Sobolev mappings to manifolds: when geometric analysis interacts with function spaces

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Sobolev spaces with values into manifolds

Let $\mathscr{N}$ be a smooth compact Riemannian manifold, isometrically embedded in $\mathbb{R}^\nu$.
Let $\Omega \subset \mathbb{R}^m$ be a smooth bounded open set, $1 \leq p < +\infty$, and $0 < s < +\infty$.

Definition

$$W^{s,p}(\Omega; \mathscr{N}) = \{u \in W^{s,p}(\Omega; \mathbb{R}^\nu) \colon u(x) \in \mathscr{N} \text{ for almost every } x \in \Omega\}$$

Reminder: classical Sobolev spaces

Let $s = k + \sigma$ with $k \in \mathbb{N}$ and $\sigma \in [0, 1)$.

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) \colon D^j u \in L^p(\Omega) \text{ for every } j \in \{1, \dots, k\}\}$$

If $\sigma \in (0, 1)$,

$$W^{\sigma,p}(\Omega) = \left\{u \in L^p(\Omega) \colon \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{m+\sigma p}} \,\mathrm{d}x \,\mathrm{d}y < +\infty\right\}.$$

If $k \geq 1$,

$$W^{s,p}(\Omega) = \{u \in W^{k,p}(\Omega) \colon D^k u \in W^{\sigma,p}(\Omega)\}.$$

A few applications

Applications in problems from physics: liquid crystals ($\mathbb{S}^2$, $\mathbb{RP}^2$), supraconductivity (Ginzburg-Landau, $\mathbb{S}^1$), biaxial liquid crystals, superfluid helium. . .

Applications in problems from numerical methods: meshing domains.

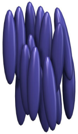

Figure: A field of liquid crystals

Figure: Meshing the earth (see the *Hextreme* project: `www.hextreme.eu`)

The strong density problem

Theorem

$C^\infty(\overline{\Omega})$ is dense in $W^{s,p}(\Omega)$

Question

Is $C^\infty(\overline{\Omega}; \mathscr{N})$ dense in $W^{s,p}(\Omega; \mathscr{N})$?

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A topological obstruction

For $2 \leq p < 3$, the map $u_0 \in W^{1,p}(\mathbb{B}^3; \mathbb{S}^2)$ defined by

$$u_0(x) = \frac{x}{|x|}$$

cannot be approached by maps in $C^\infty(\overline{\mathbb{B}^3}; \mathbb{S}^2)$.

A topological obstruction: proof

By contradiction: assume that $(u_n)_{n\in\mathbb{N}_*}$ in $C^\infty(\overline{\mathbb{B}^3};\mathbb{S}^2)$, $u_n \to u_0$ in $W^{1,p}$.

Genericity argument: up to extraction,

$$u_{n|\partial B^3_r} \xrightarrow{W^{1,p}} u_{0|\partial B^3_r} \quad \text{for a. e. } 0 < r < 1. \tag{1}$$

This comes from a Fubini–Tonelli-type argument:

$$\int_{\mathbb{B}^3} = \int_0^1 \left(\int_{\partial B^3_r} \right) \mathrm{d}r.$$

Now, Morrey–Sobolev implies that the convergence in (1) is uniform. But $u_{n|\partial B^3_r} \sim$ cte, while $u_{0|\partial B^3_r} \nsim$ cte, a contradiction.

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$$u_0(x) = \mathrm{id}_{\mathbb{S}^2}\left(\frac{x}{|x|}\right)$$

cannot be approached by maps in $C^\infty(\overline{\mathbb{B}^3}; \mathbb{S}^2)$.

Theorem (Schoen and Uhlenbeck (1983), Bethuel and Zheng (1988), Escobedo (1988))

Assume that $sp < m$. If $C^\infty(\overline{\Omega}; \mathscr{N})$ is dense in $W^{s,p}(\Omega; \mathscr{N})$, then $\pi_{\lfloor sp \rfloor}(\mathscr{N}) = \{0\}$.

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Theorem

If $sp < m$, then the class $C^\infty(\overline{\mathbb{B}^m}; \mathcal{N})$ is dense in $W^{s,p}(\mathbb{B}^m; \mathcal{N})$ if and only if $\pi_{\lfloor sp \rfloor}(\mathcal{N}) = \{0\}$.

- Case $s = 1$: Bethuel (1991), method of good and bad cubes;
- Case $0 < s < 1$: Brezis and Mironescu (2015), method of homogeneous extension;
- Case $s = 2, 3, \ldots$: Bousquet, Ponce, and Van Schaftingen (2015), method of good and bad cubes *plus* new tools for higher order spaces;
- Case $s > 1$ non-integer: D. (2023), method of good and bad cubes *plus* new tools for higher order spaces *plus* new ideas for fractional estimates.

The case of a general domain was understood by Hang and Lin (2003).

A simpler proof in a special case?

Assume that $\mathscr{N} = \mathbb{S}^1$. Exploit the fact that maps into the sphere have a phase.
Write $u = e^{i\theta}$, with $\theta \colon \mathbb{B}^m \to \mathbb{R}$.

By *classical density* theorem, get smooth maps $\theta_n \colon \mathbb{B}^m \to \mathbb{R}$ with $\theta_n \to \theta$.

By composition, $u_n = e^{i\theta_n} \to u$.

Two problems: (i) we need that Sobolev maps have a Sobolev phase, and (ii) we need continuity of the composition operator.

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A more general framework: coverings and liftings

Definition

We say that $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ is a *Riemannian covering* whenever, for every $x \in \mathcal{N}$, there exists an open neighborhood $U \subset \mathcal{N}$ of x such that $\pi^{-1}(U)$ is a disjoint union of open sets on which π restricts to an isometry.

Examples:

- $\pi : \mathbb{R}^n \to \mathbb{T}^n$, $\pi(\theta_1, \dots, \theta_n) = (e^{i\theta_1}, \dots, e^{i\theta_n})$;
- $\pi : \mathbb{S}^2 \to \mathbb{RP}^2$, $\pi(x) = [x]$.

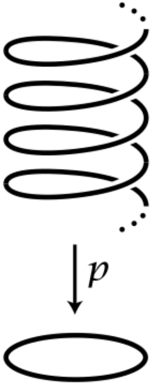

Figure: Hatcher (2002)

The lifting problem

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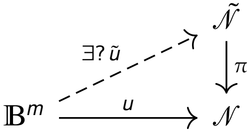
If $\pi\colon \widetilde{\mathcal{N}} \to \mathcal{N}$ is a Riemannian covering, then any continuous map $u\colon \mathbb{B}^m \to \mathcal{N}$ admits a continuous lifting $\tilde{u}\colon \mathbb{B}^m \to \widetilde{\mathcal{N}}$, i.e., such that $u = \pi \circ \tilde{u}$.



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Does every $u \in W^{s,p}(\mathbb{B}^m; \mathcal{N})$ have a lifting $\tilde{u} \in W^{s,p}(\mathbb{B}^m; \widetilde{\mathcal{N}})$?

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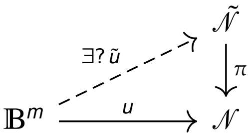

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Why do we care about liftings?

Going from $\mathscr{N}$ to $\widetilde{\mathscr{N}}$ allows to work with a *simpler* target.

A (non-exhaustive) list of applications:

- Energy bounds for Ginzburg–Landau (Bourgain, Brezis, and Mironescu (2000));
- Density problems and classification of homotopy classes for Sobolev mappings to the circle (Brezis, Mironescu (2001));
- Weak density in $W^{1,2}$ (Pakzad and Rivière (2003));
- Study of liquid crystals (Ball, Zarnescu (2011));
- Extension of traces (Mironescu and Van Schaftingen (2021)).

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The topological obstruction strikes back

For $1 \leq p < 2$, the map $u_0 \in W^{1,p}(\mathbb{B}^2; \mathbb{S}^1)$ defined by

$$u_0(x) = \frac{x}{|x|}$$

has no lifting $\tilde{u} \in W^{1,p}(\mathbb{B}^2; \mathbb{R})$.

By contradiction: assume it has a lifting $\tilde{u}_0 \in W^{1,p}(\mathbb{B}^2; \mathbb{R})$.

Genericity argument: for a.e. $0 < r < 1$, $\tilde{u}_{0|\partial B^2_r} \in W^{1,p}$ and is a lifting of $u_{0|\partial B^2_r}$.

This contradicts the fact that $\mathrm{id}_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{S}^1$ has no continuous lifting.

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Topological obstruction to lifting: the general case

Theorem (Bourgain, Brezis, and Mironescu (2000), Bethuel and Chiron (2007))

If $0 < s < +\infty$ and $1 \leq p < +\infty$ are such that $1 \leq sp < 2$, then there exists a map $u \in W^{s,p}(\mathbb{B}^m; \mathscr{N})$ that has no lifting $\tilde{u} \in W^{s,p}(\mathbb{B}^m; \widetilde{\mathscr{N}})$. (Assuming $m \geq 2$ and that the covering is non-trivial.)

Existence of the lifting when $s \geq 1$

Theorem (Bourgain, Brezis, and Mironescu (2000), Bethuel and Chiron (2007))

If $s \geq 1$ and $p \geq 1$ are such that $sp \geq 2$, then every $u \in W^{s,p}(\mathbb{B}^m; \mathscr{N})$ has a lifting $\tilde{u} \in W^{s,p}(\mathbb{B}^m; \widetilde{\mathscr{N}})$.

Here, $\widetilde{\mathscr{N}}$ is assumed to be embedded into $\mathbb{R}^{\tilde{\nu}}$. When $s > 1$, a mild assumption about this embedding is required.

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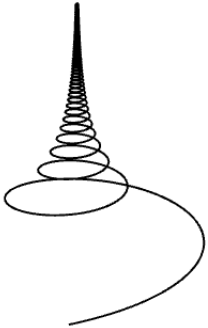
Why one should be careful about the embedding $\tilde{\mathscr{N}} \subset \mathbb{R}^{\tilde{\nu}}$



Define $j(t) = (t, \mathrm{e}^{-t}\cos \mathrm{e}^{t}, \mathrm{e}^{-t}\sin \mathrm{e}^{t})$.

Then, $\widetilde{\mathscr{N}} = j(\mathbb{R})$ is a covering of $\mathbb{S}^1$.

If $m > 2p$, then there exists a map $u \in W^{2,p}(\mathbb{B}^m; \mathbb{S}^1)$ which has no lifting $\tilde{u} \in W^{2,p}(\mathbb{B}^m; \widetilde{\mathscr{N}})$ (D. (2022)).

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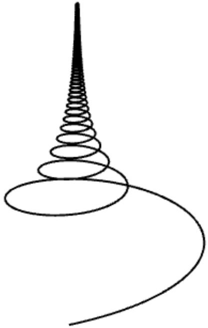

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The composition theorem

Theorem (Brezis and Mironescu (2001), Maz'ya and Shaposhnikova (2002))

Let $m = s$ if $s \in \mathbb{N}$, $m = \lfloor s \rfloor + 1$ otherwise. If f is a C^m function such that $f, f', \dots, f^m \in L^\infty$, then the operator $u \mapsto f(u)$ is continuous from $W^{s,p} \cap W^{1,sp}$ to $W^{s,p}$.

For $W^{s,p} \cap L^\infty$, this theorem was already known before (see the historical remarks in Brezis and Mironescu's paper).
However, for the application for lifting, asking $\tilde{u} \in L^\infty$ is too much.
On the other hand, $\tilde{u} \in W^{1,sp}$ is automatic if the lifting exists.

About the proof of the composition theorem I

An unsuccessful natural strategy

We have to prove that $\mathrm{D}(f(u)) = f'(u)\mathrm{D}u \in W^{s-1,p}$.

Assume $1 < s < 2$, let $s = 1 + \sigma$. We have $V = f'(u) \in W^{\sigma,\frac{1+\sigma}{\sigma}p} \cap L^{\infty}$ by interpolation, and $U = \mathrm{D}u \in W^{\sigma,p} \cap L^{(1+\sigma)p}$.

However, trying to prove that the product belongs to $W^{\sigma,p}$ by the standard trick

$$|U(x+h)V(x+h) - U(x)V(x)| \leq |U(x+h) - U(x)||V(x+h)| + |U(x)||V(x+h) - V(x)|$$

and Hölder leads to the *divergent* integral

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About the proof of the composition theorem II

A microscopic improvement in the Triebel–Lizorkin scale saves the game

Instead, we work with the refined Triebel–Lizorkin scale $F^s_{p,q}$.
In particular, $F^s_{p,p} = W^{s,p}$ for noninteger s, and $F^s_{p,q_1} \subset F^s_{p,q_2}$ if $q_1 \leq q_2$.

By a microscopic improvement in Gagliardo–Nirenberg inequality, we actually have $f'(u) \in F^{\sigma}_{\frac{1+\sigma}{\sigma}p,p}$.

This improvement is magnified by the Runst–Sickel lemma: if $0 < \frac{1}{p} = \frac{1}{p_1} + \frac{1}{r_2} = \frac{1}{p_2} + \frac{1}{r_1} < 1$, $f \in F^s_{p_1,q} \cap L^{r_1}$, $g \in F^s_{p_2,q} \cap L^{r_2}$, then $fg \in F^s_{p,q}$.

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Back to strong density: a rigorous proof at last

Assume that $s \geq 1$ and that $sp \geq 2$. Let $u \in W^{s,p}(\mathbb{B}^m; \mathbb{S}^1)$.

We have proved the existence of $\theta \in W^{s,p}(\mathbb{B}^m; \mathbb{R}) \cap W^{1,sp}(\mathbb{B}^m; \mathbb{R})$ such that $u = \mathrm{e}^{\mathrm{i}\theta}$.

By the classical density theorem, there exist $\theta_n \in C^\infty(\overline{\mathbb{B}^m}; \mathbb{R})$ such that $\theta_n \longrightarrow \theta$ in $W^{s,p} \cap W^{1,sp}$.

By the composition theorem, $\mathrm{e}^{\mathrm{i}\theta_n} \longrightarrow u$ in $W^{s,p}$.

Actually, the same argument also works for $\mathbb{T}^n$.

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Assume that $s \geq 1$ and that $sp \geq 2$. Let $u \in W^{s,p}(\mathbb{B}^m; \mathbb{S}^1)$.

We have proved the existence of $\theta \in W^{s,p}(\mathbb{B}^m; \mathbb{R}) \cap W^{1,sp}(\mathbb{B}^m; \mathbb{R})$ such that $u = \mathrm{e}^{\mathrm{i}\theta}$.

By the classical density theorem, there exist $\theta_n \in C^\infty(\overline{\mathbb{B}^m}; \mathbb{R})$ such that $\theta_n \longrightarrow \theta$ in $W^{s,p} \cap W^{1,sp}$.

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Thank you for your attention!