Density in Sobolev spaces to manifolds

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Sobolev spaces into manifolds

Let

- $1 \leq p < +\infty$;
- $0 < s < +\infty$;
- $\Omega \subset \mathbb{R}^m$ be a bounded open set;
- $\mathcal{N} \subset \mathbb{R}^\nu$ be a compact Riemannian manifold.

Definition

The *Sobolev space* of maps *with values into* $\mathcal{N}$ is defined by

$$W^{s,p}(\Omega;\mathcal{N}) = \{u \in W^{s,p}(\Omega;\mathbb{R}^\nu) : u(x) \in \mathcal{N} \text{ for a.e. } x \in \Omega\}.$$

The strong density problem

The classical density theorem

If Ω is sufficiently smooth, then $C^\infty(\overline{\Omega};\mathbb{R})$ is dense in $W^{s,p}(\Omega;\mathbb{R})$.

A natural question

Is $C^\infty(\overline{\Omega};\mathcal{N})$ dense in $W^{s,p}(\Omega;\mathcal{N})$?

The topological obstruction

The map

$$u(x) = \frac{x}{|x|} \in W^{1,p}(\mathbb{B}^2;\mathbb{S}^1) \quad (1 \leq p < 2)$$

cannot be approximated by smooth maps (Schoen and Uhlenbeck (1983)).
Proof: by a degree argument.

Generalized by Bethuel and Zheng (1988), and Escobedo (1988):

Necessary condition for density

If $\pi_{\lfloor sp \rfloor}(\mathcal{N}) \neq \{0\}$, then $C^\infty(\overline{\Omega};\mathcal{N})$ is not dense in $W^{s,p}(\Omega;\mathcal{N})$.

Here, $\pi_{\lfloor sp \rfloor}(\mathcal{N})$ is the $\lfloor sp \rfloor$-th homotopy group of $\mathcal{N}$.

A class of almost smooth maps

Definition

The class $\mathcal{R}_i(\Omega;\mathcal{N})$ is the set of all maps u such that there exists a finite union of i-submanifolds $\mathcal{S} = \mathcal{S}_u \subset \mathbb{R}^m$ such that $u \in C^\infty(\overline{\Omega} \setminus \mathcal{S};\mathcal{N})$ and

$$|D^j u(x)| \leq C \frac{1}{\operatorname{dist}(x,\mathcal{S})^j} \quad \text{for every } x \in \Omega \text{ and } j \in \mathbb{N}_*,$$

where $C > 0$ is a constant depending on u and j.

The strong density theorem

Theorem

Assume that $sp < m$.
The class $C^\infty(\overline{Q^m};\mathcal{N})$ is dense in $W^{s,p}(Q^m;\mathcal{N})$ if and only if $\pi_{\lfloor sp \rfloor}(\mathcal{N}) = \{0\}$.
The class $\mathcal{R}_{m-\lfloor sp \rfloor-1}(Q^m;\mathcal{N})$ is always dense in $W^{s,p}(Q^m;\mathcal{N})$.

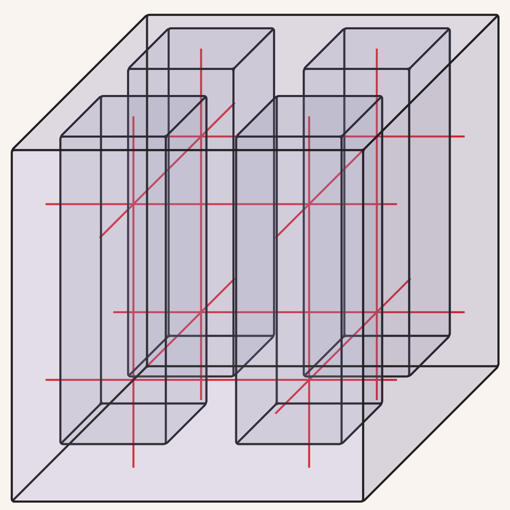
- Case $s = 1$: Bethuel (1991), method of good and bad cubes.
- Case $0 < s < 1$: Brezis and Mironescu (2015), method of homogeneous extension.
- Case $s = 2, 3, \ldots$: Bousquet, Ponce, and Van Schaftingen (2015), method of good and bad cubes + new tools.
- Case $s > 1$ non-integer: new, method of good and bad cubes + new tools + fractional estimates.

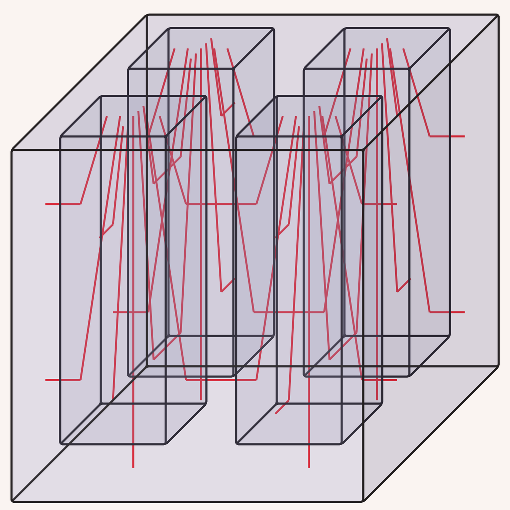
When Ω is more complex than the cube Q^m, the topology of the domain also plays a role (Hang and Lin (2003)).

Not the end of the story

Improving the class $\mathcal{R}$

Can we get $\mathcal{S}_u$ to be made of only one submanifold, that is, not to exhibit crossings?





Theorem: always true if $\Omega = Q^m$.

Which maps can be approximated?

When $\pi_{\lfloor sp \rfloor}(\mathcal{N}) \neq \{0\}$, can we characterize the closure of $C^\infty(\overline{Q^m};\mathcal{N})$ in $W^{s,p}(Q^m;\mathcal{N})$?
One good candidate: the *Jacobian*. Base idea: $Ju = \mathrm{d}(u^\sharp\omega)$.
Goal: $Ju = 0$ if and only if $u \in \overline{C^\infty(\overline{Q^m};\mathcal{N})}^{W^{s,p}}$.
See e.g. the work of Bethuel (1990); Bethuel, Coron, Demengel, and Helein (1991); Bourgain, Brezis, and Mironescu (2005); Bousquet (2007); Bousquet and Mironescu (2014); Mucci (2022); and the theory of scans by Hardt and Rivière.

Work in progress with Petru Mironescu and Kai Xiao.

The weak density problem

Is $C^\infty(\overline{\Omega};\mathcal{N})$ (sequentially) weakly dense in $W^{s,p}(\Omega;\mathcal{N})$?
Here, weak convergence means that $u_n \to u$ almost everywhere and $(u_n)_{n\in\mathbb{N}}$ is bounded in $W^{s,p}$.

When $sp \notin \mathbb{N}$, same necessary and sufficient condition as for strong condition.

- Always true in $W^{1,1}$; Pakzad (2003).
- Always true in $W^{1,2}$; Pakzad and Rivière (2003).
- $C^\infty(\overline{\mathbb{B}^4};\mathbb{S}^2)$ is not weakly dense in $W^{1,3}(\mathbb{B}^4;\mathbb{S}^2)$; Bethuel (2020).