GENERIC NON-UNIQUENESS OF MINIMIZING HARMONIC MAPS FROM A BALL TO A SPHERE

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Abstract. In this note, we study non-uniqueness for minimizing harmonic maps from B^3 to $\mathbb{S}^2$. We show that every boundary map can be modified to a boundary map that admits multiple minimizers of the Dirichlet energy by a small $W^{1,p}$-change for $p < 2$. This strengthens a remark by the second-named author and Strzelecki. The main novel ingredient is a homotopy construction, which is the answer to an easier variant of a challenging question regarding the existence of a norm control for homotopies between $W^{1,p}$ maps.

Contents

1. Introduction

Minimizing harmonic maps from B^3 to $\mathbb{S}^2$ are defined as mappings with the least Dirichlet energy

$$E(u) \coloneqq \int_{B^3} |\nabla u|^2 \,\mathrm{d}x \tag{1.1}$$

among maps $u \in W^{1,2}(B^3, \mathbb{S}^2)$ with fixed boundary datum $u\big|_{\partial B^3} = \varphi \in W^{\frac{1}{2},2}(\partial B^3, \mathbb{S}^2)$. Here, we minimize in the class of Sobolev maps with values in a manifold (in our case, a sphere); for $s > 0$ and $p \geq 1$, this space is defined as

$$W^{s,p}(\mathcal{M}, \mathcal{N}) \coloneqq \{v \in W^{s,p}(\mathcal{M}, \mathbb{R}^L) \colon v(x) \in \mathcal{N} \text{ for a.e. } x \in \mathcal{M}\},$$

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where $\mathcal{N} \subset \mathbb{R}^L$ is a Riemannian manifold embedded into $\mathbb{R}^L$ (in our case, $\mathcal{N} = \mathbb{S}^2$) and $\mathcal{M}$ is a compact Riemannian manifold (in our case, $\mathcal{M} = B^3$ or $\mathcal{M} = \mathbb{S}^2$).

The space $W^{1,2}(B^3, \mathbb{S}^2)$ is not a linear space, but it is nevertheless a complete metric space endowed with the metric defined by

$$\operatorname{dist}(u, v) = \|u - v\|_{W^{1,2}(B^3)} .$$

We emphasize that, although being a subset of it, the class $W^{1,2}(B^3, \mathbb{S}^2)$ exhibits some striking qualitative differences with the linear space $W^{1,2}(B^3, \mathbb{R}^3)$. For example, not every mapping $u \in W^{1,2}(B^3, \mathbb{S}^2)$ can be approximated by smooth maps $u_i \in C^\infty(B^3, \mathbb{S}^2)$ in the strong topology of $W^{1,2}$; see [14, Section 4]. However, maps $\varphi \in W^{1,2}(\mathbb{S}^2, \mathbb{S}^2)$ can be approximated in $W^{1,2}$ by smooth maps $\varphi_i \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$; see [13, Section 3].

For $\varphi \in W^{\frac{1}{2},2}(\partial B^3, \mathbb{S}^2)$, we also define the space

$$W^{1,2}_\varphi(B^3, \mathbb{S}^2) \coloneqq \{v \in W^{1,2}(B^3, \mathbb{S}^2) \colon v = \varphi \text{ on } \partial B^3 \text{ in the trace sense}\}$$

and note that this space is always nonempty. For instance, for a given smooth boundary datum $\varphi \in C^\infty(\partial B^3, \mathbb{S}^2)$, one can easily construct an extension $u \in W^{1,2}(B^3, \mathbb{S}^2)$ of φ, simply by considering $u(x) = \varphi\big(\frac{x}{|x|}\big)$. More generally, any boundary map $\varphi \in W^{\frac{1}{2},2}(\partial B^3, \mathbb{S}^2)$ admits an extension $u \in W^{1,2}(B^3, \mathbb{S}^2)$; see [6, Theorem 6.2]. Once again, we emphasize that this is not an immediate consequence of the analogue property of linear Sobolev spaces. For example, there exists a boundary datum $\varphi \in W^{\frac{1}{2},2}(\partial B^3, \mathbb{S}^1)$ which has *no* extension $u \in W^{1,2}(B^3, \mathbb{S}^1)$; see [6, 6.3].

Minimizing harmonic maps satisfy the following system of Euler–Lagrange equations

$$\begin{cases} -\Delta u = |\nabla u|^2 u & \text{in } B^3, \\ u = \varphi & \text{on } \partial B^3. \end{cases} \tag{1.2}$$

It is known that for every non-constant boundary datum, the system (1.2) admits infinitely many solutions; see [12]. Minimizers of (1.1) are not the only solutions to (1.2) (see, e.g., [5, Section 3]). However, even in the class of minimizing harmonic maps, we do not have uniqueness for a given boundary datum $\varphi \colon B^3 \to \mathbb{S}^2$; there are many known examples. To list a few:

- in [3, Section 3], there is an example of a planar boundary datum which admits two different minimizers, one with values on the southern hemisphere and the other one with values on the northern hemisphere;
- in [4, 2.2. Corollary], there is an example of a boundary datum for which there exists a 1-parameter family of distinct energy minimizing maps;
- in [7, Section 5], there is an example of a boundary map which serves as a boundary datum for at least two minimizers, one singular and the other one regular;
- in [1, 5.5 Theorem], there is an example of a boundary datum with mirror symmetry for which there are at least two different minimizers without the mirror symmetry.

Nevertheless, in the class of minimizing harmonic maps, we have the following *generic uniqueness* result ([1] attributes this theorem to Almgren).

Theorem 1.1 ([1, Theorem 4.1])**.** *Let $\varphi \in W^{1,2}(\mathbb{S}^2, \mathbb{S}^2)$. For every $\varepsilon > 0$, there exists $\psi \in W^{1,2}(\mathbb{S}^2, \mathbb{S}^2)$ such that $\|\varphi - \psi\|_{W^{1,2}(\mathbb{S}^2)} < \varepsilon$ and for which there exists exactly one energy minimizer $u\colon B^3 \to \mathbb{S}^2$ having boundary datum ψ. Moreover, ψ coincides with φ outside of $B_\varepsilon(x) \cap \mathbb{S}^2$, for some $x \in \mathbb{S}^2$.*

In [11], the second-named author and Strzelecki suspected that *generic non-uniqueness* occurs, when taking into account small perturbation of the boundary datum in the topology of the space $W^{1,p}$ for $p < 2$. The main result of this note is the strengthening of [11, Remark 4.1].

Theorem 1.2. *Let $\varphi \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$. For every $\varepsilon > 0$, there exists $\psi \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ such that $\|\varphi - \psi\|_{W^{1,p}(\mathbb{S}^2,\mathbb{S}^2)} < \varepsilon$ which serves as a boundary datum for at least two energy minimizing maps from B^3 to $\mathbb{S}^2$ having a different number of singularities.*

Otherwise stated, Theorem 1.2 asserts that boundary data for which non-uniqueness occurs are dense in $W^{1,p}(\mathbb{S}^2, \mathbb{S}^2)$. This strengthens [7, Section 5] and [11, Remark 4.1], which provide existence of *one* boundary map for which non-uniqueness occurs. To be precise, as it is stated, Theorem 1.2 only asserts that boundary data subjected to non-uniqueness are dense in $C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ with respect to the $W^{1,p}$ topology. In turn, $C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ is dense in $W^{1,p}(\mathbb{S}^2, \mathbb{S}^2)$ (see e.g. [2, Theorem 1]), which ensures the density of boundary data for which non-uniqueness occurs in the whole $W^{1,p}(\mathbb{S}^2, \mathbb{S}^2)$.

Both Theorem 1.1 and Theorem 1.2 are in line with the *stability* results: On one hand, it is known that small perturbations of boundary data (for which there is a unique minimizer) in the $W^{1,2}$ norm do not change the number of singularities for corresponding minimizers (see [7] for perturbations in the $W^{1,\infty}$ norm, [10] and [8] for perturbations in the $W^{1,2}$ norm). On the other hand, small perturbations of the boundary datum in the $W^{1,p}$ norm for $p < 2$ can change the number of singularities for corresponding minimizers [11].

We prove Theorem 1.2 in Section 3. To do so, roughly speaking, we follow an example by Hardt–Lin [7, Section 5]. We start with any smooth boundary datum and use the construction of a boundary map (homotopic to the original one) of [11] (see [9] for necessary modifications) for which a *Lavrentiev gap phenomenon* occurs. In Section 2, we show that a homotopy between these two maps can be chosen small in $W^{1,p}$-norm for $p < 2$, which is the novelty of this note, and prove that within this homotopy, there is a boundary datum with the required properties.

As we explained, our key contribution in this note, which allows the transition from the existence to the density of boundary data where non-uniqueness occurs, is the homotopy construction presented in Section 2. We conclude this introduction with some extra comments concerning this construction.

Assume that one is given $1 \leq p < 2$ and two maps φ and $\psi \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ that have the same topological degree. Therefore, there exists a continuous, and even smooth homotopy connecting φ to ψ. A natural question is whether or not, knowing that φ and ψ are close with respect to the $W^{1,p}$ distance, one can choose the homotopy between φ and ψ to remain close to φ and ψ all along the deformation. More precisely, one could for instance expect that there exists a constant $C > 0$ depending on p such that a homotopy $H \in C^\infty(\mathbb{S}^2 \times [0,1], \mathbb{S}^2)$ between φ and ψ can be chosen so that

$$\|\varphi - H_t\|_{W^{1,p}(\mathbb{S}^2)} \leq C \|\varphi - \psi\|_{W^{1,p}(\mathbb{S}^2)} \quad \text{for every } 0 \leq t \leq 1. \tag{1.3}$$

Here, H_t stands for the map $H(\cdot, t)$. The question is already interesting if we assume in addition that φ and ψ coincide outside of a small disk. For instance, one could ask whether or not a homotopy such that (1.3) holds can be found under the additional assumption that $\varphi = \psi$ outside of a ball of radius r, for some $r > 0$ sufficiently small, possibly depending on the map φ that would be fixed in advance.

We are not able to solve this question, and a precise statement of the problem in a more general context is given as Open Problem 2.3. However, we are able to solve a weaker version of this problem, which is nevertheless sufficient for our purposes. Namely, we prove that, if the maps φ and ψ coincide outside of a small ball, then a smooth homotopy between them can be found such that $\|\varphi - H_t\|_{W^{1,p}(\mathbb{S}^2)}$ is controlled, not by the distance between φ and ψ, but by the sum of their norms on a neighborhood of the region where they differ. This is the content of the main result of Section 2, Proposition 2.1. This allows us to deduce that, for a fixed φ and a given $\varepsilon > 0$, one can choose the radius $r > 0$ sufficiently small such that, for any map ψ sufficiently close to φ such that $\varphi = \psi$ outside of $B_r(x)$, a homotopy H connecting φ to ψ can be found such that

$$\|\varphi - H_t\|_{W^{1,p}(\mathbb{S}^2)} \leq \varepsilon \quad \text{for every } 0 \leq t \leq 1;$$

see Corollary 2.2. This is sufficient to prove our main result, Theorem 1.2, but does not solve Open Problem 2.3, as in our proof the radius $r > 0$ of the ball outside of which the maps φ and ψ are required to coincide has to depend on ε, ruling out the possibility of controlling $\|\varphi - H_t\|_{W^{1,p}(\mathbb{S}^2)}$ uniformly in t solely by $\|\varphi - \psi\|_{W^{1,p}(\mathbb{S}^2)}$ with our argument.

Notation. We denote by B^3 a Euclidean unit ball in $\mathbb{R}^3$. We will write $\mathbb{S}^n$ for the unit n-dimensional sphere. For a point $x \in \mathbb{S}^n$ and $r > 0$, we will write $B_r(x)$ for a geodesic ball of radius r around x. We will write $A \precsim B$ whenever there is a constant C (independent of all crucial quantities) such that $A \leq CB$. Throughout this paper, the term *minimizer* will always refer to an $\mathbb{S}^2$-valued mapping minimizing the Dirichlet energy with given boundary datum.

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2. Homotopy construction

We will assume in this section that $\mathcal{N}$ is a (non necessarily compact) Riemannian manifold. We work on the sphere $\mathbb{S}^n$, but the result may be readily extended to an arbitrary domain, either an open subset of $\mathbb{R}^n$ or a Riemannian manifold $\mathcal{M}$ of dimension n. We also always assume that $p < n$.

Proposition 2.1. *Let $\varphi \in C^\infty(\mathbb{S}^n, \mathcal{N})$ and $p < n$. For every $r > 0$, for every $x \in \mathbb{S}^n$, and every $\psi \in C^\infty(\mathbb{S}^n, \mathcal{N})$ homotopic to φ and satisfying $\varphi = \psi$ on $\mathbb{S}^n \setminus B_r(x)$, there exists a homotopy $H \in C^\infty(\mathbb{S}^n \times [0,1], \mathcal{N})$ from φ to ψ such that*

$$\sup_{0\le t\le 1} \|\varphi - H_t\|_{W^{1,p}(\mathbb{S}^n)} \le C \left(\|\varphi\|_{W^{1,p}(B_{2r}(x))} + \|\psi\|_{W^{1,p}(B_{2r}(x))} \right),$$

for some constant $C > 0$ depending only on n and p.

This proposition can be used in combination with Lebesgue's lemma to obtain a homotopy which remains close to φ in $W^{1,p}$. Indeed, choosing r sufficiently small, depending on φ, we may ensure that $\|\varphi\|_{W^{1,p}(B_{2r}(x))}$ is as small as we want, uniformly with respect to r. Since $\|\psi\|_{W^{1,p}(B_{2r}(x))} \le \|\varphi\|_{W^{1,p}(B_{2r}(x))} + \|\varphi - \psi\|_{W^{1,p}(\mathbb{S}^n)}$, assuming in addition that $\|\varphi - \psi\|_{W^{1,p}(\mathbb{S}^n)}$ is small, we can make $\sup_{0\le t\le 1} \|\varphi - H_t\|_{W^{1,p}(\mathbb{S}^n)}$ as small as we want. This yields the following corollary.

Corollary 2.2. *Let $\varphi \in C^\infty(\mathbb{S}^n, \mathcal{N})$ and $p < n$. For every $\varepsilon > 0$, there exists $r > 0$ sufficiently small, depending on φ, and there exists $\delta > 0$ such that, for every $x \in \mathbb{S}^n$ and every $\psi \in C^\infty(\mathbb{S}^n, \mathcal{N})$ homotopic to φ and satisfying $\varphi = \psi$ on $\mathbb{S}^n \setminus B_r(x)$ and $\|\varphi - \psi\|_{W^{1,p}(\mathbb{S}^n)} \le \delta$, there exists a homotopy $H \in C^\infty(\mathbb{S}^n \times [0,1], \mathcal{N})$ from φ to ψ such that*

$$\sup_{0\le t\le 1} \|\varphi - H_t\|_{W^{1,p}(\mathbb{S}^n)} \le \varepsilon.$$

Proof of Proposition 2.1. Let $G \in C^\infty(\mathbb{S}^n \times [0,1], \mathcal{N})$ be any homotopy connecting φ to ψ with $G_0 = \varphi$ and $G_1 = \psi$. Since $\varphi = \psi$ outside of $B_r(x)$, we may assume that G is stationary outside of $B_r(x)$, i.e., for each $t \in [0,1]$, we have $G_t = \varphi = \psi$ on $\mathbb{S}^n \setminus B_r(x)$. This claim can be proved with a by-hand construction, that we sketch below. We denote by $\hat{x}$ the point at the antipode of x. Let $\Psi \colon \mathbb{S}^n \to \mathbb{S}^n$ be a smooth map such that $\Psi = \mathrm{id}$ outside of $B_r(x)$, and such that Ψ maps the annulus $B_r(x) \setminus \overline{B}_{r/2}(x)$ diffeomorphically onto the annulus $\mathbb{S}^n \setminus (\overline{B}_r(x) \cup \{\hat{x}\})$, the circle $\partial B_{r/2}(x)$ onto $\{\hat{x}\}$, and the ball $B_{r/2}(x)$ diffeomorphically onto $\mathbb{S}^n \setminus \{\hat{x}\}$. It is readily observed that $\Psi \sim \mathrm{id}$, through a homotopy stationary outside of $B_r(x)$. Therefore, the maps $u \circ \Psi$ and $v \circ \Psi$ are homotopic to u and

v respectively, through a homotopy stationary outside of $B_r(x)$. Now, given a homotopy G' connecting u to v, a homotopy G'' connecting $u \circ \Psi$ to $v \circ \Psi$ can be constructed by prescribing that G'' is stationary outside of B_r, by letting $G''_t = G'_t \circ \Psi$ on $\overline{B}_{r/2}$ — which corresponds to rescaling G' from $\mathbb{S}^n \setminus \{\hat{x}\}$ to $B_r(x)$ — and extending smoothly on the annulus $B_r \setminus \overline{B}_{r/2}$. This is readily done by combining the observations that (i) $u \circ \Psi$ and $v \circ \Psi$ coincide also on $B_r \setminus B_{r/2}(x)$ and are constant on $\partial B_{r/2}(x)$, and (ii) G''_t is constant on $\partial B_{r/2}(x)$. The required homotopy G stationary outside of $B_r(x)$ is then obtained by patching the three above homotopies, from u to $u \circ \Psi$, from $u \circ \Psi$ to $v \circ \Psi$, and from $v \circ \Psi$ to v.

Consider $\tau > 0$, which will be chosen sufficiently small at a later stage. We are going to rescale G, φ, and ψ from $B_r(x)$ to a smaller ball $B_\tau(x)$, while keeping them unchanged outside of $B_{2r}(x)$. More specifically, let $(\Phi_t)_{0 \leq t \leq 1}$ be a family of smooth diffeomorphisms of $\mathbb{S}^n$ such that $\Phi_t = \mathrm{id}$ outside of $B_{2r}(x)$ and such that, on $B_{2r}(x)$, in the local chart given by the exponential map around x, Φ_t is expressed as

$$\begin{cases} \frac{rx}{(1-t)r+t\tau} & \text{if } |x| \leq (1-t)\,r + t\tau, \\ \frac{x}{|x|}\left(\frac{r}{2r-(1-t)r-t\tau}\left(|x| - (1-t)\,r - t\tau\right) + r\right) & \text{if } (1-t)\,r + t\tau \leq |x| \leq 2r. \end{cases}$$

We define $H \in C^\infty(\mathbb{S}^n \times [0,1], \mathcal{N})$ by

$$H_t \coloneqq \begin{cases} \varphi \circ \Phi_{3t} & \text{if } 0 \leq t \leq \frac{1}{3}, \\ G_{3(t-1/3)} \circ \Phi_1 & \text{if } \frac{1}{3} \leq t \leq \frac{2}{3}, \\ \psi \circ \Phi_{1-3(t-2/3)} & \text{if } \frac{2}{3} \leq t \leq 1. \end{cases}$$

Of course, H is a homotopy from φ to ψ. It remains to show that, if $\tau > 0$ is suitably small, then H satisfies the required estimate.

For $0 \leq t \leq \frac{1}{3}$, we note that $\varphi - H_t = 0$ outside $B_{2r}(x)$. We readily obtain bounds on the Jacobian and the derivatives of Φ_t, so that the change of variable theorem combined with $n - p > 0$ implies that

$$\|\varphi - H_t\|_{W^{1,p}(\mathbb{S}^n)} \leq \|\varphi\|_{W^{1,p}(B_{2r}(x))} + \|\varphi \circ \Phi_{3t}\|_{W^{1,p}(B_{2r}(x))} \lesssim \|\varphi\|_{W^{1,p}(B_{2r}(x))}.$$

Similarly, for $\frac{2}{3} \leq t \leq 1$, we have

$$\|\varphi - H_t\|_{W^{1,p}(\mathbb{S}^n)} \leq \|\varphi\|_{W^{1,p}(B_{2r}(x))} + \|\psi \circ \Phi_{3t}\|_{W^{1,p}(B_{2r}(x))} \lesssim \|\varphi\|_{W^{1,p}(B_{2r}(x))} + \|\psi\|_{W^{1,p}(B_{2r}(x))}.$$

Concerning $\frac{1}{3} \leq t \leq \frac{2}{3}$, we estimate

$$\begin{aligned} \|\varphi - H_t\|_{W^{1,p}(\mathbb{S}^n)} &\leq \|\varphi\|_{W^{1,p}(B_{2r}(x))} + \left\|G_{3(t-1/3)} \circ \Phi_1\right\|_{W^{1,p}(B_{2r}(x))} \\ &\lesssim \|\varphi\|_{W^{1,p}(B_{2r}(x))} + \left\|G_{3(t-1/3)}\right\|_{W^{1,p}(B_{2r}(x)\setminus B_r(x))} + \tau^{\frac{n-p}{p}} \left\|G_{3(t-1/3)}\right\|_{W^{1,p}(B_{2r}(x))}. \end{aligned}$$

Since the homotopy G has been assumed to be stationary outside of $B_r(x)$, we know that $\left\|G_{3(t-1/3)}\right\|_{W^{1,p}(B_{2r}(x)\setminus B_r(x))} = \|\varphi\|_{W^{1,p}(B_{2r}(x)\setminus B_r(x))}$. On the other hand, by compactness, we have

$$\sup_{0\le t\le 1} \|G_t\|_{W^{1,p}(B_{2r}(x))} \le C_1$$

for some possibly large constant $C_1 > 0$. We may assume that either $\|\varphi\|_{W^{1,p}(B_{2r}(x))} \neq 0$ or $\|\psi\|_{W^{1,p}(B_{2r}(x))} \neq 0$. Indeed, if $\|\varphi\|_{W^{1,p}(B_{2r}(x))} = 0 = \|\psi\|_{W^{1,p}(B_{2r}(x))}$, this implies that both φ and ψ are identically zero — note that this may only happen if $0 \in \mathcal{N}$ — and we may directly conclude by choosing H to be constantly zero. As $p < n$, we may therefore choose $\tau > 0$ sufficiently small, depending on C_1, so that

$$\tau^{\frac{n-p}{p}} \left\|G_{3(t-1/3)}\right\|_{W^{1,p}(B_{2r}(x))} \le \|\varphi\|_{W^{1,p}(B_{2r}(x))} + \|\psi\|_{W^{1,p}(B_{2r}(x))} \quad \text{for every } \frac{1}{3} \le t \le \frac{2}{3}.$$

Hence, we deduce that

$$\|\varphi - H_t\|_{W^{1,p}(\mathbb{S}^n)} \lesssim \|\varphi\|_{W^{1,p}(B_{2r}(x))} + \|\psi\|_{W^{1,p}(B_{2r}(x))} \quad \text{for every } \frac{1}{3} \le t \le \frac{2}{3}.$$

This concludes the proof. □

In Corollary 2.2, both the $\delta > 0$ controlling $\|\varphi - \psi\|_{W^{1,p}(\mathbb{S}^n)}$ *and* the $r > 0$ depend on ε. A very natural question is whether or not one may find a homotopy H so that $\sup_{0\le t\le 1} \|\varphi - H_t\|_{W^{1,p}(\mathbb{S}^n)}$ is controlled only by $\|\varphi - \psi\|_{W^{1,p}(\mathbb{S}^n)}$. More precisely, we formulate the following open question (cf. [11, Problem, p.11]).

Open Problem 2.3. *Let $\varphi \in C^\infty(\mathbb{S}^n, \mathcal{N})$. Does there exist some $r > 0$, possibly depending on φ, such that for every $x \in \mathbb{S}^n$ and every $\psi \in C^\infty(\mathbb{S}^n, \mathcal{N})$ homotopic to φ and satisfying $\varphi = \psi$ on $\mathbb{S}^n \setminus B_r(x)$, there exists a homotopy $H \in C^\infty(\mathbb{S}^n \times [0,1], \mathcal{N})$ from φ to ψ such that*

$$\sup_{0\le t\le 1} \|\varphi - H_t\|_{W^{1,p}(\mathbb{S}^n)} \le \omega\left(\|\varphi - \psi\|_{W^{1,p}(\mathbb{S}^n)}\right),$$

where ω is a modulus of continuity satisfying $\omega(t) \to 0$ as $t \to 0$.

One may expect ω to be linear in t, but any modulus of continuity would already be of interest. The question is already interesting for maps $\mathbb{S}^2 \to \mathbb{S}^2$.

3. Proof of the generic non-uniqueness

Proof of Theorem 1.2. Fix $\varepsilon > 0$ and $\varphi \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$. We note first that, by Theorem 1.1 combined with Hölder's inequality, we may find another mapping $\varphi_0 \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ which admits exactly one energy minimizer $u_0 \colon B^3 \to \mathbb{S}^2$ among all maps having boundary datum φ_0, and such that φ_0 differs from φ only on a set $B_{\frac{\varepsilon}{2}}(x_0)$ for some $x_0 \in \mathbb{S}^2$ and is such that

$$\|\varphi - \varphi_0\|_{W^{1,p}(\mathbb{S}^2)} < \frac{\varepsilon}{2}. \tag{3.1}$$

We recall that, combining the regularity result [13, Theorem II] with the boundary regularity [14, Theorem 2.7] of Schoen–Uhlenbeck, u_0 can have only a finite number of singularities; let us denote this number by $M = \#\operatorname{sing} u$ (possibly $M = 0$).

Next, we apply Corollary 2.2 to $\varphi_0 \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$. We obtain the existence of a $\delta = \delta(\varepsilon) > 0$ and an $r = r(\varphi_0, \varepsilon) > 0$ such that for any $\psi \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ that differs from φ_0 only on the set $B_r(x_0)$ and such that $\|\varphi_0 - \psi\|_{W^{1,p}(\mathbb{S}^2)} < \delta$, there exists a homotopy $H \in C^\infty(\mathbb{S}^2 \times [0,1], \mathbb{S}^2)$ with

$$\sup_{0\leq t\leq 1} \|\varphi_0 - H_t\|_{W^{1,p}(\mathbb{S}^2)} < \frac{\varepsilon}{2}. \tag{3.2}$$

Let $\varepsilon_1 \coloneqq \min\{\delta, r, \frac{\varepsilon}{2}\}$. By [9, Theorem 2.3.1], we construct $\varphi_1 \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ with the properties:

(1) $\deg \varphi_0 = \deg \varphi_1$;
(2) $\|\varphi_0 - \varphi_1\|_{W^{1,p}} < \varepsilon_1$ and $\varphi_0 = \varphi_1$ except on $B_{\varepsilon_1}(x)$ for some point $x \in \mathbb{S}^2$;
(3) φ_1 admits only one energy minimizer $u_1 \colon B^3 \to \mathbb{S}^2$ having at least $M+1$ singularities.

To be precise, the statement [9, Theorem 2.3.1] gives only that $\mathcal{H}^2(\{x \in \mathbb{S}^2 \colon \varphi_0(x) \neq \varphi_1(x)\}) < \varepsilon_1$, but following the lines of the proof, we may deduce that $\varphi_0 = \varphi_1$ except on $B_{\varepsilon_1}(x)$ for some point $x \in \mathbb{S}^2$.

Now, let us take the homotopy H_t between φ_0 and φ_1 constructed in Corollary 2.2. Let

$$\begin{aligned}\tau \coloneqq \sup\{t \in [0,1] \colon\ &\text{each energy minimizer with boundary datum } H_t\\ &\text{has at most } M \text{ singular points in } B^3\}.\end{aligned}$$

We argue like in [11, Remark 4.1] (which is a modified argument from [7, Section 5]). For the convenience of the reader, we state here the main lines of the reasoning. First, we note that from the Stability Theorem [7], see also [10, Theorem 8.9], we have $\tau \in (0,1)$.

Now take $s_i \nearrow \tau$ and a sequence of minimizing harmonic maps $u_i \in W^{1,2}(B^3, \mathbb{S}^2)$ with $u_i\big|_{\partial B^3} = H_{s_i}$ and $\#\operatorname{sing} u_i \leq M$. Let us also take $t_i \searrow \tau$ and a sequence of minimizing harmonic maps $v_i \in W^{1,2}(B^3, \mathbb{S}^2)$ with $v_i\big|_{\partial B^3} = H_{t_i}$ and $\#\operatorname{sing} v_i > M$. Since $\sup_i \big([H_{s_i}]_{W^{1,2}(\mathbb{S}^2)} + [H_{t_i}]_{W^{1,2}(\mathbb{S}^2)}\big) < \infty$, we may deduce from the strong convergence of minimizers, see [1, Theorem 1.2 (4)] (see also [10, Theorem 6.1 (3)]), that up to a subsequence we have

$$\begin{aligned} u_i &\to u \quad \text{strongly in } W^{1,2}(B^3, \mathbb{S}^2),\\ v_i &\to v \quad \text{strongly in } W^{1,2}(B^3, \mathbb{S}^2),\end{aligned}$$

and both u and v are energy minimizers with $u\big|_{\partial B^3} = v\big|_{\partial B^3} = H_\tau$. We claim that $\#\operatorname{sing} u \leq M$. Indeed, assume on the contrary that $\#\operatorname{sing} u > M$. Then, by [1, Theorem 1.8 (2)] (see also [10, Theorem 2.10]), we would obtain that for each $y \in \operatorname{sing} u$ and for sufficiently large i, there would exist $y_i \in \operatorname{sing} u_i$ with $y_i \to y$ as $i \to \infty$, a contradiction.

Moreover, $\#\operatorname{sing} v > M$. To see this, let us again assume by contradiction that $\#\operatorname{sing} v \leq M$. Let now $z_{i,j} \in \operatorname{sing} v_i$ for $j \in \{1, \dots, M+1\}$ be distinct singular points of v_i. Now let us observe that for sufficiently large i, we know that that H_{t_i} and H_τ are close in

C^∞. Hence, by uniform boundary regularity [1, Theorem 1.10 (2)] (see also [10, Theorem 7.4]), there is a uniform neighborhood of the boundary ∂B^3 which contains no singularities of v and v_i, say $\operatorname{dist}(z, \partial B^3) \geq \lambda > 0$ for any $z \in \bigcup_i \operatorname{sing} v_i \cup \operatorname{sing} v$. Since singular points converge to singular points, we deduce from [1, Theorem 1.8 (1)] (see also [10, Theorem 2.5]) that for each j, we have $z_{i,j} \to z_j$ as $i \to \infty$ and $z_j \in \#\operatorname{sing} v$. The only possibility for $\#\{z_1, \ldots, z_{M+1}\} < M + 1$ is that two singularities of v_i converge to the same singularity of v. This, however, is impossible, because by the uniform distance between singularities [1, Theorem 2.1] (see also [10, Theorem 2.12]), there exists a universal constant C (independent of the minimizer) such that no singularity can occur next to $z_{i,j}$ at a distance $C \operatorname{dist}(z_{i,j}, \partial B^3) \geq C\lambda$.

Hence, $H_\tau \colon \mathbb{S}^2 \to \mathbb{S}^2$ serves as a boundary condition for at least two minimizers u and v having a different number of singularities. Combining (3.2) with (3.1), we obtain

$$\|\varphi - H_\tau\|_{W^{1,p}(\mathbb{S}^2)} \leq \|\varphi - \varphi_0\|_{W^{1,p}(\mathbb{S}^2)} + \|\varphi_0 - H_\tau\|_{W^{1,p}(\mathbb{S}^2)} < \frac{\varepsilon}{2} + \varepsilon_1 \leq \varepsilon.$$

This finishes the proof. □

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