The strong density problem for Sobolev spaces with values into manifolds

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Sobolev spaces into manifolds

Let

- $1 \leq p < +\infty$;
- $0 < s < +\infty$;
- $\Omega \subset \mathbb{R}^m$ be a bounded open set;
- $\mathcal{N} \subset \mathbb{R}^\nu$ be a compact Riemannian manifold.

Definition

The *Sobolev space* of maps *with values into* $\mathcal{N}$ is defined by

$$W^{s,p}(\Omega;\mathcal{N}) = \{u \in W^{s,p}(\Omega;\mathbb{R}^\nu) : u(x) \in \mathcal{N} \text{ for a.e. } x \in \Omega\}.$$

Applications

In physics:

- nematic liquid crystals ($\mathcal{N} = \mathbb{S}^2, \mathbb{RP}^2$);
- biaxial liquid crystals ($\mathcal{N} = \mathbb{S}^3/H$);
- Ginzburg–Landau theory ($\mathcal{N} = \mathbb{S}^1$).

In numerical methods:

- meshing domains;
- cross fields.

(See www.hextreme.eu.)

The strong density problem

The classical density theorem

If Ω is sufficiently smooth, then $C^\infty(\overline{\Omega};\mathbb{R})$ is dense in $W^{s,p}(\Omega;\mathbb{R})$.

A natural question

Is $C^\infty(\overline{\Omega};\mathcal{N})$ dense in $W^{s,p}(\Omega;\mathcal{N})$?

The easy case: $sp \geq m$

When $sp \geq m$, $C^\infty(\overline{\Omega};\mathcal{N})$ is always dense in $W^{s,p}(\Omega;\mathcal{N})$.

Proved by Schoen and Uhlenbeck (1983), clarified by Brezis and Nirenberg (1995) in the case $sp = m$.

Proof: $W^{s,p}$ maps are continuous (or VMO when $sp = m$).
One brings back to the classical case using the *nearest point projection*.

The topological obstruction

The map

$$u(x) = \frac{x}{|x|} \in W^{1,p}(\mathbb{B}^2;\mathbb{S}^1) \quad (1 \leq p < 2)$$

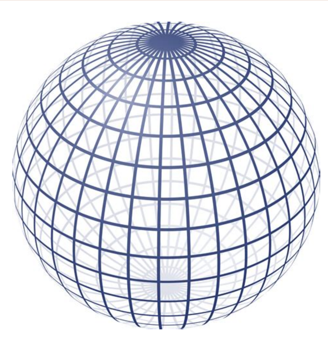
cannot be approximated by smooth maps (Schoen and Uhlenbeck (1983)).
Proof: by a degree argument.

Generalized by Bethuel and Zheng (1988), and Escobedo (1988): if $\pi_{[sp]}(\mathcal{N}) \neq \{0\}$, then $C^\infty(\overline{\Omega};\mathcal{N})$ is not dense in $W^{s,p}(\Omega;\mathcal{N})$.
Here, $\pi_{[sp]}(\mathcal{N})$ is the $[sp]$-th homotopy group of $\mathcal{N}$.

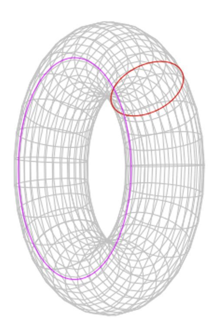
A complete answer to the strong density problem

Theorem

If $sp < m$, then $C^\infty(\overline{Q}^m;\mathcal{N})$ is dense in $W^{s,p}(Q^m;\mathcal{N})$ if and only if $\pi_{[sp]}(\mathcal{N}) = \{0\}$.



$\pi_1(\mathcal{N}) = \{0\}$



$\pi_1(\mathcal{N}) \neq \{0\}$

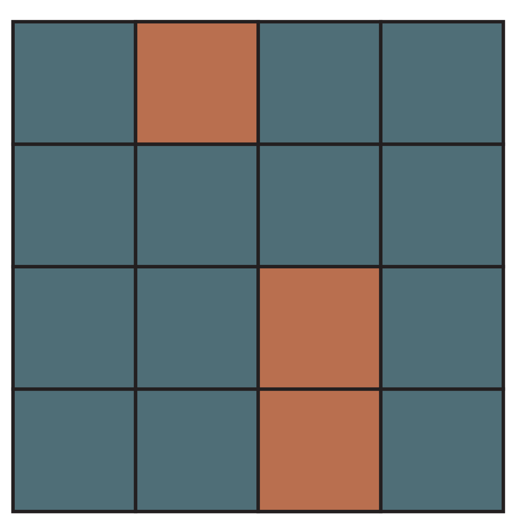
- Case $s = 1$: Bethuel (1991), method of good and bad cubes.
- Case $0 < s < 1$: Brezis and Mironescu (2015), method of homogeneous extension.
- Case $s = 2, 3, \dots$: Bousquet, Ponce, and Van Schaftingen (2015), method of good and bad cubes + new tools.
- Case $s > 1$ non-integer: new, method of good and bad cubes + new tools + fractional estimates.

When Ω is more complex than the cube Q^m, the topology of the domain also plays a role (Hang and Lin (2003)).

A unified proof covering the full range $0 < s < +\infty$

Approach from Bousquet, Ponce, and Van Schaftingen (2015), based on the method of good and bad cubes introduced by Bethuel (1991).
Start with $u \in W^{s,p}(Q^m;\mathcal{N})$.

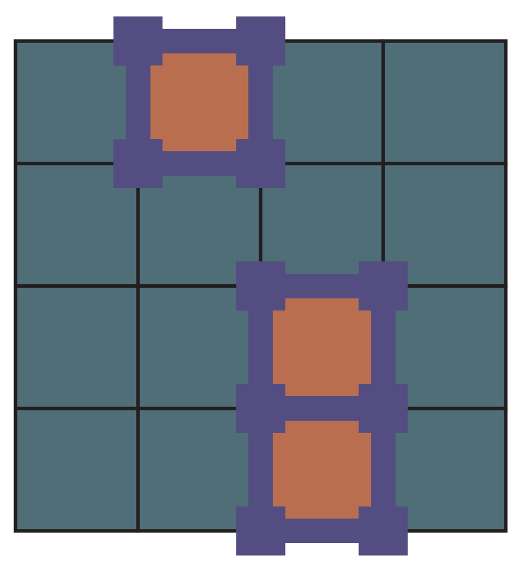
Good and bad cubes



Introduced by Bethuel for the case $s = 1$ (1991).

- The map u has low energy on good cubes.
- There are not too many bad cubes.

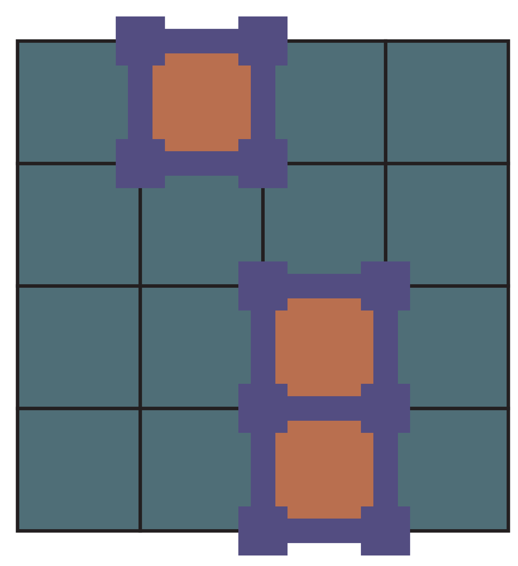
Opening



Introduced by Brezis and Li (2001).

- The map u becomes VMO in the blue region.

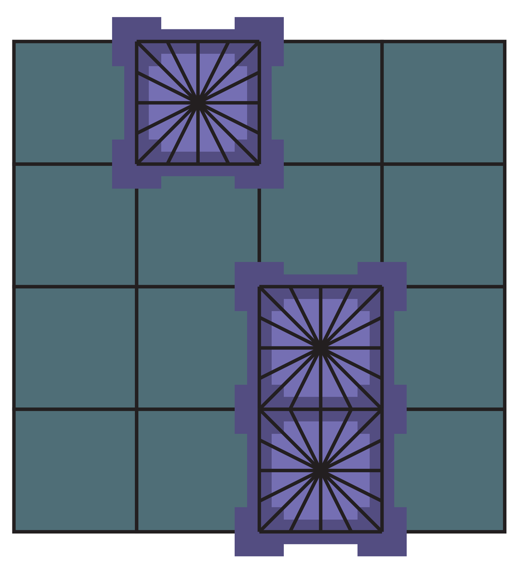
Adaptative smoothing



Popularized by Schoen and Uhlenbeck (1982).

- The map u becomes smooth everywhere.
- The map u is close to $\mathcal{N}$ outside of the red region.

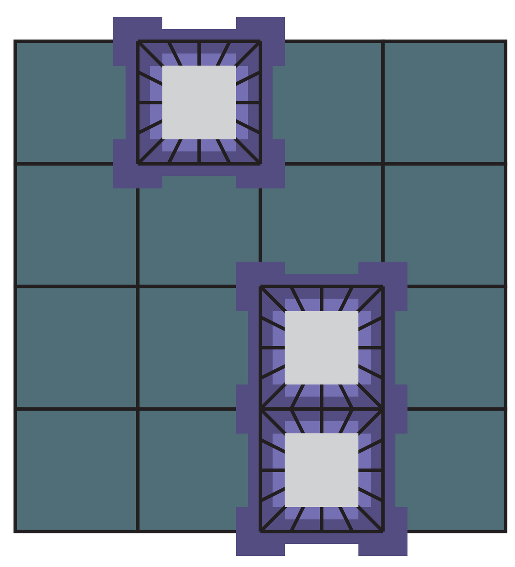
Thickening



Introduced by Bousquet, Ponce, and Van Schaftingen (2015) based on homogeneous extension.

- The map u becomes close to $\mathcal{N}$ everywhere.
- This process creates singularities.

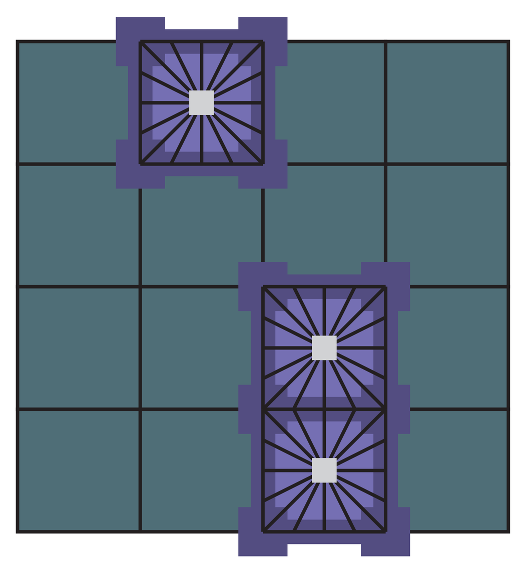
Topological extension



At this step, we use the assumption $\pi_{[sp]}(\mathcal{N}) = \{0\}$.

- The singularities created by thickening are removed.
- The construction comes with no control on the energy of the resulting map.

Shrinking



Introduced by Bousquet, Ponce, and Van Schaftingen (2015) based on a scaling argument already used by Bethuel for the case $s = 1$ (1991).

- Produces a better map with control of energy.