu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where ν satisfies the condition

$$\nu \leq 4\pi\mathcal{H}^{N-2}. \tag{C}$$

Theorem (Vázquez (1983), Bartolucci, Leoni, Orsina, and Ponce (2005))

Let $N \geq 2$ and $\Omega \subset \mathbb{R}^N$ be a smooth bounded open set. If ν is a finite measure in Ω that satisfies (C), then there exists a function u in the Sobolev space $W_0^{1,1}(\Omega)$ such that $e^u \in L^1(\Omega)$ and

$$-\Delta u + (e^u - 1) = \nu \quad \text{in the sense of distributions in } \Omega.$$

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A first step: when the measure satisfies a density bound

Lemma

The theorem holds when $\nu \leq \alpha \mathcal{H}^{N-2}_{\delta}$ for some $\alpha < 4\pi$ and $0 < \delta \leq +\infty$.

This stronger assumption implies that

$$e^{\mathcal{N}\nu} \in L^1_{\mathrm{loc}}(\mathbb{R}^N).$$

First solve the regularized equation

$$\begin{cases} -\Delta u_k + (e^{u_k} - 1) = \rho_k * \nu & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega. \end{cases}$$

Then pass to the limit $k \to +\infty$ using properties of Dirichlet problems with an absorption term.

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A localized version of the decomposition theorem

Lemma

If $\mu \leq \mathcal{H}^s$, then there exist a non-increasing sequence of open sets $(U_j)_{j\in\mathbb{N}}$ and a sequence of positive numbers $(\delta_j)_{j\in\mathbb{N}}$ such that $\mu(U_j) \to 0$ and

$$\mu\lfloor_{\mathbb{R}^N\setminus U_j} \leq \mathcal{H}^s_{\delta_j} \quad \text{for every } j \in \mathbb{N}.$$

Pick $(\beta_j)_{j\in\mathbb{N}}$ increasing to 1, and solve the truncated equation

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Then pass to the limit $j \to +\infty$ using monotonicity.

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Then pass to the limit $j \to +\infty$ using monotonicity.

Thank you for your attention !

The crucial step to the proof: an extraction result

Proposition

If μ is a nonzero finite Borel measure on $\mathbb{R}^N$ satisfying $\mu \leq \mathcal{H}^s$, then there exists a Borel set $E \subset \mathbb{R}^N$ such that $\mu(E) > 0$ and $\mu\lfloor_E \leq \mathcal{H}^s_\infty$.

Proof of the main theorem: by exhaustion.

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Precise density bound

Proposition

If μ is a finite Borel measure on $\mathbb{R}^N$ satisfying $\mu \leq \mathcal{H}^s$, then, for every $\varepsilon > 0$, there exists a Borel set $A \subset \mathbb{R}^N$ such that

(1) *for every $\beta > 1$, there exists $\delta > 0$ such that $\mu\lfloor_A \leq \beta \mathcal{H}^s_\delta$,*

(2) $\mu(\mathbb{R}^N \setminus A) \leq \varepsilon$.

The construction of the extracted set E

- Pick well-chosen sequences $(\varepsilon_j)_{j\in\mathbb{N}}$ and $(r_j)_{j\in\mathbb{N}}$ decreasing to 0.
- For each $j \in \mathbb{N}_*$, partition

$$A = \bigcup_{i\in\mathbb{N}} A_{i,j},$$

with $\operatorname{diam} A_{i,j} \leq r_{j+1}$.

- Extract $E_{i,j} \subset A_{i,j}$ with

$$\mu(E_{i,j}) = (1 - \varepsilon_{j-1})\mu(A_{i,j}).$$

- Set

$$E = \bigcap_{j\in\mathbb{N}_*} \bigcup_{i\in\mathbb{N}} E_{i,j}.$$