

A complete answer to the strong density problem for Sobolev spaces with values into manifolds

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Sobolev spaces with values into manifolds

Let \mathcal{N} be a smooth Riemannian manifold, compact, connected, without boundary, isometrically embedded in \mathbb{R}^{ν} .

Let $\Omega \subset \mathbb{R}^m$ be a smooth bounded open set, $1 \leq p < +\infty$, $0 < s < +\infty$.

Definition

$$W^{s,p}(\Omega; \mathcal{N}) = \{u \in W^{s,p}(\Omega; \mathbb{R}^{\nu}) : u(x) \in \mathcal{N} \text{ for almost every } x \in \Omega\}$$

Reminder: classical Sobolev spaces

Let $s = k + \sigma$ with $k \in \mathbb{N}$ and $\sigma \in [0, 1)$.

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^j u \in L^p(\Omega) \text{ for every } j \in \{1, \dots, k\}\}$$

If $\sigma \in (0, 1)$,

$$W^{\sigma,p}(\Omega) = \left\{ u \in L^p(\Omega) : \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{m+\sigma p}} dx dy < +\infty \right\}.$$

If $k \geq 1$,

$$W^{s,p}(\Omega) = \{u \in W^{k,p}(\Omega) : D^k u \in W^{\sigma,p}(\Omega)\}.$$

A few applications

Applications in problems from physics: liquid crystals (S^2 , \mathbb{RP}^2), supraconductivity (Ginzburg-Landau, S^1), biaxial liquid crystals, superfluid helium. . .

Applications in problems from numerical methods: meshing domains.



Figure: A field of liquid crystals



Figure: Meshing the earth (see the *Hextreme* project: www.hextreme.eu)

The strong density problem

Theorem

$C^\infty(\overline{\Omega})$ is dense in $W^{s,p}(\Omega)$

Question

Is $C^\infty(\overline{\Omega}; \mathcal{N})$ dense in $W^{s,p}(\Omega; \mathcal{N})$?

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A topological obstruction

For $1 \leq p < 2$, the map $u \in W^{1,p}(\mathbb{B}^2; \mathbb{S}^1)$ defined by

$$u(x) = \frac{x}{|x|}$$

cannot be approached by maps in $C^\infty(\overline{\mathbb{B}^2}; \mathbb{S}^1)$.

Theorem (Schoen-Uhlenbeck (1983), Bethuel-Zheng (1988), Escobedo (1988))

Assume that $sp < m$. If $C^\infty(\overline{\Omega}; \mathcal{N})$ is dense in $W^{s,p}(\Omega; \mathcal{N})$, then $\pi_{\lfloor sp \rfloor}(\mathcal{N}) = \{0\}$.

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The easy case $sp \geq m$

Proposition

If $sp \geq m$, then $C^\infty(\overline{\Omega}; \mathcal{N})$ is dense in $W^{s,p}(\Omega; \mathcal{N})$.

There exists $\iota > 0$ such that the *nearest point projection* $\Pi: \mathcal{N} + B_\iota^V \rightarrow \mathcal{N}$ is well-defined and smooth.

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The strong density problem: $sp < m$

Question

Is $C^\infty(\overline{\Omega}; \mathcal{N})$ dense in $W^{s,p}(\Omega; \mathcal{N})$ when $\pi_{[sp]}(\mathcal{N}) = \{0\}$?

Yes (when $\Omega = Q^m$) if

- $s = 1$ (Bethuel (1991));
- $s \in \mathbb{N}_*$ (Bousquet-Ponce-Van Schaftingen (2015));
- $0 < s < 1$ (Brezis-Mironescu (2015)).

Remaining case: $s \geq 1, s \notin \mathbb{N}_*$.

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A complete answer to the density problem

Theorem

For every $1 \leq p < +\infty$ and $0 < s < +\infty$ such that $sp < m$, $C^\infty(\overline{Q}^m; \mathcal{N})$ is dense in $W^{s,p}(Q^m; \mathcal{N})$ if and only if $\pi_{\lfloor sp \rfloor}(\mathcal{N}) = \{0\}$.

A unified proof of this result for the whole range $0 < s < +\infty$ *via* the method of good and bad cubes.

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Opening

Introduced by Brezis and Li (2001).

Let $v \in L^p(\mathbb{B}^m)$.

Can we modify v to make it constant on B_η^m ?

Idea: take $\Phi: \mathbb{B}^m \rightarrow \mathbb{B}^m$ smooth, $\Phi = 0$ on B_η^m , $\Phi(x) = x$ on $\mathbb{B}^m \setminus B_{2\eta}^m$.

Consider $v_\eta^{\text{op}} = v \circ \Phi$.

Then v_η^{op} is constant on B_η^m and $v_\eta^{\text{op}} = v$ on $\mathbb{B}^m \setminus B_{2\eta}^m$.

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Consider $v_\eta^{\text{op}} = v \circ \Phi_a$, where $\Phi_a(x) = \Phi(x - a) + a$ for $a \in B_\eta^m$.

Then v_η^{op} is constant on B_η^m and $v_\eta^{\text{op}} = v$ on $\mathbb{B}^m \setminus B_{4\eta}^m$.

Adaptative smoothing

Popularized by Schoen and Uhlenbeck (1982).

Let $\varphi \in C_c^\infty(\mathbb{B}^m)$ be a mollifier:

$$\varphi \geq 0 \quad \text{on } \mathbb{B}^m \quad \text{and} \quad \int_{\mathbb{B}^m} \varphi = 1.$$

Let $\psi \in C^\infty(\Omega)$ with $\psi > 0$.

For $v \in W^{s,p}(\Omega)$, define

$$\begin{aligned} \varphi_\psi * v(x) &= \int_{\mathbb{B}^m} \varphi(z) v(x + \psi(x)z) \, dz \\ &= \frac{1}{\psi(x)^m} \int_{B_{\psi(x)}^m(x)} \varphi\left(\frac{y-x}{\psi(x)}\right) v(y) \, dy. \end{aligned}$$

Thickening

Let $v \in C^\infty(\partial\mathbb{B}^m)$.

Can we propagate the values of v inside \mathbb{B}^m ?

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Shrinking

Let $v \in W^{1,p}(\mathbb{B}^m; \mathcal{N})$ and let $f \in C^\infty(\mathbb{B}^m; \mathcal{N})$ be such that $v = f$ on $\mathbb{B}^m \setminus B_\mu^m$.

Can we obtain a better extension with control on the energy?

Idea: define

$$v_\tau(x) = \begin{cases} f(x) = v(x) & \text{if } |x| \geq 2\mu, \\ f\left(\frac{x}{\tau}\right) & \text{if } |x| \leq \tau\mu, \\ f\left(\frac{x}{|x|} \left(\frac{1}{2-\tau}(|x| - \tau\mu) + \mu\right)\right) & \text{otherwise.} \end{cases}$$

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Thank you for your attention!