Analytical obstructions to the weak approximation of Sobolev mappings into manifolds

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The goal of our journey

Theorem (D. and Van Schaftingen (2024))

For every $p \in \mathbb{N} \setminus \{0, 1\}$, there exists a compact Riemannian manifold $\mathscr{N}$ such that, if $\dim \mathscr{M} > p$, then there exists a mapping $u \in W^{1,p}(\mathscr{M}; \mathscr{N})$ which is not a weak limit of a sequence of smooth mappings $\mathscr{M} \to \mathscr{N}$.

On our path:

- What is $W^{1,p}(\mathscr{M}; \mathscr{N})$ and why study it?
- Why we care about weak approximation?
- Why $p \in \mathbb{N}$?

Sobolev spaces with values into manifolds

Let $\mathscr{N}$ be a smooth compact Riemannian manifold, isometrically embedded in $\mathbb{R}^\nu$.
Let $\mathscr{M}$ be a smooth compact Riemannian manifold of dimension m and $1 \leq p < +\infty$.

Definition

$$W^{1,p}(\mathscr{M};\mathscr{N}) = \{u \in W^{1,p}(\mathscr{M};\mathbb{R}^\nu) : u(x) \in \mathscr{N} \text{ for almost every } x \in \mathscr{M}\}$$

A few applications

Applications in problems from physics: liquid crystals ($\mathbb{S}^2$, $\mathbb{RP}^2$), supraconductivity (Ginzburg–Landau, $\mathbb{S}^1$), biaxial liquid crystals, superfluid helium. . .

Applications in problems from numerical methods: meshing domains.

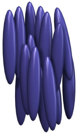

Figure: A field of liquid crystals

Figure: Meshing the earth (see the *Hextreme* project: `www.hextreme.eu`)

The strong density problem

Theorem

The space $C^\infty(\mathscr{M})$ is dense in $W^{1,p}(\mathscr{M})$.

Question

Is $C^\infty(\mathscr{M};\mathscr{N})$ dense in $W^{1,p}(\mathscr{M};\mathscr{N})$?

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A topological obstruction

For $2 \leq p < 3$, the map $u_0 \in W^{1,p}(\mathbb{B}^3; \mathbb{S}^2)$ defined by

$$u_0(x) = \frac{x}{|x|}$$

cannot be approached by maps in $C^\infty(\mathbb{B}^3; \mathbb{S}^2)$.

Theorem (Schoen and Uhlenbeck (1983), Bethuel and Zheng (1988))

Assume that $p < m$. If $C^\infty(\mathscr{M}; \mathscr{N})$ is dense in $W^{1,p}(\mathscr{M}; \mathscr{N})$, then $\pi_{\lfloor p \rfloor}(\mathscr{N}) = \{0\}$.

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The strong density theorem

Theorem (Bethuel (1991))

Assume that $p < m$. Then, $C^\infty(\mathbb{B}^m; \mathscr{N})$ is dense in $W^{1,p}(\mathbb{B}^m; \mathscr{N})$ if and only if $\pi_{\lfloor p \rfloor}(\mathscr{N}) = \{0\}$.

Extensions to $W^{s,p}$: Brezis and Mironescu (2015, $0 < s < 1$); Bousquet, Ponce, and Van Schaftingen (2015, $s = 2, 3, \ldots$); D. (2023, $s > 1$ noninteger).

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Let's be less demanding: weak approximation

We say that $(u_n)_{n\in\mathbb{N}}$ *weakly converges* to u in $W^{1,p}$, and we write $u_n \rightharpoonup u$, whenever $u_n \to u$ almost everywhere and

$$\sup_{n\in\mathbb{N}} \int_{\mathscr{M}} |Du_n|^p < +\infty.$$

Define

$$H^{1,p}_W(\mathscr{M};\mathscr{N}) = \{u \in W^{1,p}(\mathscr{M};\mathscr{N}) \colon \text{there exists } (u_n)_{n\in\mathbb{N}} \text{ in } C^\infty(\mathscr{M};\mathscr{N}) \text{ such that } u_n \rightharpoonup u\}.$$

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Does it hold that $H^{1,p}_W(\mathscr{M};\mathscr{N}) = W^{1,p}(\mathscr{M};\mathscr{N})$?

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Theorem (Bethuel (1991))

If $p \notin \mathbb{N}$, then $H^{1,p}_W(\mathscr{M}; \mathscr{N}) = H^{1,p}_S(\mathscr{M}; \mathscr{N})$.

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A new phenomenon: the case $p \in \mathbb{N}$

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More generally, we have:

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- $H^{1,p}_W(\mathscr{M}; \mathscr{N}) = W^{1,p}(\mathscr{M}; \mathscr{N})$ whenever $\mathscr{N}$ is $(p-1)$-connected (Hajłasz (1994));
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Obstructions strike back: the analytical obstruction

Theorem (Bethuel (2020))

If $m \geq 4$, then $H^{1,3}_W(\mathscr{M}; \mathbb{S}^2) \subsetneq W^{1,3}(\mathscr{M}; \mathbb{S}^2)$.

Global topological obstruction were already known (Hang and Lin (2003)).
Here, the obstruction is local: it arises already if $\mathscr{M} = \mathbb{B}^4$.

Ingredients involve: the Hopf invariant, Pontryagin construction, the theory of scans by Hardt and Rivière (2003), and branched optimal transportation.

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An analytical obstruction for every $p \in \mathbb{N} \setminus \{0, 1\}$

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Key procedure in the proof: superlinear energy growth

The *relaxed energy* is defined as

$$\mathcal{E}^{1,p}_{\mathrm{rel}}(u,\mathscr{M}) = \inf \liminf_{n\to+\infty} \int_{\mathscr{M}} |\mathrm{D}u_n|^p,$$

where the inf is over all sequences of $C^\infty(\mathscr{M};\mathscr{N})$ maps converging a.e. to u.

We construct a sequence $(u_n)_{n\in\mathbb{N}}$ such that

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A second family of analytical obstructions

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For every $n \in \mathbb{N}_*$, if $\dim \mathscr{M} > 4n-1$, then

$$H_W^{1,4n-1}(\mathscr{M};\mathbb{S}^{2n}) \subsetneq W^{1,4n-1}(\mathscr{M};\mathbb{S}^{2n}).$$

The key ingredient is a periodic construction using a Whitehead product.

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Thank you for your attention!