A glimpse at the marvelous world of Sobolev mappings into manifolds

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Sobolev spaces with values into manifolds

Let $\mathcal{N}$ be a smooth compact Riemannian manifold, isometrically embedded in $\mathbb{R}^\nu$.
Let $\Omega \subset \mathbb{R}^m$ be a smooth bounded open set, and $1 \leq p < +\infty$.

Definition

$$W^{1,p}(\Omega;\mathcal{N}) = \{u \in W^{1,p}(\Omega;\mathbb{R}^\nu) \colon u(x) \in \mathcal{N} \text{ for almost every } x \in \Omega\}$$

Applications in physics: liquid crystals ($\mathbb{S}^2$, $\mathbb{RP}^2$), supraconductivity (Ginzburg-Landau, $\mathbb{S}^1$), biaxial liquid crystals, superfluid helium. . .

Applications in numerical methods: meshing domains, see project *Hextreme*.

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The strong density problem

Theorem

The space $C^\infty(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$.

Question

Is $C^\infty(\overline{\Omega}; \mathcal{N})$ dense in $W^{1,p}(\Omega; \mathcal{N})$?

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A topological obstruction

For $2 \leq p < 3$, the map $u_0 \in W^{1,p}(\mathbb{B}^3; \mathbb{S}^2)$ defined by

$$u_0(x) = \frac{x}{|x|}$$

cannot be approached by maps in $C^\infty(\overline{\mathbb{B}^3}; \mathbb{S}^2)$.

A topological obstruction: proof

By contradiction: assume that $(u_n)_{n\in\mathbb{N}_*}$ in $C^\infty(\overline{\mathbb{B}^3};\mathbb{S}^2)$, $u_n \to u_0$ in $W^{1,p}$.

Genericity argument: up to extraction,

$$u_{n|\partial B^3_r} \xrightarrow{W^{1,p}} u_{0|\partial B^3_r} \quad \text{for a. e. } 0 < r < 1. \tag{1}$$

This comes from a Fubini–Tonelli-type argument:

$$\int_{\mathbb{B}^3} = \int_0^1 \left(\int_{\partial B^3_r} \right) \mathrm{d}r.$$

Now, Morrey–Sobolev implies that the convergence in (1) is uniform. But $u_{n|\partial B^3_r} \sim$ cte, while $u_{0|\partial B^3_r} \nsim$ cte, a contradiction.

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cannot be approached by maps in $C^\infty(\overline{\mathbb{B}^3}; \mathbb{S}^2)$.

Theorem (Schoen-Uhlenbeck (1983), Bethuel-Zheng (1988))

Assume that $p < m$. If $C^\infty(\overline{\Omega}; \mathscr{N})$ is dense in $W^{1,p}(\Omega; \mathscr{N})$, then $\pi_{\lfloor p \rfloor}(\mathscr{N}) = \{0\}$.

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The strong density theorem

Theorem (Bethuel (1991))

Assume that $p < m$. *Then,* $C^{\infty}(\overline{\mathbb{B}^m}; \mathcal{N})$ *is dense in* $W^{1,p}(\mathbb{B}^m; \mathcal{N})$ *if and only if* $\pi_{\lfloor p \rfloor}(\mathcal{N}) = \{0\}$.

Extensions to $W^{s,p}$: Brezis and Mironescu (2015, $0 < s < 1$); Bousquet, Ponce, and Van Schaftingen (2015, $s = 2, 3, \ldots$); D. (2023, $s > 1$ noninteger).

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A new question

When $C^\infty(\overline{\Omega};\mathcal{N})$ is not dense in $W^{1,p}(\Omega;\mathcal{N})$, can we characterize $\overline{C^\infty(\overline{\Omega};\mathcal{N})}^{W^{1,p}}$?

A good starting point: the *Jacobian*.
Let $u \in W^{1,2}(\mathbb{B}^3;\mathbb{S}^2)$. We want to define $Ju = \mathrm{d}(u^\sharp\omega_{\mathbb{S}^2})$.
This is well-defined in the sense of distributions:

$$\langle Ju, \alpha\rangle = -\int_{\mathbb{B}^3} \mathrm{d}\alpha \wedge u^\sharp\omega_{\mathbb{S}^2} \quad \text{for every } \alpha \in C^\infty_c(\mathbb{B}^3).$$

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A Jacobian computation

By the Leibniz rule,

$$\mathrm{d}(\alpha \wedge u^\sharp \omega_{\mathbb{S}^2}) = \mathrm{d}\alpha \wedge u^\sharp \omega_{\mathbb{S}^2} + (-1)^* \alpha \wedge \mathrm{d}(u^\sharp \omega_{\mathbb{S}^2}) = \mathrm{d}\alpha \wedge u^\sharp \omega_{\mathbb{S}^2}.$$

By Stokes's formula,

$$\langle Ju_0, \alpha \rangle = -\lim_{\varepsilon \to 0} \int_{\mathbb{B}^3 \setminus B^3_\varepsilon} \mathrm{d}(\alpha \wedge u_0^\sharp \omega_{\mathbb{S}^2}) = \lim_{\varepsilon \to 0} \int_{\partial B^3_\varepsilon} \alpha \wedge u_0^\sharp \omega_{\mathbb{S}^2}.$$

Since α is smooth and u_0 is homogeneous,

$$\int_{\partial B^3_\varepsilon} \alpha \wedge u_0^\sharp \omega_{\mathbb{S}^2} \approx \alpha(0) \int_{\partial B^3_\varepsilon} u_0^\sharp \omega_{\mathbb{S}^2} = \alpha(0) \int_{\mathbb{S}^2} \omega_{\mathbb{S}^2} = 4\pi\alpha(0).$$

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Characterizing the closure of smooth maps with the Jacobian

We have computed that $Ju_0 = 4\pi\delta_0$.

On the other hand, if $u \in \overline{C^\infty(\overline{\mathbb{B}^3};\mathbb{S}^2)}^{W^{1,2}}$, then $Ju = 0$.

Theorem (Bethuel (1990))

$$\overline{C^\infty(\overline{\mathbb{B}^3};\mathbb{S}^2)}^{W^{1,2}} = \{u \in W^{1,2}(\mathbb{B}^3;\mathbb{S}^2) : Ju = 0\}$$

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Possible ways of generalization

- Identify the topological singularities as ∂G_u; see Giaquinta, Modica, and Souček.
- Define $\operatorname{Sing} u$ as a flat chain with values into a group; see Pakzad and Rivière.
- Identify the topological singularities using *scans*; see Hardt and Rivière.
- Extension to $W^{s,p}$ with $0 < s < 1$; see Bourgain, Brezis, and Mironescu; Bousquet and Mironescu; Mucci; and work in progress.

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The weak density problem

Definition

We say that $(u_n)_{n\in\mathbb{N}}$ in $W^{1,p}(\Omega;\mathscr{N})$ *converges weakly* to $u \in W^{1,p}(\Omega;\mathscr{N})$, and we write $u_n \rightharpoonup u$, whenever $u_n \to u$ almost everywhere and there exists $C > 0$ such that $\int_\Omega |Du_n|^p \leq C$ for every $n \in \mathbb{N}$.

Question

When is $C^\infty(\overline{\Omega};\mathscr{N})$ sequentially weakly dense in $W^{1,p}(\Omega;\mathscr{N})$?

A few results

- When $p \notin \mathbb{N}$, same as strong density.
- Always true when $\mathscr{N}$ is $(p-1)$-connected; see Hajłasz (1994).
- True for more general manifolds when $p = 2$; see Pakzad and Rivière (2003).
- Counterexample in $W^{1,3}(\mathbb{B}^4; \mathbb{S}^2)$; see Bethuel (2020).

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Thank you for your attention!