The strong density problem in Sobolev spaces of maps with values into manifolds

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Sobolev spaces with values into manifolds

Let $\mathscr{N}$ be a smooth Riemannian manifold, compact, connected, without boundary, isometrically embedded in $\mathbb{R}^\nu$.
Let $\Omega \subset \mathbb{R}^m$ be a smooth bounded open set, $1 \leq p < +\infty$, $0 < s < +\infty$.

Definition

$$W^{s,p}(\Omega;\mathscr{N}) = \{u \in W^{s,p}(\Omega;\mathbb{R}^\nu) : u(x) \in \mathscr{N} \text{ for almost every } x \in \Omega\}$$

Reminder: classical Sobolev spaces

Let $s = k + \sigma$ with $k \in \mathbb{N}$ and $\sigma \in [0, 1)$.

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) \colon D^j u \in L^p(\Omega) \text{ for every } j \in \{1, \dots, k\}\}$$

If $\sigma \in (0, 1)$,

$$W^{\sigma,p}(\Omega) = \left\{u \in L^p(\Omega) \colon \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{m+\sigma p}} \, \mathrm{d}x \, \mathrm{d}y < +\infty\right\}.$$

If $k \geq 1$,

$$W^{s,p}(\Omega) = \{u \in W^{k,p}(\Omega) \colon D^k u \in W^{\sigma,p}(\Omega)\}.$$

A few applications

Applications in problems from physics: liquid crystals ($\mathbb{S}^2$, $\mathbb{RP}^2$), supraconductivity (Ginzburg-Landau, $\mathbb{S}^1$), biaxial liquid crystals, superfluid helium...

Applications in problems from numerical methods: meshing domains.

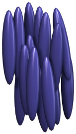

Figure: A field of liquid crystals

Figure: Meshing the earth (see the *Hextreme* project: `www.hextreme.eu`)

The strong density problem

Theorem

$C^\infty(\overline{\Omega})$ is dense in $W^{s,p}(\Omega)$

Question

Is $C^\infty(\overline{\Omega}; \mathscr{N})$ dense in $W^{s,p}(\Omega; \mathscr{N})$?

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A topological obstruction

For $1 \leq p < 2$, the map $u \in W^{1,p}(\mathbb{B}^2; \mathbb{S}^1)$ defined by

$$u(x) = \frac{x}{|x|}$$

cannot be approached by maps in $C^\infty(\overline{\mathbb{B}}^2; \mathbb{S}^1)$.

Theorem (Schoen-Uhlenbeck (1983), Bethuel-Zheng (1988), Escobedo (1988))

Assume that $sp < m$. If $C^\infty(\overline{\Omega}; \mathcal{N})$ is dense in $W^{s,p}(\Omega; \mathcal{N})$, then $\pi_{\lfloor sp \rfloor}(\mathcal{N}) = \{0\}$.

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The easy case $sp \geq m$

Proposition

If $sp \geq m$, then $C^\infty(\overline{\Omega}; \mathcal{N})$ is dense in $W^{s,p}(\Omega; \mathcal{N})$.

There exists $\iota > 0$ such that the *nearest point projection* $\Pi \colon \mathcal{N} + B^\nu_\iota \to \mathcal{N}$ is well-defined and smooth.

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The strong density problem: $sp < m$

Question

Is $C^\infty(\overline{\Omega}; \mathscr{N})$ dense in $W^{s,p}(\Omega; \mathscr{N})$ when $\pi_{\lfloor sp \rfloor}(\mathscr{N}) = \{0\}$?

Theorem

For every $1 \leq p < +\infty$ and $0 < s < +\infty$ such that $sp < m$, $C^\infty(\overline{Q}^m; \mathscr{N})$ is dense in $W^{s,p}(Q^m; \mathscr{N})$ if and only if $\pi_{\lfloor sp \rfloor}(\mathscr{N}) = \{0\}$.

Case of a general domain: Hang and Lin (2003).

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A class of almost smooth maps

Definition

The class $\mathscr{R}_i(Q^m;\mathscr{N})$ is the set of all maps u such that there exists a finite union of i-submanifolds $\mathcal{S} = \mathcal{S}_u \subset \mathbb{R}^m$ such that $u \in C^\infty(\overline{Q}^m \setminus \mathcal{S};\mathscr{N})$ and

$$|D^j u(x)| \leq C \frac{1}{\operatorname{dist}(x,\mathcal{S}_u)^j} \quad \text{for every } x \in Q^m \text{ and } j \in \mathbb{N}_*,$$

where $C > 0$ is a constant depending on u and j.

Theorem

For every $1 \leq p < +\infty$ and $0 < s < +\infty$ such that $sp < m$, $\mathscr{R}_{m-\lfloor sp \rfloor -1}(Q^m;\mathscr{N})$ is dense in $W^{s,p}(Q^m;\mathscr{N})$.

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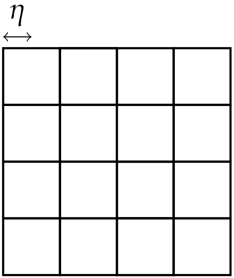
For every $1 \leq p < +\infty$ *and* $0 < s < +\infty$ *such that* $sp < m$, $C^{\infty}(\overline{Q}^m; \mathscr{N})$ *is dense in* $W^{s,p}(Q^m; \mathscr{N})$ *if and only if* $\pi_{\lfloor sp \rfloor}(\mathscr{N}) = \{0\}$, *and* $\mathscr{R}_{m-\lfloor sp \rfloor-1}(Q^m; \mathscr{N})$ *is always dense in* $W^{s,p}(Q^m; \mathscr{N})$.

- Case $s = 1$: Bethuel (1991), method of good and bad cubes.
- Case $0 < s < 1$: Brezis and Mironescu (2015), method of homogeneous extension.
- Case $s = 2, 3, \dots$: Bousquet, Ponce, and Van Schaftingen (2015), method of good and bad cubes + new tools.
- Case $s > 1$ non-integer: D. (2023), method of good and bad cubes + new tools + fractional estimates.

Good and bad cubes



$$\int_{\mathscr{K}_\eta^{m-1}} |Du|^p \lesssim \frac{1}{\eta} \int_Q |Du|^p$$

Good cubes: $\frac{1}{\eta^{m-p}} \int_\sigma |Du|^p \leq \kappa^p$, $\frac{1}{\eta^{m-p-1}} \int_{\partial\sigma} |Du|^p \leq \kappa^p$.

Bad cubes: $|\mathscr{B}_\eta^m| \lesssim \eta^p \kappa^{-p} \int_Q |Du|^p$.

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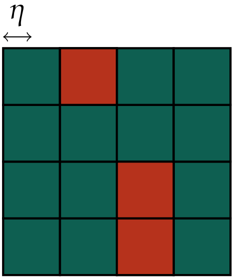

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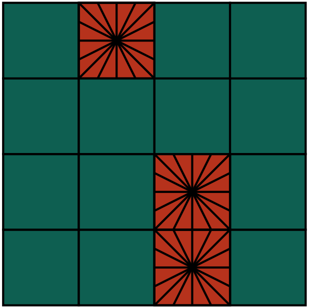
On bad cubes: homogeneous extension



Model case: $\sigma = Q_\eta$.

Define $u_\eta(x) = u\left(\eta \frac{x}{|x|_\infty}\right)$.

$$\int_{Q_\eta} |Du_\eta|^p \leq \frac{\eta}{m-p} \int_{\partial Q_\eta} |Du|^p$$

On good cubes: regularization by convolution

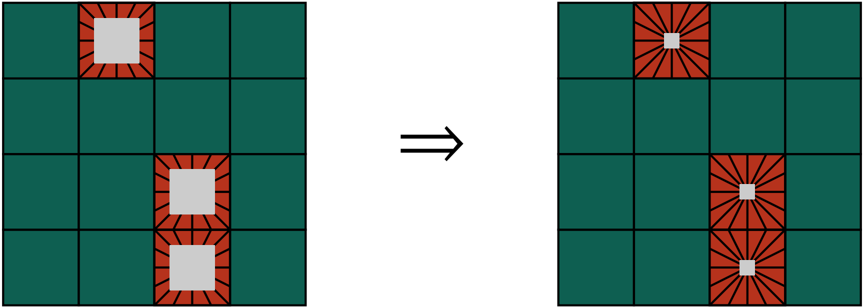
Lemma

There exists $w_\eta \in W^{1,p}(\mathscr{G}^m_\eta; \mathbb{R}^\nu)$ *such that*

1. *for every good cube* σ*, there exists* $y_\sigma \in \mathscr{N}$ *such that* $w_\eta \in B_\iota(y_\sigma)$ *on* σ*;*
2. $w_\eta = u$ *on* $\partial\sigma$*;*
3. *there exists* A_η *with* $|A_\eta| \lesssim \kappa^p \iota^{-p}$ *such that*

$$\int_{\mathscr{G}^m_\eta} |w_\eta - u|^p + \int_{\mathscr{G}^m_\eta} |Dw_\eta - Du|^p \lesssim \int_{A_\eta} |Du|^p.$$

Removing the singularities



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Can we improve the class $\mathscr{R}_i$?

Fact

If $\pi_{\lfloor sp \rfloor}(\mathscr{N}) \neq \{0\}$, then the class $\mathscr{R}_i(Q^m; \mathscr{N})$ is dense in $W^{s,p}(Q^m; \mathscr{N})$ if and only if $i = m - \lfloor sp \rfloor - 1$.

Definition

The class $\mathscr{R}_i^{\mathrm{uncr}}(Q^m; \mathscr{N})$ is the set of all $u \in \mathscr{R}_i(Q^m; \mathscr{N})$ such that the singular set $\mathscr{S}$ is a closedly embedded i-dimensional submanifold of $\mathbb{R}^m$.

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Density of the class $\mathscr{R}^{\mathrm{uncr}}$

Theorem (D. (2024))

Assume that $sp < m$. The class $\mathscr{R}^{\mathrm{uncr}}_{m-\lfloor sp \rfloor -1}(Q^m; \mathscr{N})$ is dense in $W^{s,p}(Q^m; \mathscr{N})$.

Previously known cases using the *method of singular projection* (Federer and Fleming (1960), adapted by Hardt and Lin (1987)):

- $W^{1,p}(\mathbb{B}^m; \mathbb{S}^{m-1})$ (Bethuel and Zheng (1988));
- $W^{\frac{1}{2},2}(\mathbb{S}^2; \mathbb{S}^1)$ (Rivière (2000));
- $W^{s,p}(\mathbb{S}^m; \mathbb{S}^{m-1})$ with $0 < s < 1$ and $sp < m$ (Bourgain, Brezis, and Mironescu (2005));
- $W^{s,p}(\mathbb{S}^m; \mathbb{S}^1)$ with $1 \leq sp < 2$ (Bousquet (2007));
- $(\lfloor sp \rfloor - 1)$-connected target with $0 < s < 1$ (Bousquet, Ponce, and Van Schaftingen (2014)).

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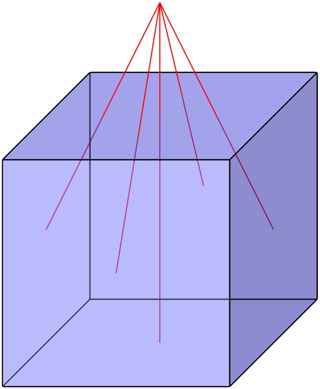
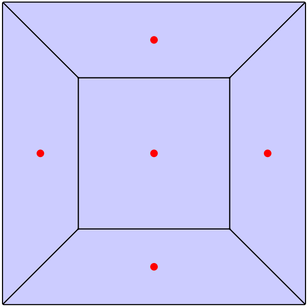
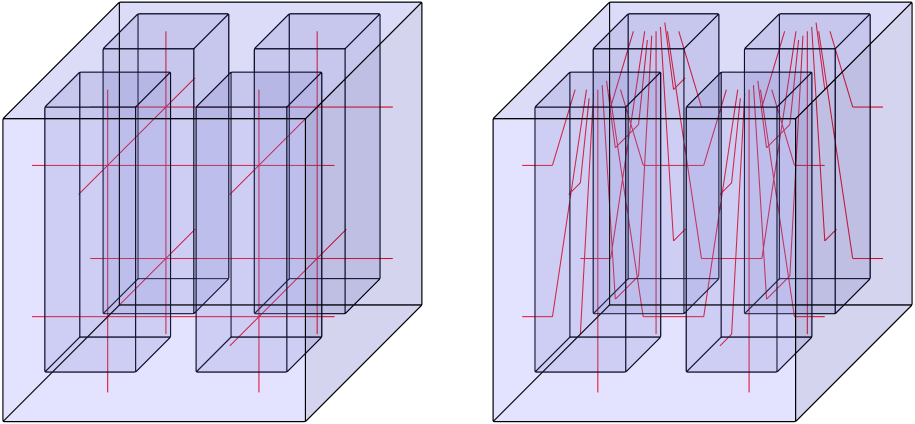
The crossings removal procedure: basic idea

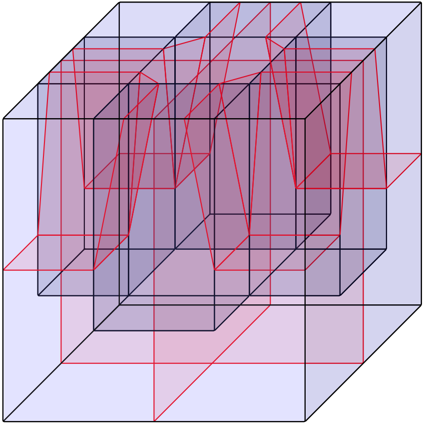


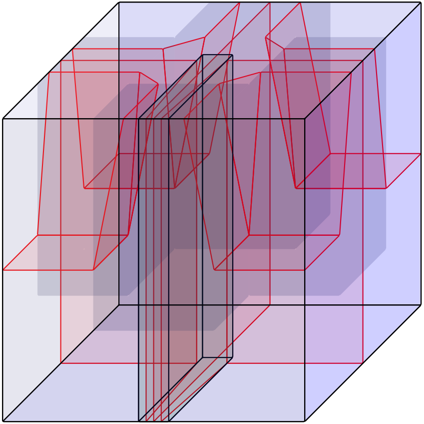


The crossings removal procedure: uncrossing lines



The crossings removal procedure: uncrossing planes





Thank you for your attention!