Harmonic maps into the sphere: how bad can they be?

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A classical problem

Reminder: the Dirichlet problem

$$\min\left\{\frac{1}{2}\int_{\mathbb{B}^3}|Du|^2 : u \in W^{1,2}(\mathbb{B}^3), u = \varphi \text{ on } \mathbb{B}^3 \text{ with } \varphi \in W^{\frac{1}{2},2}(\partial\mathbb{B}^3)\right\}.$$

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Has an important history: model problem of calculus of variations, motivated the study of Sobolev spaces among other.

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Important properties

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- there exists a minimizer (direct method of the calculus of variations);
- the minimizer is unique (strict convexity of the functional being minimized);
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A problem under a geometric constraint

We consider a variant of the Dirichlet problem:

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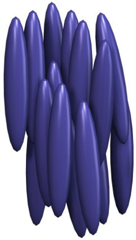

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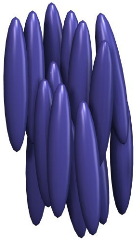

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Regularity fails I

Theorem

The mapping $u_0 \colon \mathbb{B}^3 \to \mathbb{S}^2$ defined by

$$u_0(x) = \frac{x}{|x|}$$

is the unique minimizer of the Dirichlet energy associated to the identity boundary datum.

Due to Brezis, Coron, and Lieb (1986). See also Hong (2001) and the references therein for a more general result and the history of the problem.

Regularity fails II

Even for zero degree boundary data, singularities can occur.

Theorem

Given $\varphi \in C^\infty(\mathbb{S}^2; \mathbb{S}^2)$, $\varepsilon > 0$, $1 \leq p < 2$, and $M \in \mathbb{N}$, there exists $\psi \in C^\infty(\mathbb{S}^2; \mathbb{S}^2)$ such that $\deg \varphi = \deg \psi$, $\|\varphi - \psi\|_{W^{1,p}(\mathbb{S}^2)} \leq \varepsilon$, $\varphi = \psi$ outside of $B_\varepsilon(q)$, and ψ admits a unique minimizer, with at least $\deg \varphi + M$ singularities.

Version due to Mazowiecka (2018), with ideas already in Almgren and Lieb (1988).

Regularity theory

It turns out that isolated singularities are the worst that may happen.

Theorem (Schoen, Uhlenbeck (1982, 1983))

The minimizers of the Dirichlet energy with values into the sphere are continuous (and even analytic) outside of a discrete set of singularities.
If the boundary datum is smooth, then minimizers are smooth in a neighborhood of the boundary.

Uniqueness fails

- There exists a planar boundary datum with at least two associated minimizers, one with values in each hemisphere (Hardt, Kinderlehrer (1988)).
- There exists a boundary datum with mirror symmetry with at least two associated minimizers without the mirror symmetry (Almgren, Lieb (1988)).
- There exists a boundary datum with at least two associated minimizers, one smooth and one not (Hardt, Lin (1989)).
- There exists a boundary datum with a continuous 1-parameter family of minimizers (Hardt, Kinderlehrer, Lin (1990)).

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A generic uniqueness theorem

Despite this, "most of" boundary data admit a unique associated minimizer.

Almgren's generic uniqueness theorem (Almgren, Lieb (1988))

Given $\varphi \in C^\infty(\mathbb{S}^2;\mathbb{S}^2)$ and $\varepsilon > 0$, there exists $\psi \in C^\infty(\mathbb{S}^2;\mathbb{S}^2)$ such that $\|\varphi - \psi\|_{W^{1,2}(\mathbb{S}^2)} \leq \varepsilon$ which has a unique associated minimizer. Moreover, $\varphi = \psi$ outside of $B_\varepsilon(q)$.

A generic non-uniqueness theorem

If we take $1 \leq p < 2$, it turns out that also "most of" boundary data exhibit non-uniqueness.

Theorem (D., Mazowiecka (2024))

Given $\varphi \in C^\infty(\mathbb{S}^2; \mathbb{S}^2)$, $\varepsilon > 0$, and $1 \leq p < 2$, there exists $\psi \in C^\infty(\mathbb{S}^2; \mathbb{S}^2)$ such that $\|\varphi - \psi\|_{W^{1,p}(\mathbb{S}^2)} \leq \varepsilon$ which has at least two associated minimizers with a different number of singularities.

Some useful results I

Stability theorem (Hardt, Lin (1989))

Let $\varphi \in W^{1,2}(\mathbb{S}^2;\mathbb{S}^2)$ that admits a unique associated minimizer. There exists $\delta > 0$ such that, if $\psi \in W^{1,2}(\mathbb{S}^2;\mathbb{S}^2)$ satisfies $\|\varphi - \psi\|_{W^{1,2}(\mathbb{S}^2;\mathbb{S}^2)} \leq \delta$, then every minimizer for ψ has the same number of singularities as the minimizer for φ.

Some useful results II

Strong convergence of minimizers (Almgren, Lieb (1988))

Let $(u_i)_{i\in\mathbb{N}}$ be a sequence of minimizers with boundary datum φ_i. Assume moreover that $(\varphi_i)_{i\in\mathbb{N}}$ is bounded in $W^{1,2}(\mathbb{S}^2)$. Then, up to extraction of a subsequence, $u_i \to u$ strongly in $W^{1,2}(\mathbb{B}^3)$, and u is a minimizer.

Some useful results III

Singularities are limits of singularities (Almgren, Lieb (1988))

Let $(u_i)_{i\in\mathbb{N}}$ be a sequence of minimizers that converges strongly in $W^{1,2}$ to a minimizer u. If u has a singularity at y, then for sufficiently large i, u_i has a singularity at some point y_i with $y_i \longrightarrow y$.

Some useful results IV

Singularities converge to singularities (Almgren, Lieb (1988))

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Uniform boundary regularity (Almgren, Lieb (1988))

Under control of the $W^{1,2}$ energy of the boundary datum on balls, there exists a uniform neighborhood of $\partial\mathbb{B}^3$ on which minimizers do not have singularities.

Uniform distance between singularities (Almgren, Lieb (1988))

There exists a universal constant $C > 0$ such that, if y is a singularity for a minimizer u, then u has no other singularity at distance less than $C\,\mathrm{dist}(y, \partial\mathbb{B}^3)$ from y.

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A key ingredient: a homotopy construction

Proposition

Let $\varphi \in C^\infty(\mathbb{S}^2; \mathbb{S}^2)$ and $1 \leq p < 2$. For every $r > 0$ and $q \in \mathbb{S}^2$, and for every $\psi \in C^\infty(\mathbb{S}^2; \mathbb{S}^2)$ homotopic to φ with $\varphi = \psi$ outside of $B_r(q)$, there exists a homotopy $H \in C^\infty(\mathbb{S}^2 \times [0,1]; \mathbb{S}^2)$ between φ and ψ such that

$$\sup_{0 \leq t \leq 1} \|\varphi - H_t\|_{W^{1,p}(\mathbb{S}^2)} \lesssim \|\varphi\|_{W^{1,p}(B_{2r}(q))} + \|\psi\|_{W^{1,p}(B_{2r}(q))}.$$

Open problems

1. In the homotopy construction, can one get $\sup_{0\leq t\leq 1}\|\varphi - H_t\|_{W^{1,p}(\mathbb{S}^2)} \leq \omega(\|\varphi-\psi\|_{W^{1,p}(\mathbb{S}^2)})$, with ω a suitable modulus of continuity?
2. Can one get genericity in the sense of Baire (for uniqueness or non-uniqueness) ?

Thank you for your attention!