Density problems for Sobolev maps into manifolds

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Sobolev spaces with values into manifolds

Let $\mathscr{N}$ be a smooth compact Riemannian manifold, isometrically embedded in $\mathbb{R}^\nu$.
Let $\mathscr{M}$ be a smooth compact Riemannian manifold of dimension m, and $1 \leq p < +\infty$.

Definition

$$W^{1,p}(\mathscr{M}; \mathscr{N}) = \{u \in W^{1,p}(\mathscr{M}; \mathbb{R}^\nu) \colon u(x) \in \mathscr{N} \text{ for almost every } x \in \mathscr{M}\}$$

Applications in physics: liquid crystals ($\mathbb{S}^2$, $\mathbb{RP}^2$), supraconductivity (Ginzburg–Landau, $\mathbb{S}^1$), biaxial liquid crystals, superfluid helium...

Applications in numerical methods: meshing domains, see project *Hextreme*.

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The strong density problem

Theorem

The space $C^\infty(\mathscr{M})$ is dense in $W^{1,p}(\mathscr{M})$.

Question

Is $C^\infty(\mathscr{M};\mathscr{N})$ dense in $W^{1,p}(\mathscr{M};\mathscr{N})$?

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The strong density theorem

Theorem (Bethuel (1991))

Assume that $p < m$. Then, $C^\infty(\overline{\mathbb{B}^m}; \mathscr{N})$ is dense in $W^{1,p}(\mathbb{B}^m; \mathscr{N})$ if and only if $\pi_{\lfloor p \rfloor}(\mathscr{N}) = \{0\}$.

Extensions to $W^{s,p}$: Brezis and Mironescu (2015, $0 < s < 1$); Bousquet, Ponce, and Van Schaftingen (2015, $s = 2, 3, \ldots$); D. (2023, $s > 1$ noninteger).

The case where $\mathscr{M}$ is topologically non-trivial was explored by Hang and Lin (2003).

Typical obstruction: $u(x) = f(x/|x|)$, where $f : \mathbb{S}^{\lfloor p \rfloor} \to \mathscr{N}$ is not homotopic to a constant.

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Never think that a topic is exhausted!

Some of the (many) possible further questions:

- When $C^\infty(\mathscr{M};\mathscr{N})$ is not dense in $W^{1,p}(\mathscr{M};\mathscr{N})$, can we characterize $\overline{C^\infty(\mathscr{M};\mathscr{N})}^{W^{1,p}}$?
- Can we find a nice class of "almost smooth maps" that would always be dense?
- Do we recover density if we weaken the notion of convergence?

Characterizing the closure of smooth maps

A good starting point: the *Jacobian*.

Let $u \in W^{1,2}(\mathbb{B}^3; \mathbb{S}^2)$. We want to define $Ju = \mathrm{d}(u^\sharp \omega_{\mathbb{S}^2})$.

This is well-defined in the sense of distributions:

$$\langle Ju, \alpha \rangle = -\int_{\mathbb{B}^3} \mathrm{d}\alpha \wedge u^\sharp \omega_{\mathbb{S}^2} \quad \text{for every } \alpha \in C^\infty_c(\mathbb{B}^3).$$

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A Jacobian computation

By the Leibniz rule, away from 0,

$$\mathrm{d}(\alpha \wedge u^{\sharp}\omega_{\mathbb{S}^2}) = \mathrm{d}\alpha \wedge u^{\sharp}\omega_{\mathbb{S}^2} + (-1)^{*}\alpha \wedge \mathrm{d}(u^{\sharp}\omega_{\mathbb{S}^2}) = \mathrm{d}\alpha \wedge u^{\sharp}\omega_{\mathbb{S}^2}.$$

By Stokes's formula,

$$\langle Ju_0, \alpha\rangle = -\lim_{\varepsilon\to 0}\int_{\mathbb{B}^3\setminus B^3_\varepsilon} \mathrm{d}(\alpha \wedge u_0^{\sharp}\omega_{\mathbb{S}^2}) = \lim_{\varepsilon\to 0}\int_{\partial B^3_\varepsilon} \alpha \wedge u_0^{\sharp}\omega_{\mathbb{S}^2}.$$

Since α is smooth and u_0 is homogeneous,

$$\int_{\partial B^3_\varepsilon} \alpha \wedge u_0^{\sharp}\omega_{\mathbb{S}^2} \approx \alpha(0)\int_{\partial B^3_\varepsilon} u_0^{\sharp}\omega_{\mathbb{S}^2} = \alpha(0)\int_{\mathbb{S}^2}\omega_{\mathbb{S}^2} = 4\pi\alpha(0).$$

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Characterizing the closure of smooth maps with the Jacobian

We have computed that $Ju_0 = 4\pi\delta_0$.

On the other hand, if $u \in \overline{C^\infty(\overline{\mathbb{B}^3}; \mathbb{S}^2)}^{W^{1,2}}$, then $Ju = 0$.

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Possible ways of generalization

- Identify the topological singularities as the boundary of the graph of the map in the sense of distributions; see Giaquinta, Modica, Souček, and collaborators.
- Define Sing u as a flat chain with values into a group; see Pakzad and Rivière; and Canevari and Orlandi.
- Identify the topological singularities using *scans*; see Hardt and Rivière.
- Extension to $W^{s,p}$ with $0 < s < 1$; see Bourgain, Brezis, and Mironescu; Bousquet and Mironescu; Mucci; and D., Mironescu, and Xiao.

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The Jacobian of maps with fractional regularity by extension

A formal computation: if $U \in C^\infty(\mathbb{B}^3 \times (0,+\infty); \mathbb{R}^3)$ is an extension of u and $\tilde{\alpha}$ an extension of α, then

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Strategy: (i) show that this computation is valid for $u \in W^{1,p}$; (ii) show that the right-hand side makes sense and is continuous for $u \in W^{s,p}$; (iii) show that Ju indeed detects the closure of smooth maps.

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Thank you for your attention!