

Low regularity maps on Lipschitz manifolds: lifting, pullback of forms, and approximation

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Abstract

We present some contributions connecting various notions from geometry and topology in a low regularity setting. Our framework is the one of fractional Sobolev maps from a Lipschitz manifold to a complete Riemannian manifold. Our main result is a new approach to define the pullback in the sense of distributions of a closed form by a Sobolev map relying on a lifting procedure to “kill” the cohomology class of the form whose pullback is taken. On the way, we establish several results of independent interest regarding low regularity mappings defined on Lipschitz manifolds, including a strong approximation result by “almost smooth” maps as well as existence, (essential) uniqueness, and selection principles for liftings of Sobolev and vanishing mean oscillation maps. Our motivation comes from the study of classical density problems for Sobolev mappings between manifolds, to which such tools have proved to be instrumental.

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1 Introduction

The goal of this paper is to bring together two *a priori* unrelated notions from differential geometry, namely liftings and pullbacks. These notions are classical and well-understood in the smooth setting. Our goal here is to investigate how to define the pullback of a differential form by a map with *low regularity*, and to obtain some insight about how working with liftings allows to define robust pullbacks for maps less regular than expected.

Let us start by briefly recalling some well-known definition and facts. If $f: \mathcal{M} \rightarrow \mathcal{N}$ is a smooth map between two Riemannian manifolds $\mathcal{M}$ and $\mathcal{N}$, $\mathcal{M}$ being simply connected, and if $\pi: \mathcal{E} \rightarrow \mathcal{N}$ is a Riemannian covering, then there exists a smooth map $\varphi: \mathcal{M} \rightarrow \mathcal{E}$ such that $f = \pi \circ \varphi$, called a *lifting* of f . Moreover, such a map is unique once its value at one point is prescribed. The importance of liftings comes from the fact that, in general, a covering space has a simpler geometry and topology than the base space. For instance, one can consider the *universal covering*, which by construction is always simply connected. The most striking example is the covering space $\pi: \mathbb{R} \rightarrow \mathbb{S}^1$, where the covering map π is defined by $\pi(x) = e^{2\pi i x}$. In this case, passing to the universal

covering $\mathbb{R}$ *completely* kills the geometry and the topology of the base space, allowing to be reduced to a linear setting.

Meanwhile, again if $f: \mathcal{M} \rightarrow \mathcal{N}$ is a smooth map, and if $\omega \in \Lambda^k(\mathcal{N})$ is a smooth k -form on $\mathcal{N}$, then one can define the *pullback* of ω by f , denoted by $f^*\omega$, which is a k -form on $\mathcal{M}$. Pullback of forms are of crucial importance in differential geometry, in that they provide a great source of topological invariants. For instance, if $\mathcal{M} = \mathbb{S}^k = \mathcal{N}$ and $\omega = \omega_{\mathbb{S}^k}$ is the volume form on $\mathbb{S}^k$, then $f^*\omega_{\mathbb{S}^k}$ is the Brouwer degree of the map f [20]. Similarly, the Hopf invariant for maps $f: \mathbb{S}^3 \rightarrow \mathbb{S}^2$ can be defined *via* an integral formula involving the pullback of $\omega_{\mathbb{S}^2}$ by f [44]. This applies more generally to any topological invariant encoded by rational homotopy classes, as follows from the work of Sullivan [41] and Novikov [34, 35, 36]; see also Hardt and Rivière [23].

Our contribution focuses on a *low regularity* setting. More specifically, we work with $f \in W^{s,p}(\mathcal{M}; \mathcal{N})$, where $0 < s \leq 1$, $1 \leq p < \infty$, where $\mathcal{N}$ is assumed to be isometrically embedded into $\mathbb{R}^n$ — without loss of generality, by virtue of the Nash isometric embedding theorem [32, 33] — and where

$$W^{s,p}(\mathcal{M}; \mathcal{N}) := \{f \in W^{s,p}(\mathcal{M}; \mathbb{R}^n) : f(x) \in \mathcal{N} \text{ for a.e. } x \in \mathcal{M}\}$$

is the Sobolev space of mappings from $\mathcal{M}$ to $\mathcal{N}$; see the definition in (3.1).

In this setting, the existence of both liftings and pullbacks becomes a challenging issue. Regarding liftings, as $W^{s,p}$ maps need not be continuous when $sp \leq k := \dim \mathcal{M}$, the classical theorems from topology cannot be applied to deduce that they admit a lifting. This is not just an issue related to the proof. Indeed, even if $\mathcal{M}$ itself is simply connected, since $W^{s,p}$ regularity allows for singularities, it is possible to construct a $W^{s,p}$ map whose restriction to the complement of its singular set, which will be a non-simply connected set, is a continuous map without lifting. The first example to such an obstruction to the existence of a lifting for Sobolev mappings is due to Demengel [17]; see also Bourgain, Brezis, and Mironescu [8].

Concerning pullbacks of forms, the very first challenge is how to even *define* $f^*\omega$ for $f \in W^{s,p}(\mathcal{M}; \mathcal{N})$. Indeed, the classical definition of $f^*\omega$ for a smooth f involves taking products of k partial derivatives of f . If $f \in W^{1,k}(\mathcal{M}; \mathcal{N})$, this can be adapted *mutatis mutandis*, replacing the classical derivative by the weak derivative and using Hölder's inequality to ensure that the resulting object is well-defined and belongs to L^1 . If $f \in W^{1,p}(\mathcal{M}; \mathcal{N})$ with $p < k$, a first challenge arises from the fact that the resulting object, which makes sense in the a.e. sense, needs not be defined *a priori* as an L^1 function, or even as a distribution. Moreover, being able to *define* $f^*\omega$ as a distribution leads to a new problem: It needs not coincide with the definition in the a.e. sense. An observation of this fact goes back to Ball [3, Section 6] (see also Müller [31] for related results, and

Brezis, Mawhin, and Mironescu [14] for the history of this problem and related issues). But the real challenge arises when $f \in W^{s,p}(\mathcal{M}; \mathcal{N})$ with $0 < s < 1$. Indeed, in this case, f does not even have a full weak derivative, and hence it is not clear how to even *define* $f^*\omega$, as the usual formula does not make sense anymore.

The lifting problem for Sobolev mappings is by now quite well-understood, and there is a full characterization of the values of s , p , and of the targets $\mathcal{N}$ such that every map $f \in W^{s,p}(\mathcal{M}; \mathcal{N})$ admits a lifting $\varphi \in W^{s,p}(\mathcal{M}; \mathcal{N})$ [7, 8, 6, 30].

On the other hand, the exact conditions that allow to define $f^*\omega$ in a robust way for $W^{s,p}(\mathcal{M}; \mathcal{N})$ remain yet to be understood. It is known that, when ω is a closed k -form, $\int_{\mathcal{M}} f^*\omega$ is well-defined and it only depends on the *homotopy class* of f and the *de Rham cohomology classes* of ω . When $\mathcal{M}$ and f smooth and $\mathcal{N} = \mathbb{S}^k$, this is classical. For a proof, see, e.g., Dincă and Mawhin [20]. Under the natural “weakest possible” assumptions, this was proved in our recent work with Mironescu [19], relying on a generalization of the VMO homotopy theory introduced by Brezis and Nirenberg [15]. However, this method based on homotopical invariants fails, even in the smooth setting, when it comes to defining $\int_{\mathcal{M}} \xi f^*\omega$ against an arbitrary “test” function ξ . This generalization is investigated by Brezis and Nguyen [13] when $\mathcal{M} = \mathbb{S}^k = \mathcal{N}$ and, in a general setting, in [19] for $f \in (W^{s,p} \cap W^{1-1/k,k})(\mathcal{M}; \mathcal{N})$.

Our goal in this article is to explain a possible strategy to use liftings in order to push further the realm in which pullbacks of differential forms by Sobolev mappings can be given a robust meaning. Apart from the low regularity setting induced by the mappings themselves, we also work in a lower regularity setting as what is usually done for our *domain*. Indeed, in the sequel, we do not assume the domain $\mathcal{M}$ to be a *smooth* manifold, but only a *Lipschitz manifold*; see Definition 2.1. Our main motivation for studying pullback of forms in such a low regularity setting of fractional Sobolev maps on a Lipschitz manifold comes from the strong approximation problem for manifold-valued Sobolev mappings, and more specifically, from our recent work with Mironescu [19]. Indeed, in [19], we developed a rigorous framework for pullback of forms by low regularity maps, which allows us to provide an analytic characterization of the $\mathcal{N}$ -valued $W^{s,p}$ maps that are strong limits of *smooth* $\mathcal{N}$ -valued maps. Instrumental to us was to be able to work with pullbacks of forms by maps defined on boundary of cubes, as strong approximation arguments make intensive use of decompositions of the domain into cubes. Lipschitz manifolds provide a unified framework that contains this example as a special case. In some sense, our work is a continuation of [19]. More specifically, our main goal is to pursue the study of the distribution $f^*\omega$ as in [19, Section 5], and especially as in [19, Sections 5.3 and 5.4].

Our main result in this work is as follows. Consider a normal covering $\pi: \mathcal{E} \rightarrow \mathcal{N}$ with $\mathcal{E}$ compact. Assume that ω is a closed form, and that its pullback is exact,

i.e., $\pi^*\omega = d\gamma$ for some $(k-1)$ -form γ on $\mathcal{E}$. Then, we can “naturally” extend the map $W^{1,k}(\mathcal{M}; \mathcal{N}) \ni f \mapsto \int_{\mathcal{M}} \xi f^*\omega$ to $W^{1-1/k,k}(\mathcal{M}; \mathcal{N})$; see Theorem 4.11. Our result, as compared to [19], does not require $f \in W^{s,p}(\mathcal{M}; \mathcal{N})$ with $sp = k$, thanks to the assumption on ω . The extension, which depends on the choice of γ (more precisely, on the $(k-1)$ -th *de Rham cohomology class* of γ ; see Proposition 4.12), may have independent interest when studying the density of $\text{Lip}(\mathcal{M}; \mathcal{N})$ in $W^{s,p}(\mathcal{M}; \mathcal{N})$.

On our way towards Theorem 4.11, we establish several intermediate results related to lifting and approximation, which we believe are of independent interest.

Lifting in VMO A space of crucial importance when dealing with problems with a topological and geometric flavour in a low regularity context is the space VMO of functions with *vanishing mean oscillation*; see the definition in (2.6). This comes notably from the fact that VMO is a slightly larger space than C^0 , and in some sense the largest space in which many invariants defined for continuous functions can still be defined in a robust way. This includes notably, but not only, homotopy classes. We refer the reader to the seminal work of Brezis and Nirenberg [15].

In Section 2, we establish a general theory of lifting for VMO maps defined on a Lipschitz manifold. Our results include existence of a lifting, uniqueness up to a deck transformation, and continuity of the lifting operation with respect to the convergence in VMO. When the domain is a smooth manifold and the target is $\mathbb{S}^1$, with its universal covering given by $\mathbb{R}$ along with the exponential map, these results were established by Brezis and Nirenberg [15]. Our contribution here is to handle the case where the domain is merely a Lipschitz manifold, and more importantly to be able to work with very general classes of covering spaces. A quite delicate aspect of our analysis is to prevent the liftings we construct to “escape at infinity”. When working with the covering space $\mathbb{R} \rightarrow \mathbb{S}^1$ via the exponential map as in [15], this can easily be avoided by translation. Here on the other hand, we need to provide a careful selection of deck transformations to thwart this phenomenon. This only relies on the polynomial control of the geodesic distance by the ambient distance, and on the normality of the covering.

A strong approximation result A very convenient way to prove the existence of liftings or pullbacks for Sobolev maps is by approximation. Namely, starting from $f \in W^{s,p}(\mathcal{M}; \mathcal{N})$, one would ideally like to first consider a sequence of maps $f_\ell \in C^\infty(\mathcal{M}; \mathcal{N})$ that strongly converges towards f as $\ell \rightarrow +\infty$, define their lifting or pullback using the classical theory for smooth maps, and then show that the resulting object passes to the limit, hence providing a definition valid for the limiting map f .

However, in the context of manifold-valued maps, this is not always possible. Indeed, smooth $\mathcal{N}$ -valued maps *need not* be dense in the space of $\mathcal{N}$ -valued Sobolev maps. This

was first observed by Schoen and Uhlenbeck [40]. However, when the domain $\mathcal{M}$ is a smooth manifold, one can instead work with a suitable class of “almost smooth” maps, as devised originally by Bethuel and Zheng [7], and further developed by Bethuel [4], see also Hang and Lin [22]. More specifically, define the class $\mathcal{R}_i^\infty(\mathcal{M}; \mathcal{N})$ as the set of those maps $f: \mathcal{M} \rightarrow \mathcal{N}$ which are smooth outside of a singular set $\mathcal{S} = \mathcal{S}_f$ made of a finite union of i -dimensional submanifolds of $\mathcal{M}$, and such that

$$|\nabla f|(x) \leq \frac{C}{\text{dist}(x, \mathcal{S}_f)},$$

for some finite constant C depending on f . Then, the class $\mathcal{R}_{m-[sp]-1}^\infty(\mathcal{M}; \mathcal{N})$ is dense in $W^{s,p}(\mathcal{M}; \mathcal{N})$.

Working with a domain being a Lipschitz manifold features several new difficulties. First, the definition of the appropriate class of almost smooth maps itself is not obvious, as the proof of the density of $\mathcal{R}_i^\infty$ involves dual skeletons associated with cubulations, which may not exist in the Lipschitz case, see below. However, when $i = 0$, the required adaptation is pretty clear: It is natural to work with maps that are locally Lipschitz outside of a singular set made of finitely many point singularities. This is the choice we make in this work, at the beginning of Section 3, to define the class $\mathcal{R}_0$ we work with.

Another difficulty comes from the proof itself. Indeed, the standard approach by Bethuel [4] and Hang and Lin [22] relies on decompositions of the domain into grids. However, unlike smooth manifolds, Lipschitz manifolds *need not* admit a triangulation. Indeed, when $k \geq 5$, nonexistence follows from the fact that (j) there exist closed topological manifolds that cannot be triangulated; see Kirby and Siebenman [26] (see also Manolescu [28]); and (jj) every topological manifold has a Lipschitz coordinate system; see Sullivan [42].

To bypass this issue, we rely instead on a *good and bad balls*-type approach, in the spirit of the work of Bousquet, Ponce, and Van Schaftingen [9]. The difficulties in our situation are twofold. First, in the case $0 < s < 1$, one has to deal with the non-local character of the Gagliardo seminorm, and adapt all the corresponding estimates for the constructions involved in the process, which are local by nature. Second, we include the limiting case $sp = k - 1$, where the restriction of Sobolev maps to generic $(k - 1)$ -dimensional sets need not be continuous, which raises additional technical difficulties. To the best of our knowledge, this is the first strong approximation result for manifold-valued Sobolev mappings *defined on a Lipschitz manifold*.

About our geometric assumptions In most of our work, we will require our target manifold $\mathcal{N}$ — or $\mathcal{E}$ — to be embedded into an ambient Euclidean space *via a tubed embedding*, or even in such a way that *the geodesic distance has polynomial growth with*

respect to the distance induced by the ambient space; see the beginning of Sections 2.1 and 2.2 for the precise definitions. The first assumption is instrumental in order to reproject mappings obtained, e.g., *via* an averaging procedure, back into the target manifold. The second one enables us to relate the two possible definitions of VMO, with respect to the geodesic distance or with respect to the distance induced by the ambient Euclidean metric, through the use of the John–Nirenberg inequality. When the target manifold is compact, it is automatic that it satisfies both assumptions. However, in the non-compact case, this is not always true: For instance, *no* isometric embedding of the hyperbolic space in a Euclidean space is a tubed embedding (see, e.g., the remark after the observation p.1 in [38]). There are however many examples of non-compact manifolds satisfying these assumptions. This is the case, for instance, for any universal covering of a compact manifold whose fundamental group has polynomial growth, by combining [38, Main theorem] with the Milnor–Švarc lemma [43]. This includes, for example, the case where the fundamental group of the base manifold is abelian.

It would be interesting to know whether these assumptions are just an artefact of our proofs, or whether they are really needed in order to ensure the validity of our main results. For some closely related problems involving non-compact target manifolds, examples are known where the way the target is embedded plays a decisive role. This is the case, for instance, for the problem of strong density of bounded maps; see Bousquet, Ponce, and Van Schaftingen [10]. Concerning the lifting problem, it was shown in [18, Exemple 8.4] that higher-order Sobolev maps may fail to have a lifting if the embedding of the covering space “swirls too much”; see also [29, (1.1)].

Perspectives and generalisations An arguable drawback of several of our main results is the quite restrictive assumptions that they require. Our strong approximation result, Theorem 3.1 and 3.2, is limited to the range $k - 1 \leq sp < k$, where the relevant class of almost smooth maps is made of maps having point singularities, both for technical reasons and for the lack of a clear candidate for a class of almost smooth maps with higher-dimensional singularities in the framework where the domain is a Lipschitz manifold. But most importantly, our main result regarding the pullback in the sense of distributions, Theorem 4.11, is restricted to the situation where $k \geq 3$ and we work with a compact covering space.

In this regard, the main contribution of this work is less the actual final theorems as such rather than the ideas that are developed in their proofs as well as highlighting the general method of defining distributional pullbacks *via* lifting. To be more specific, a very natural perspective of generalisation to be explored is about more general fiber maps. Consider for instance the Hopf fibration $\pi: \mathbb{S}^3 \rightarrow \mathbb{S}^2$. This map also has the “killing” property: If ω is a closed 2-form on $\mathbb{S}^2$ — for instance, the volume form of

$\mathbb{S}^2$ — then $\pi^* \omega_{\mathbb{S}^2}$ is an *exact* 2-form on $\mathbb{S}^3$, because the second cohomology group of $\mathbb{S}^3$ vanishes. Due to the Hopf fibration not being a covering map, our theory does not apply in this setting. However, in presence of a suitable theory for liftings through the Hopf fibration, the same procedure as we have devised in this work could be applied to extend the range of definition of the pullback of ω through a Sobolev map. However, unlike for covering maps, the lifting of Sobolev maps through general fibrations is still far from being understood. The special case of $W^{1,p}(\mathbb{B}^k; \mathbb{S}^2)$ and of a lifting through the Hopf fibration was studied by Bethuel and Chiron [6, Section 5] and by Bethuel [5, Section 1.6].

Use of AI No generative AI was used at any point in the preparation of this article. The original ideas featured in this work, their practical implementation, as well as all the writing choices, are exclusively the outcome of the (human) authors of this text.

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2 Lifting in VMO

In this section, we study existence, (essential) uniqueness, and selection principles for liftings of maps with *vanishing mean oscillation* (VMO). This is in line with the seminal work of Brezis and Nirenberg [15]. Here, the novelty is that we deal with domains being mere Lipschitz manifolds, and that the covering space can be quite general, not being restricted to the canonical covering $\mathbb{R} \rightarrow \mathbb{S}^1$ by the exponential map. But first, we start with some reminders about VMO and some basic facts about the types of manifolds that we will consider as targets.

2.1 Definitions

Through this paper, and unless otherwise specified, we assume that $\mathcal{M}$ is a k -dimensional connected and simply connected compact Lipschitz manifold embedded into some $\mathbb{R}^m$, with the induced k -dimensional Hausdorff measure. The definition of Lipschitz manifold that we adopt in this text is the following one.

Definition 2.1. *A k -dimensional finite chart structure on a compact set $\mathcal{M} \subset \mathbb{R}^m$ is a finite family $\{(U_i, V_i, \psi_i)\}_{i \in I}$ such that*

- (i) U_i is open in $\mathcal{M} \subset \mathbb{R}^m$, $\forall i \in I$, and $\bigcup_{i \in I} U_i$ is a cover of $\mathcal{M}$;
- (ii) V_i is an open subset of $\mathbb{R}^k$, $\forall i \in I$;
- (iii) $\psi_i: V_i \rightarrow U_i$ is bi-Lipschitz, $\forall i \in I$.

We say that $\mathcal{M}$ is a k -dimensional Lipschitz manifold whenever it can be endowed with a k -dimensional finite chart structure.

By assumption on the maps ψ_i , and since the chart structure is finite, there exists $0 < K_1 \leq K_2 < \infty$ such that

$$K_1|u - v| \leq |\psi_i(u) - \psi_i(v)| \leq K_2|u - v|, \forall i, \forall u, v \in V_i. \quad (2.1)$$

In particular, there exists r_0 such that

$$\mathcal{H}^k(B_r(x)) \sim r^k, \forall r \leq r_0, \forall x \in \mathcal{M}. \quad (2.2)$$

For further use, we also assume that r_0 is sufficiently small so that, for any $x \in \mathcal{M}$, there exists i such that

$$B_{4\sqrt{k}K_2r_0/K_1}(x) \subset U_i. \quad (2.3)$$

We also consider a smooth closed manifold $\mathcal{N}$ embedded into $\mathbb{R}^n$ and a Riemannian covering $\pi \in C^\infty(\mathcal{E}; \mathcal{N})$. We assume that (i) π is locally isometric and non-trivial ; (ii) $\mathcal{E}$ is complete, connected, and embedded into some $\mathbb{R}^v$ via a tubed embedding. The latter assumption means that there exists $\iota_{\mathcal{E}} > 0$ such that there is a well-defined and smooth nearest point projection $\mathcal{E} + B_{\iota_{\mathcal{E}}} \rightarrow \mathcal{E}$; see e.g. [38] for some results about tubed embeddings that will be instrumental to us in the sequel. Sets with a tubed embedding are also sometimes called *sets with finite reach*.

Next we define BMO. This can be done in a more general setting than the one of Lipschitz manifolds. In the following definitions, we assume that $\mathcal{M}$ is a connected,

compact doubling metric measure space equipped with a doubling measure μ . For $f \in \mathcal{L}^1(\mathcal{M}) = \mathcal{L}^1(\mathcal{M}; \mathbb{R})$, we define the seminorm

$$|f|_{\text{BMO}} = \sup_{x \in \mathcal{M}, 0 < \varepsilon \leq \varepsilon_0} \int_{B_\varepsilon(x)} \int_{B_\varepsilon(x)} |f(y) - f(z)| d\mu(y) d\mu(z),$$

and let

$$\text{BMO} = \text{BMO}(\mathcal{M}) = \text{BMO}(\mathcal{M}; \mathbb{R}) := \{f \in \mathcal{L}^1(\mathcal{M}): |f|_{\text{BMO}} < \infty\}. \quad (2.4)$$

A similar definition can be given for maps in $\mathcal{L}^1(\mathcal{M}; \mathbb{R}^n)$. Here, $\varepsilon_0 > 0$ is some fixed number. A natural choice would be to take $\varepsilon_0 = \text{diam}(\mathcal{M})$, but different choices of ε_0 give equivalent BMO seminorms, see e.g. [19, Corollary 2.2].

We now turn to VMO and its basic characterizations and properties, in the spirit of Sarason [39]. For the exact statements we present, we follow [19, Section 2.1], to which we also refer the reader for their proofs. For $f \in \text{BMO}$ and $r > 0$, we define

$$M_r(f) := \sup_{x \in \mathcal{M}, 0 < s \leq r} \int_{B_s(x)} \int_{B_s(x)} |f(y) - f(z)| d\mu(y) d\mu(z) \leq |f|_{\text{BMO}}, \quad (2.5)$$

and

$$M_0(f) := \lim_{r \searrow 0} M_r(f).$$

We denote $\text{VMO} = \text{VMO}(\mathcal{M}) = \text{VMO}(\mathcal{M}; \mathbb{R})$ the closure of continuous functions with respect to the BMO seminorm, i.e.,

$$\text{VMO} := \overline{C(\mathcal{M})/\mathbb{R}}^{\cdot|_{\text{BMO}}}. \quad (2.6)$$

The space $\text{VMO}(\mathcal{M}; \mathbb{R}^n)$ is defined similarly. By [19, Lemma 2.5], we have

$$\text{VMO} = \{f \in \text{BMO}: M_0(f) = 0\}.$$

We also denote

$$\text{VMO}(\mathcal{M}; \mathcal{N}) := \{f \in \text{VMO}(\mathcal{M}; \mathbb{R}^n): f(x) \in \mathcal{N} \text{ for a.e. } x \in \mathcal{M}\}. \quad (2.7)$$

We next recall an approximation procedure adapted to the study of VMO, devised in

[19]. For $x, y \in \mathcal{M}$ and $\varepsilon > 0$, let

$$\rho(x, \varepsilon, y) := [\varepsilon - \text{dist}(x, y)]_+, \quad K(x, \varepsilon) := \left(\int_{\mathcal{M}} \rho(x, \varepsilon, y) \, d\mu(y) \right)^{-1},$$

and set

$$f_\varepsilon(x) := K(x, \varepsilon) \int_{\mathcal{M}} \rho(x, \varepsilon, y) f(y) \, d\mu(y) = \int_{\mathcal{M}} f \, d[\rho(x, \varepsilon, \cdot)\mu].$$

As averaging processes are not compatible with the nonlinear constraint induced by $\mathcal{N}$, the map f_ε needs not take its values into $\mathcal{N}$ even if f does. An instrumental tool to remedy this issue is the nearest point projection on $\mathcal{N}$, which we introduce below.

Definition 2.2. For $\delta = \delta(\mathcal{N})$ sufficiently small, we let Π be the nearest point projection from $\mathcal{N}_{4\delta} := \{z \in \mathbb{R}^n : \text{dist}(z, \mathcal{N}) \leq 4\delta\}$ to $\mathcal{N}$.

Here, δ is chosen such that Π is well-defined, smooth, and has bounded derivatives.

By [19, Lemma 2.9], for each $f \in \text{VMO}(\mathcal{M}; \mathcal{N})$, there exists some $\varepsilon_1 = \varepsilon_1(f)$ such that

$$f_\varepsilon(x) \in \mathcal{N}_\delta, \forall x \in \mathcal{M}, \forall 0 < \varepsilon \leq \varepsilon_1,$$

and hence the map f^ε given by

$$f^\varepsilon := \Pi \circ f_\varepsilon : \mathcal{M} \rightarrow \mathcal{N} \tag{2.8}$$

is well-defined, $\forall 0 < \varepsilon \leq \varepsilon_1$, and the mapping

$$(0, \varepsilon_1] \ni \varepsilon \mapsto f^\varepsilon \in C(\mathcal{M}; \mathcal{N})$$

is continuous.

2.2 The space $\text{VMO}(\mathcal{M}; \mathcal{E})$ for a target $\mathcal{E}$ with polynomial growth

Having in mind the objective of working with mappings with values into a covering space, considering only maps with values into a compact manifold would be an unnatural and overly restrictive assumption. In this section, we introduce the setting to work with mappings into the covering space $\mathcal{E}$. An important issue in this endeavour is the presence of two, non equivalent for a non compact manifold, ways of measuring the distance on $\mathcal{E}$: the geodesic distance on $\mathcal{E}$, or the Euclidean distance induced by the ambient space $\mathbb{R}^V$. We overcome this issue by introducing an extra *polynomial growth* assumption on $\mathcal{E}$, under which the BMO spaces defined using both these distances are

equivalent; see Lemma 2.3.

More specifically, in this section, $\mathcal{M}$ is a connected, compact Lipschitz manifold as in Definition 2.1, and $\mathcal{E}$ is a complete, connected Riemannian manifold isometrically embedded into some $\mathbb{R}^v$ in such a way that *the geodesic distance has polynomial growth with respect to the distance induced by the ambient space*:

$$\text{dist}(e_1, e_2) = \text{dist}_{\mathcal{E}}(e_1, e_2) \leq a_1|e_1 - e_2| + a_2|e_1 - e_2|^{p_0}, \forall e_1, e_2 \in \mathcal{E} \subset \mathbb{R}^v. \quad (2.9)$$

for some finite constants a_1 and a_2 and some $p_0 \in \mathbb{N}_+$.

In this context, a natural definition of the $\mathcal{L}^1(\mathcal{M}; \mathcal{E})$ norm is

$$\|f\|_{\mathcal{L}^1} := \int_{\mathcal{M}} \text{dist}(f(x), e) d\mu(x)$$

with some fixed $e \in \mathcal{E}$. We let

$$\mathcal{L}^1(\mathcal{M}; \mathcal{E}) := \{f: \mathcal{M} \rightarrow \mathcal{E} \text{ measurable} : \|f\|_{\mathcal{L}^1} < \infty\}.$$

Clearly, this space does not depend on the choice of e . Let us emphasize that this definition is not specific to our manifold context. It is tailored for a measured space $\mathcal{M}$ equipped with a finite measure μ and a metric space $\mathcal{E}$.

For a map $f \in \mathcal{L}^1(\mathcal{M}; \mathcal{E})$, we define its BMO seminorm by

$$|f|_{\text{BMO}} := \sup_{x \in \mathcal{M}, 0 < \varepsilon \leq \varepsilon_0} \int_{B_\varepsilon(x)} \int_{B_\varepsilon(x)} \text{dist}(f(y), f(z)) d\mu(y) d\mu(z),$$

and let

$$\text{BMO}(\mathcal{M}; \mathcal{E}) := \{f \in \mathcal{L}^1(\mathcal{M}; \mathcal{E}) : |f|_{\text{BMO}} < \infty\}. \quad (2.10)$$

In principle, the above BMO space depends on the choice of dist . The next statement, which is not so obvious, asserts that, in the case of polynomial growth, the two natural definitions of BMO yield the same space.

Lemma 2.3. *Assume that $\mathcal{M}$ is a connected, compact Lipschitz manifold, and that the target manifold $\mathcal{E}$ is complete, connected, and satisfies (2.9). Then, both definitions (2.4) and (2.10) for $\text{BMO}(\mathcal{M}; \mathcal{E})$ yield the same space.*

Proof. Since $\mathcal{M}$ is compact, given a map f in the space defined in (2.10), we have

$$\int_{\mathcal{M}} |f| d\mathcal{H}^k \leq \int_{\mathcal{M}} \text{dist}(f(x), e) d\mathcal{H}^k(x) + |e| \mathcal{H}^k(\mathcal{M}) < \infty,$$

and

$$|f(x) - f(y)| \leq \text{dist}(f(x), f(y)), \forall x, y \in \mathcal{M},$$

so that f belongs to the space defined in (2.4).

Conversely, we first notice using (2.9) that

$$\begin{aligned} \int_{\mathcal{M}} \text{dist}(f(x), e) d\mathcal{H}^k(x) &\lesssim \int_{\mathcal{M}} |f(x) - e| d\mathcal{H}^k(x) + \int_{\mathcal{M}} |f(x) - e|^{p_0} d\mathcal{H}^k(x) \\ &\lesssim \int_{\mathcal{M}} |f(x) - e| d\mathcal{H}^k(x) + \left(\int_{\mathcal{M}} |f(x) - e| d\mathcal{H}^k(x) \right)^{p_0} \\ &\quad + \int_{\mathcal{M}} \left| f(x) - \oint_{\mathcal{M}} f \right|^{p_0} d\mathcal{H}^k(x). \end{aligned} \quad (2.11)$$

The first two of the last three terms in (2.11) are finite, so that it only remains to estimate the last one. For this purpose, we first recall that r_0 is assumed to satisfy (2.3). For simplicity, we further assume that $K_1 = 1$ and $\psi_i^{-1}(x) = 0$ — which can always be achieved by rescaling and translation of the charts. Therefore, going back to local charts, we have

$$\begin{aligned} &\oint_{B_{r_0}(x)} \oint_{B_{r_0}(x)} |f(y) - f(z)|^{p_0} d\mathcal{H}^k(y) d\mathcal{H}^k(z) \\ &\lesssim \oint_{B_{r_0}(0)} \oint_{B_{r_0}(0)} |f \circ \psi_i(u) - f \circ \psi_i(v)|^{p_0} du dv \\ &\lesssim \frac{1}{r_0^{2k}} \int_{Q_{r_0}(0)} \int_{Q_{r_0}(0)} |f \circ \psi_i(u) - f \circ \psi_i(v)|^{p_0} du dv, \end{aligned}$$

where $Q_{r_0}(0) := (-r_0, r_0)^k$. Thus, using the John–Nirenberg inequality (see, e.g., [15, (B.3)]), we have

$$\begin{aligned} &\oint_{B_{r_0}(x)} \oint_{B_{r_0}(x)} |f(y) - f(z)|^{p_0} d\mathcal{H}^k(y) d\mathcal{H}^k(z) \\ &\lesssim \frac{1}{r_0^{2k}} |f \circ \psi_i|_{\text{BMO}(Q_{r_0}(0))}^{p_0-1} \int_{Q_{r_0}(0)} \int_{Q_{r_0}(0)} |f \circ \psi_i(u) - f \circ \psi_i(v)| du dv \\ &\lesssim |f \circ \psi_i|_{\text{BMO}(Q_{r_0}(0))}^{p_0-1} \oint_{B_{\sqrt{k}r_0}(0)} \oint_{B_{\sqrt{k}r_0}(0)} |f \circ \psi_i(u) - f \circ \psi_i(v)| du dv \\ &\lesssim |f|_{\text{BMO}(B_{\sqrt{k}K_2r_0}(x))}^{p_0-1} \oint_{B_{\sqrt{k}K_2r_0}(x)} \oint_{B_{\sqrt{k}K_2r_0}(x)} |f(y) - f(z)| d\mathcal{H}^k(y) d\mathcal{H}^k(z) \\ &\leq [M_{\sqrt{k}K_2r_0}(f)]^{p_0} \lesssim [M_{r_0}(f)]^{p_0}, \end{aligned} \quad (2.12)$$

where, in the last inequality, we use [15, Lemma A.14] (there, the result is for a smooth $\mathcal{M}$, but both Lemma A.14 and its proof have straightforward versions on Lipschitz manifolds).

Let us consider a maximal collection of disjoint balls $\{B_{\sqrt{k}K_2r_0/2}(x_j)\}_{j=1}^T$, so that the family $\{B_{\sqrt{k}K_2r_0}(x_j)\}_{j=1}^T$ is a covering of $\mathcal{M}$. Since $\mathcal{M}$ is connected, there exists a chain in $\{B_{\sqrt{k}K_2r_0}(x_j)\}_{j=1}^T$ that connects x_j and $x_{j'}$. For simplicity, we relabel the chain as $\{B_{r'_0}(y_\ell)\}_{\ell=1}^L$, with (i) $r'_0 := \sqrt{k}K_2r_0$ and $L \leq T$; (ii) $y_1 = x_j$, $y_L = x_{j'}$; and (iii) $B_{r'_0}(y_\ell) \cap B_{r'_0}(y_{\ell+1}) \neq \emptyset$ for $1 \leq \ell \leq L-1$. Repeating the argument in (2.12), we obtain

$$\begin{aligned} & \left[\int_{B_{r'_0}(y_1)} \int_{B_{r'_0}(y_L)} |f(y) - f(z)|^{p_0} d\mathcal{H}^k(y) d\mathcal{H}^k(z) \right]^{1/p_0} \\ & \leq \sum_{\ell=1}^{L-1} \left[\int_{B_{r'_0}(y_\ell)} \int_{B_{r'_0}(y_{\ell+1})} |f(y) - f(z)|^{p_0} d\mathcal{H}^k(y) d\mathcal{H}^k(z) \right]^{1/p_0} \\ & \leq \sum_{\ell=1}^{L-1} \left[\int_{B_{3r'_0}(y_\ell)} \int_{B_{3r'_0}(y_{\ell+1})} |f(y) - f(z)|^{p_0} d\mathcal{H}^k(y) d\mathcal{H}^k(z) \right]^{1/p_0} \\ & \lesssim TM_{r_0}(f). \end{aligned}$$

This implies that

$$\begin{aligned} & \int_{\mathcal{M}} \left| f(x) - \int_{\mathcal{M}} f \right|^{p_0} d\mathcal{H}^k(x) \\ & \lesssim \int_{\mathcal{M}} \int_{\mathcal{M}} |f(x) - f(y)|^{p_0} d\mathcal{H}^k(y) d\mathcal{H}^k(x) \\ & \lesssim (r'_0)^{2k} \sum_j \sum_{j'} \int_{B_{r'_0}(x_j)} \int_{B_{r'_0}(x_{j'})} |f(y) - f(z)|^{p_0} d\mathcal{H}^k(y) d\mathcal{H}^k(z) \\ & \lesssim T^{2+p_0} r_0^{2k} [M_{r_0}(f)]^{p_0} < \infty. \end{aligned} \tag{2.13}$$

Combining (2.13) and (2.11), we have

$$\int_{\mathcal{M}} \text{dist}(f(x), e) d\mathcal{H}^k(x) < \infty.$$

Arguing similarly as for (2.12), one obtains that, for $r \leq r_0$, it holds

$$\begin{aligned}
& \int_{B_r(x)} \int_{B_r(x)} \text{dist}(f(y), f(z)) \, d\mathcal{H}^k(y) \, d\mathcal{H}^k(z) \\
& \lesssim \int_{B_r(x)} \int_{B_r(x)} |f(y) - f(z)| \, d\mathcal{H}^k(y) \, d\mathcal{H}^k(z) \\
& \quad + \int_{B_r(x)} \int_{B_r(x)} |f(y) - f(z)|^{p_0} \, d\mathcal{H}^k(y) \, d\mathcal{H}^k(z) \\
& \lesssim M_r(f) + [M_r(f)]^{p_0}.
\end{aligned} \tag{2.14}$$

Using (2.13) when $r_0 \leq r \leq \varepsilon_0$, respectively (2.14) when $0 < r \leq r_0$, we find that f belongs to the BMO space defined in (2.10). $\square$

Remark 2.4. The above proof also shows that, under the assumptions of Lemma 2.3, the definition of $\text{BMO}(\mathcal{M}; \mathcal{E})$ in (2.10) does not depend on the choice of ε_0 . $\square$

Remark 2.5. In this setting, we can define $M_r(f)$ by replacing the Euclidean distance by the geodesic distance in (2.5). Moreover, this allows us to define $\text{VMO}(\mathcal{M}; \mathcal{E})$ in the following sense:

$$\text{VMO}(\mathcal{M}; \mathcal{E}) := \{f \in \text{BMO}(\mathcal{M}; \mathcal{E}) : M_0(f) = 0\}. \tag{2.15}$$

The above proof also shows that, under the assumptions of Lemma 2.3, both definitions (2.7) and (2.15) for $\text{VMO}(\mathcal{M}; \mathcal{E})$ yield the same space. $\square$

2.3 Basic properties of liftings

After having defined the suitable function spaces to work with, we now turn to the interaction between them and the covering map. In this section, $\mathcal{M}$ is a finite doubling metric measure space, $\mathcal{N}$ is a smooth closed manifold isometrically embedded into $\mathbb{R}^n$, $\mathcal{E}$ is a complete, connected Riemannian manifold embedded into $\mathbb{R}^V$ via a tubed embedding, and $\pi: \mathcal{E} \rightarrow \mathcal{N}$ is a Riemannian covering.

We first recall the definition of deck transformations.

Definition 2.6. Let $\pi: \mathcal{E} \rightarrow \mathcal{N}$ be a covering map. Its group of deck transformations (or group of covering transformations or Galois group) is the group

$$\text{Aut}(\pi) = \{\tau: \mathcal{E} \rightarrow \mathcal{E} : \tau \text{ is a homeomorphism and } \pi \circ \tau = \pi\}.$$

When π is a Riemannian covering, any $\tau \in \text{Aut}(\pi)$ is a global isometry of $\mathcal{E}$ (see, e.g., [37, Section 1.3.3]).

Next, we provide some properties of covering spaces and liftings. The first property is the connection between the geodesic distance on $\mathcal{E}$ and $\mathcal{N}$.

Lemma 2.7. [30, Lemma 2.1] *Let $\text{Inj}(\mathcal{N}) > 0$ be the injectivity radius of $\mathcal{N}$. For every $e_1, e_2 \in \mathcal{E}$ such that $\text{dist}_{\mathcal{E}}(e_1, e_2) \leq \text{Inj}(\mathcal{N})$, we have $\text{dist}_{\mathcal{E}}(e_1, e_2) = \text{dist}_{\mathcal{N}}(\pi \circ e_1, \pi \circ e_2)$.*

The second property is the “non-intersection” between two VMO liftings.

Lemma 2.8. *If $\varphi, \varphi' \in \text{VMO}(\mathcal{M}; \mathcal{E})$ are two liftings of $f \in \text{VMO}(\mathcal{M}; \mathcal{N})$, then either (i) $\varphi = \varphi'$ a.e. or (ii) $\varphi \neq \varphi'$ a.e.*

Proof. Let $g: [0, \infty) \rightarrow \mathbb{R}$ be a Lipschitz map such that

$$g(t) := \min\left\{\frac{t}{\min\{\text{Inj}(\mathcal{N}), 1\}}, 1\right\}.$$

Define $h: \mathcal{M} \rightarrow \mathbb{R}$ by $h(x) = g \circ (\text{dist}_{\mathcal{E}}(\varphi(x), \varphi'(x)))$. For any $x, y \in \mathcal{M}$, we have

$$\begin{aligned} |h(x) - h(y)| &\leq |g|_{\text{Lip}} |\text{dist}_{\mathcal{E}}(\varphi(x), \varphi'(x)) - \text{dist}_{\mathcal{E}}(\varphi(y), \varphi'(y))| \\ &\leq |g|_{\text{Lip}} (\text{dist}_{\mathcal{E}}(\varphi(x), \varphi(y)) + \text{dist}_{\mathcal{E}}(\varphi'(x), \varphi'(y))). \end{aligned}$$

Hence, the VMO property of φ and φ' implies that $h \in \text{VMO}(\mathcal{M}; \mathbb{R})$. By the definition of h and Lemma 2.7, we have $h(\mathcal{M}) \subset \{0, 1\}$. Otherwise stated, if $A := \{x \in \mathcal{M} : h(x) = 1\}$, then $h = \chi_A$.

To prove Lemma 2.8, it suffices to prove that either $\mathcal{H}^k(A) = 0$, or $\mathcal{H}^k(\mathcal{M} \setminus A) = 0$. We argue by contradiction, and assume that $0 < \mathcal{H}^k(A) < \mathcal{H}^k(\mathcal{M})$. For $0 < r < r_0$ fixed, we define

$$\Psi_r \in C(\mathcal{M}), \Psi_r(x) := \frac{\mathcal{H}^k(B_r(x) \cap A)}{\mathcal{H}^k(B_r(x))}, \forall x \in \mathcal{M}.$$

First we give the following claim.

Claim 2.9. *There exists $0 < \eta < 1/2$ such that for r sufficiently small, there exists $x_0 = x_0(r) \in \mathcal{M}$ satisfying*

$$\eta \leq \Psi_r(x_0) \leq 1 - \eta.$$

Taking Claim 2.9 for granted, we conclude the proof as follows. For r sufficiently small, we have

$$\begin{aligned} M_r(h) &\geq \int_{B_r(x_0)} \int_{B_r(x_0)} |h(x) - h(y)| d\mathcal{H}^k(x) d\mathcal{H}^k(y) \\ &= 2\Psi_r(x_0)(1 - \Psi_r(x_0)) \geq 2\eta^2. \end{aligned}$$

Letting $r \rightarrow 0$, we get a contradiction to the fact that $h = \chi_A$ is in VMO.

We finish the proof by proving Claim 2.9. Again we argue by contradiction. By continuity of Ψ_r , it suffices to assume that $\Psi_r < \eta$ for all $x \in \mathcal{M}$, which yields

$$\int_{\mathcal{M}} \chi_{B_r(x) \cap A}(y) d\mathcal{H}^k(y) \leq \eta \int_{\mathcal{M}} \chi_{B_r(x)}(y) d\mathcal{H}^k(y). \quad (2.16)$$

Integrating over $\mathcal{M}$ with respect to x on both side of (2.16), and using Fubini–Tonelli’s theorem, we have

$$\int_A \mathcal{H}^k(B_r(y)) d\mathcal{H}^k(y) \leq \eta \int_{\mathcal{M}} \mathcal{H}^k(B_r(y)) d\mathcal{H}^k(y). \quad (2.17)$$

Using (j) the fact that $\mathcal{H}^k(B_r(y)) \sim r^k$, $\forall y \in \mathcal{M}$, $0 < r < r_0$, with r_0 as in (2.2); and (jj) (2.17), we deduce that

$$\eta \geq C \frac{\mathcal{H}^k(A)}{\mathcal{H}^k(\mathcal{M})},$$

with the constant C comes from the relationship between $\mathcal{H}^k(B_r(y))$ and r^k . Thus, we get a contradiction by letting $\eta = C \mathcal{H}^k(A)/(2\mathcal{H}^k(\mathcal{M}))$. $\square$

2.4 Existence and uniqueness of a VMO lifting

We now turn to existence and uniqueness of liftings of VMO maps through a Riemannian covering. Here, $\mathcal{M}$ is a connected, compact k -dimensional Lipschitz manifold embedded into $\mathbb{R}^m$, $\mathcal{N}$ is a closed connected Riemannian manifold isometrically embedded into $\mathbb{R}^n$, $\mathcal{E}$ is a complete, connected Riemannian manifold, isometrically embedded into $\mathbb{R}^v$ via a tubed embedding, in such a way that the geodesic distance has polynomial growth with respect to the distance induced by the ambient space, and $\pi: \mathcal{E} \rightarrow \mathcal{N}$ is a Riemannian covering.

Theorem 2.10 (Existence). *For $f \in \text{VMO}(\mathcal{M}; \mathcal{N})$, there exists $\varphi \in \text{VMO}(\mathcal{M}; \mathcal{E})$ such that $f = \pi \circ \varphi$. Moreover, there exist a number $\delta_1 > 0$ and a constant C such that*

$$M_r(f) \leq \delta_1, \forall r \leq \tilde{\varepsilon}_0, \text{ for some } \tilde{\varepsilon}_0 := \tilde{\varepsilon}_0(f) \implies M_r(\varphi) \leq CM_r(f), \forall r \leq \tilde{\varepsilon}_0. \quad (2.18)$$

Proof. The proof is similar to the one of [15, Theorem 3], with a slight adaptation to the case of a general target $\mathcal{N}$. We divide the proof into several steps.

Step 1. Existence of φ .

Set $F(x, \varepsilon) := f^\varepsilon(x)$, where f^ε is defined in (2.8). In particular, $F \in C(\mathcal{M} \times (0, \varepsilon_1); \mathcal{N})$.

Since $\pi_1(\mathcal{M} \times (0, \varepsilon_1)) = 0$, there exists a map $\Phi \in C(\mathcal{M} \times (0, \varepsilon_1); \mathcal{E})$ such that

$$F(x, \varepsilon) = \pi \circ \Phi(x, \varepsilon).$$

Let $\varphi_\varepsilon(x) = \Phi(x, \varepsilon)$. By construction, $\varphi_\varepsilon \in C(\mathcal{M}; \mathcal{E})$ and $f^\varepsilon = \pi \circ \varphi_\varepsilon$. Since $f^\varepsilon \rightarrow f$ in $(\text{BMO} \cap \mathcal{L}^1)(\mathcal{M}; \mathcal{N})$, then, up to extraction of a subsequence,

$$f^\varepsilon(x) \rightarrow f(x) \text{ as } \varepsilon \rightarrow 0 \text{ for a.e. } x \in \mathcal{M}. \quad (2.19)$$

By (2.19), we have $\lim_{\varepsilon \rightarrow 0} \pi \circ \Phi(x, \varepsilon) = f(x)$. Viewing $\Phi(x, \cdot): (0, \varepsilon_1) \rightarrow \mathcal{N}$ as a continuous map and using Lemma 2.7, we have that the limit $\lim_{\varepsilon \rightarrow 0} \varphi_\varepsilon(x)$ exists along the subsequence and it is one of the point in the set $\pi^{-1}\{f(x)\}$, denoted by $\varphi(x)$. Moreover, we have

$$f(x) = \pi \circ \varphi(x) \text{ for a.e. } x \in \mathcal{M}.$$

Step 2. There exist a constant $C_1 \geq 1$ independent of f and $\varepsilon'_0 > 0$ such that

$$M_r(\varphi_\varepsilon) \leq C_1 M_r(f^\varepsilon) + C_1 [M_r(\varphi_\varepsilon)]^2 + C_1 [M_r(\varphi_\varepsilon)]^{2p_0}, \forall r \leq \varepsilon'_0, \varepsilon \leq \varepsilon_1. \quad (2.20)$$

Indeed, Lemma 2.7 and the polynomial growth assumption (2.9) imply that

$$\begin{aligned} & \text{dist}_{\mathcal{E}}(\varphi_\varepsilon(y), \varphi_\varepsilon(z)) \\ & \leq \text{dist}_{\mathcal{N}}(f^\varepsilon(y), f^\varepsilon(z)) + (\text{Inj}(\mathcal{N}))^{-1} [\text{dist}_{\mathcal{E}}(\varphi_\varepsilon(y), \varphi_\varepsilon(z))]^2 \\ & \lesssim |f^\varepsilon(y) - f^\varepsilon(z)| + |\varphi_\varepsilon(y) - \varphi_\varepsilon(z)|^2 + |\varphi_\varepsilon(y) - \varphi_\varepsilon(z)|^{2p_0}. \end{aligned} \quad (2.21)$$

Again, let us work with $r \leq r_0$, with r_0 as in (2.2) and (2.3). We estimate

$$\int_{B_r(x)} \int_{B_r(x)} |\varphi_\varepsilon(y) - \varphi_\varepsilon(z)|^{2p_0} d\mathcal{H}^k(y) d\mathcal{H}^k(z),$$

and the term involving $|\varphi_\varepsilon(y) - \varphi_\varepsilon(z)|^2$ on the right-hand side of (2.21) is a special case when $p_0 = 1$. Repeating the calculations as in (2.12), we have

$$\int_{B_r(x)} \int_{B_r(x)} |\varphi_\varepsilon(y) - \varphi_\varepsilon(z)|^{2p_0} d\mathcal{H}^k(y) d\mathcal{H}^k(z) \lesssim M_r^{2p_0}(\varphi_\varepsilon). \quad (2.22)$$

Combining (2.22) and (2.21), we obtain (2.20).

Step 3. There exist some $\tilde{\varepsilon}_0 := \tilde{\varepsilon}_0(f)$ and $\tilde{\varepsilon}_1 := \tilde{\varepsilon}_1(f)$ such that

$$M_r(\varphi_\varepsilon) \lesssim M_r(f^\varepsilon), \forall r \leq \tilde{\varepsilon}_0, \varepsilon \leq \tilde{\varepsilon}_1. \quad (2.23)$$

As we now show, (2.23) holds provided that $\tilde{\varepsilon}_0$ and $\tilde{\varepsilon}_1$ are such that the (2.25) below holds.

To prove (2.23), it suffices to prove that there exists some $\tilde{\varepsilon}_0$ and $\tilde{\varepsilon}_1$ such that

$$M_r(\varphi_\varepsilon) \leq \frac{1}{4C_1}, \forall r \leq \tilde{\varepsilon}_0, \varepsilon \leq \tilde{\varepsilon}_1. \quad (2.24)$$

Indeed, granted (2.24), by (2.20), we have

$$M_r(\varphi_\varepsilon) \lesssim M_r(f^\varepsilon) + \frac{1}{4}M_r(\varphi_\varepsilon) + \frac{1}{4^{2p_0-1}C_1^{2p_0-2}}M_r(\varphi_\varepsilon),$$

whence (2.23), absorbing the terms involving φ_ε back into the left-hand side.

Now we prove (2.24). Since $\varphi_\varepsilon \in C(\mathcal{M}; \mathcal{E})$, we have $M_r(\varphi_\varepsilon) \rightarrow 0$ as $r \rightarrow 0$. Combining this and the fact that $M_r(\varphi_\varepsilon)$ is continuous with respect to r , it suffices to prove that there exists some $\tilde{\varepsilon}_0$ and $\tilde{\varepsilon}_1$ such that

$$M_r(f^\varepsilon) \leq \frac{1}{9C_1}, \forall r \leq \tilde{\varepsilon}_0, \varepsilon \leq \tilde{\varepsilon}_1. \quad (2.25)$$

Indeed, we argue by contradiction and assume that $M_r(\varphi_\varepsilon) > 1/(4C_1)$ for some $r \leq \tilde{\varepsilon}_0$ and $\varepsilon \leq \tilde{\varepsilon}_1$. Then, there exists $r' < r$ such that $M_{r'}(\varphi_\varepsilon) = 1/(4C_1)$, which implies, by (2.20) and (2.25), that $M_{r'}(\varphi_\varepsilon) \leq 1/(9C_1) + 1/(8C_1) < 1/(4C_1)$, a contradiction. Let us mention that this relies on the continuity of $M_r(\varphi_\varepsilon)$ with respect to r , which follows readily from Lebesgue's lemma.

In order to prove (2.25), we first notice that

$$M_r(f^\varepsilon) \leq \|f - f^\varepsilon\|_{\text{BMO}} + M_r(f). \quad (2.26)$$

Then we conclude the existence of $\tilde{\varepsilon}_0$ and $\tilde{\varepsilon}_1$ by using (2.26) and the fact that: (j) $f^\varepsilon \rightarrow f$ in $\text{BMO} \cap \mathcal{L}^1(\mathcal{M}; \mathcal{N})$; and (jj) $f \in \text{VMO}$.

Step 4. The map φ is in $\text{VMO}(\mathcal{M}; \mathcal{E})$.

By Fatou's lemma, it suffices to prove that

$$M_r(\varphi_\varepsilon) \lesssim M_{2\varepsilon}(f) + M_r(f), \forall r \leq \tilde{\varepsilon}_0, \varepsilon \leq \tilde{\varepsilon}_1/2. \quad (2.27)$$

By (2.23), it suffices to prove that

$$M_r(f^\varepsilon) \lesssim M_{2\varepsilon}(f) + M_r(f), \forall r \leq \tilde{\varepsilon}_0, \varepsilon \leq \tilde{\varepsilon}_1/2.$$

We start with the estimate

$$M_r(f^\varepsilon) \leq M_r(f) + M_r(f - f^\varepsilon), \quad (2.28)$$

and it remains to upper bound $M_r(f - f^\varepsilon)$. For any $x \in \mathcal{M}$ and $y, z \in B_r(x)$, we first notice that

$$\begin{aligned} & |f(y) - f^\varepsilon(y) - (f(z) - f^\varepsilon(z))| \\ & \leq |\Pi \circ f(y) - \Pi \circ f_\varepsilon(y) - (\Pi \circ f(z) - \Pi \circ f_\varepsilon(z))| \\ & \leq |\Pi \circ [f_\varepsilon(y) - f(y) + f(y)] - \Pi \circ [f_\varepsilon(z) - f(z) + f(z)]| \\ & \quad + |\Pi \circ f(y) - \Pi \circ f(z)| \\ & \lesssim [|f(y) - f_\varepsilon(y) - (f(z) - f_\varepsilon(z))| + |f(y) - f(z)|]. \end{aligned}$$

Hence, we have

$$\begin{aligned} & \int_{B_r(x)} \int_{B_r(x)} |f(y) - f^\varepsilon(y) - (f(z) - f^\varepsilon(z))| d\mathcal{H}^k(y) d\mathcal{H}^k(z) \\ & \lesssim M_r(f) + M_r(f - f_\varepsilon) \lesssim M_r(f) + M_{2\varepsilon}(f), \end{aligned} \quad (2.29)$$

where in the last inequality we use the fact that (see, e.g., [19, Lemma 2.5])

$$|f - f_\varepsilon|_{\text{BMO}} \lesssim M_{2\varepsilon}(f), \forall f \in \text{BMO}, \forall 0 < \varepsilon \leq \varepsilon_0/2.$$

We conclude (2.27), and thus Step 4, via (2.28) and (2.29).

Step 5. The map φ satisfies (2.18).

Repeating Step 2 and Step 3 and replacing φ_ε , respectively f^ε , by φ , respectively f , we conclude the existence of a constant C and a number $\delta_1 > 0$ such that, if

$$M_r(f) \leq \delta_1, \forall r \leq \tilde{\varepsilon}_0,$$

then

$$M_r(\varphi) \leq CM_r(f), \forall r \leq \tilde{\varepsilon}_0.$$

This concludes the proof. $\square$

We next show the uniqueness of liftings in VMO. To state our uniqueness result, we recall the definition of a *normal* covering map. We say that a covering map $\pi: \mathcal{E} \rightarrow \mathcal{N}$ is *normal* whenever, for every $e_1, e_2 \in \mathcal{E}$ such that $\pi(e_1) = \pi(e_2)$, there exists a deck transformation $\tau \in \text{Aut}(\pi)$ such that $\tau(e_1) = e_2$. To highlight the importance of this

assumption in our setting, let us recall that there are indeed compact Riemannian manifolds $\mathcal{N}$ admitting non-normal covering maps $\pi: \mathcal{E} \rightarrow \mathcal{N}$. Indeed, we know that $\pi: \mathcal{E} \rightarrow \mathcal{N}$ is normal if and only if $\pi_{\#}(\pi_1(\mathcal{E}))$ is a normal subgroup of $\pi_1(\mathcal{N})$ (see, e.g., [24, Proposition 1.39]), and meanwhile, for every subgroup $H \leq \pi_1(\mathcal{N})$, there exists a covering map $\pi: \mathcal{E} \rightarrow \mathcal{N}$ such that $\pi_{\#}(\pi_1(\mathcal{E})) = H$ (see, e.g., [24, Proposition 1.36]). Hence, any compact manifold whose fundamental group has a non-normal subgroup will provide an example of a compact manifold with a non-normal covering space. One can take, for instance, $\mathcal{N} = \mathbb{T}^1 \# \mathbb{T}^1$ to be the two-holed torus, which satisfies $\pi_1(\mathcal{N}) = F(a, b) *_{\mathbb{Z}} F(c, d) = \langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} = 1 \rangle$. Here, for example, the subgroup generated by a is not a normal subgroup.

Theorem 2.11 (Uniqueness). *Assume that the covering map $\pi: \mathcal{E} \rightarrow \mathcal{N}$ is normal. Then, the lifting φ of $f \in \text{VMO}(\mathcal{M}; \mathcal{N})$ is unique up to deck transformations.*

Proof. We first notice that, by Definition 2.6 of $\text{Aut}(\pi)$, if $\varphi \in \text{VMO}(\mathcal{M}; \mathcal{E})$ is a lifting of f and $\tau \in \text{Aut}(\pi)$, then $\tau \circ \varphi \in \text{VMO}(\mathcal{M}; \mathcal{E})$ is another lifting of f .

Next, assume that φ' is another lifting of f . Let $x \in \mathcal{M}$ be a Lebesgue point of φ , φ' and f .

Let $\tau \in \text{Aut}(\pi)$ be such that $\tau \circ \varphi(x) = \varphi'(x)$, which exists by virtue of the assumption that π is a normal covering map. We will prove that $\tau \circ \varphi = \varphi'$ a.e. By Lemma 2.8, it suffices to find a set $A \subset \mathcal{M}$ with positive measure such that $\tau \circ \varphi(y) = \varphi'(y)$ for all $y \in A$.

Since

$$\lim_{r \rightarrow 0} \int_{B_r(x)} [\text{dist}_{\mathcal{E}}(\varphi(y), \varphi(x)) + \text{dist}_{\mathcal{E}}(\varphi'(y), \varphi'(x))] d\mathcal{H}^k(y) = 0,$$

we can choose r sufficiently small such that

$$\int_{B_r(x)} [\text{dist}_{\mathcal{E}}(\varphi(y), \varphi(x)) + \text{dist}_{\mathcal{E}}(\varphi'(y), \varphi'(x))] d\mathcal{H}^k(y) < \frac{\text{Inj}(\mathcal{N})}{2}.$$

Define

$$A := \{y \in \mathcal{M} : \text{dist}_{\mathcal{E}}(\varphi(x), \varphi(y)) + \text{dist}_{\mathcal{E}}(\varphi'(x), \varphi'(y)) < \text{Inj}(\mathcal{N})\}.$$

Thus we have $\mathcal{H}^k(A) > 0$.

On the other hand, $\forall y \in A$, since

$$\begin{aligned} \text{dist}_{\mathcal{G}}(\tau \circ \varphi(y), \varphi'(y)) &\leq \text{dist}_{\mathcal{G}}(\tau \circ \varphi(y), \tau \circ \varphi(x)) + \text{dist}_{\mathcal{G}}(\tau \circ \varphi(x), \varphi'(y)) \\ &= \text{dist}_{\mathcal{G}}(\varphi(y), \varphi(x)) + \text{dist}_{\mathcal{G}}(\varphi'(x), \varphi'(y)) \leq \text{Inj}(\mathcal{N}), \end{aligned}$$

by Lemma 2.7, we have

$$\begin{aligned} \text{dist}_{\mathcal{G}}(\tau \circ \varphi(y), \varphi'(y)) &= \text{dist}_{\mathcal{N}}(\pi \circ \tau \circ \varphi(y), \pi \circ \varphi'(y)) \\ &= \text{dist}_{\mathcal{N}}(\pi \circ \varphi(y), \pi \circ \varphi'(y)) \\ &= \text{dist}_{\mathcal{N}}(f(y), f(y)) = 0. \end{aligned}$$

This implies that both liftings $\tau \circ \varphi$ and φ' coincide in a subset of $\mathcal{M}$ with positive measure and thus, by Lemma 2.8, coincide in $\mathcal{M}$ a.e. $\square$

2.5 A selection result

This section is devoted to a “selection” result, which asserts that, under suitable assumptions, liftings along a converging sequence can always be chosen in such a way that they converge to a lifting of the limit. This can be viewed as some kind of continuity of the lifting operation with respect to the $\text{BMO} \cap \mathcal{L}^1$ topology on VMO .

Theorem 2.12 (Continuity of φ with respect to f). *Assume that the covering map $\pi: \mathcal{E} \rightarrow \mathcal{N}$ is normal. Let $f_\ell, f \in \text{VMO}(\mathcal{M}; \mathcal{N})$ be such that $f_\ell \rightarrow f$ in $(\text{BMO} \cap \mathcal{L}^1)(\mathcal{M}; \mathcal{N})$. Then, there exists a sequence of corresponding liftings $\varphi_\ell, \varphi \in \text{VMO}(\mathcal{M}; \mathcal{E})$ such that $\varphi_\ell \rightarrow \varphi$ in $(\text{BMO} \cap \mathcal{L}^1)(\mathcal{M}; \mathcal{E})$.*

Proof. By Theorem 2.10, every f_ℓ admits a lifting φ_ℓ such that

$$M_r(\varphi_\ell) \leq C M_r(f_\ell), \forall r \leq \tilde{\varepsilon}_0(f_\ell),$$

provided that $\tilde{\varepsilon}_0(f_\ell)$ satisfies

$$M_r(f_\ell) \leq \delta_1, \forall r \leq \tilde{\varepsilon}_0(f_\ell).$$

Since

$$M_r(f_\ell) \leq M_r(f) + |f_\ell - f|_{\text{BMO}},$$

using the fact that $f_\ell \rightarrow f$ in BMO, there exists $\tilde{\varepsilon}_0 := \tilde{\varepsilon}_0(f)$ such that

$$M_r(f_\ell) \leq \delta_1, \forall r \leq \tilde{\varepsilon}_0, \forall \ell$$

and thus

$$M_r(\varphi_\ell) \leq CM_r(f_\ell) \leq C\delta_1, \forall r \leq \tilde{\varepsilon}_0. \quad (2.30)$$

By [19, Lemma 2.1],

$$\|\tau_\ell \circ \varphi_\ell - \varphi\|_1 \lesssim |\tau_\ell \circ \varphi_\ell - \varphi|_{\text{BMO}} + \left| \int_{\mathcal{M}} (\tau_\ell \circ \varphi_\ell - \varphi) \right|. \quad (2.31)$$

We divide the proof into 3 parts: First, we show that for each ℓ , all the liftings of f_ℓ have a uniform BMO bound similar to (2.30); second, we find, for each ℓ , a suitable τ_ℓ such that $\int_{\mathcal{M}} \tau_\ell \circ \varphi_\ell$ is bounded uniformly; third, based on first two steps, we can pass to the limit (up to a subsequence) and obtain a specific lifting of f , and then we modify the original sequence *via* deck transformations to make the whole sequence converge.

Step 1. Estimate of $|\tau \circ \varphi_\ell|_{\text{BMO}}$.

In this step, τ can be any element in $\text{Aut}(\pi)$. Later, in Step 2, we will take $\tau = \tau_\ell$ suitably chosen, depending on ℓ . We first recall that, $\forall \tau \in \text{Aut}(\pi)$, τ is an isometry, and therefore

$$\begin{aligned} |\tau \circ \varphi_\ell(x) - \tau \circ \varphi_\ell(y)| &\leq \text{dist}_{\mathcal{G}}(\tau \circ \varphi_\ell(x), \tau \circ \varphi_\ell(y)) \\ &= \text{dist}_{\mathcal{G}}(\varphi_\ell(x), \varphi_\ell(y)). \end{aligned} \quad (2.32)$$

Thus, by (2.14), which holds because we work under the polynomial growth assumption (2.9), and (2.30), we have

$$M_r(\tau \circ \varphi_\ell) \leq CM_r(f_\ell) + C[M_r(f_\ell)]^{p_0}, \forall \ell, \forall \tau \in \text{Aut}(\pi), \forall r \leq \tilde{\varepsilon}_0. \quad (2.33)$$

For $\tilde{\varepsilon}_0 \leq r \leq r_0$ and any $x \in \mathcal{M}$, by (2.9), (2.32), and the same calculation as in (2.13), we have

$$\begin{aligned} &\int_{B_r(x)} \int_{B_r(x)} |\tau \circ \varphi_\ell(y) - \tau \circ \varphi_\ell(z)| d\mathcal{H}^k(y) d\mathcal{H}^k(z) \\ &\leq C(\tilde{\varepsilon}_0)[M_{\tilde{\varepsilon}_0}(\varphi_\ell) + (M_{\tilde{\varepsilon}_0}(\varphi_\ell))^{p_0}]. \end{aligned}$$

Hence, by (2.30), we have

$$\sup_{\ell} \sup_{\tau \in \text{Aut}(\pi)} |\tau \circ \varphi_{\ell}|_{\text{BMO}} < \infty. \quad (2.34)$$

Step 2. In this step, we will prove that, for a suitable choice of $\tau_{\ell} \in \text{Aut}(\pi)$,

$$\sup_{\ell} \left| \int_{\mathcal{M}} \tau_{\ell} \circ \varphi_{\ell} \right| < \infty. \quad (2.35)$$

Without loss of generality, we assume that $\tilde{\varepsilon}_0 \leq r_0$, with r_0 defined in (2.2) and (2.3). To avoid confusion, in this proof geodesic balls on $\mathcal{E}$ are distinguished by a tilde.

Consider a covering of $\mathcal{M}$ as in the proof of Lemma 2.3, with the radius r_0 replaced with $\tilde{\varepsilon}_0$, and pick one ball, denoted by B , from the covering. Using the fact that τ_{ℓ} is an isometry and the polynomial growth condition (2.9), similarly to (2.13) and (2.14), we have

$$\begin{aligned} & \left| \int_{\mathcal{M}} \tau_{\ell} \circ \varphi_{\ell} \right| \\ & \leq \int_{\mathcal{M}} \left| \tau_{\ell} \circ \varphi_{\ell} - \int_B \tau_{\ell} \circ \varphi_{\ell} \right| + C_1 \left| \int_B \tau_{\ell} \circ \varphi_{\ell} \right| \\ & \leq \int_{\mathcal{M}} \int_B \text{dist}_{\mathcal{E}}(\tau_{\ell} \circ \varphi_{\ell}(x), \tau_{\ell} \circ \varphi_{\ell}(y)) d\mathcal{H}^k(y) d\mathcal{H}^k(x) + C_1 \left| \int_B \tau_{\ell} \circ \varphi_{\ell} \right| \\ & = \int_{\mathcal{M}} \int_B \text{dist}_{\mathcal{E}}(\varphi_{\ell}(x), \varphi_{\ell}(y)) d\mathcal{H}^k(y) d\mathcal{H}^k(x) + C_1 \left| \int_B \tau_{\ell} \circ \varphi_{\ell} \right| \\ & \leq C(\tilde{\varepsilon}_0)[M_{\tilde{\varepsilon}_0}(\varphi_{\ell}) + (M_{\tilde{\varepsilon}_0}(\varphi_{\ell}))^{p_0}] + C_1 \left| \int_B \tau_{\ell} \circ \varphi_{\ell} \right|. \end{aligned} \quad (2.36)$$

By (2.30), the first term in the last line of (2.36) is bounded uniformly. We now turn to the second term. We first notice that, since deck transformations are isometries, for any $y \in B$ and $\tau_{\ell} \in \text{Aut}(\pi)$,

$$\left| \int_B \tau_{\ell} \circ \varphi_{\ell} \right| \leq \int_B \text{dist}_{\mathcal{E}}(\varphi_{\ell}(x), \varphi_{\ell}(y)) d\mathcal{H}^k(x) + \left| \tau_{\ell} \circ \varphi_{\ell}(y) \right|. \quad (2.37)$$

Therefore, it suffices to find, for each ℓ , some $y_{\ell} \in B$ and $\tau_{\ell} \in \text{Aut}(\pi)$ such that the right-hand side of (2.37) has a uniform bound.

Since $\mathcal{N}$ is compact, we have $\rho := \text{Inj}(\mathcal{N}) > 0$. Consider a finite covering $\{B_{\rho}(\mathfrak{n}_t)\}_{t=1}^T$ of $\mathcal{N}$. For each $\mathfrak{n}_t$, pick $\mathfrak{e}_t \in \pi^{-1}(\{\mathfrak{n}_t\})$, and let $\tilde{B}_{R_0}(\mathfrak{e}_0)$ be a ball on $\mathcal{E}$ that covers $\bigcup_{t=1}^T \tilde{B}_{\rho}(\mathfrak{e}_t)$.

Let $t := t(\ell)$ be such that

$$\mathcal{H}^k(f_\ell^{-1}(B_\rho(n_t)) \cap B) \geq \mathcal{H}^k(B)/T. \quad (2.38)$$

For simplicity, with an abuse of notation, we let $n_\ell := n_{t(\ell)}$. By Lemma 2.7, we have

$$\pi^{-1}(B_\rho(n_\ell)) = \bigsqcup_{j \geq 0} \widetilde{B}_\rho(e_\ell^j), \quad (2.39)$$

where the e_ℓ^j are those points in $\mathcal{E}$ such that $\pi(e_\ell^j) = n_\ell$. Without loss of generality, we assume that $e_\ell^0 \in \widetilde{B}_{R_0}(e_0)$.

In order to deal with the two terms on the right-hand side of (2.37), we define

$$A_\ell := \left\{ y \in B : \int_B \text{dist}_{\mathcal{E}}(\varphi_\ell(x), \varphi_\ell(y)) d\mathcal{H}^k(x) \leq bC\delta_1 \right\},$$

with $b > 1$ a constant to be chosen later, and

$$E_\ell^j := \varphi_\ell^{-1}(\widetilde{B}_\rho(e_\ell^j)) \cap B. \quad (2.40)$$

First, by definition of A_ℓ and (2.30), we have

$$\mathcal{H}^k(A_\ell)/\mathcal{H}^k(B) \geq (b-1)/b. \quad (2.41)$$

Next, we claim that

$$\inf_\ell \sup_j \mathcal{H}^k(E_\ell^j) > 0. \quad (2.42)$$

The proof of this claim is deferred to Step 4.

Granted (2.42) and using (2.41), we can choose b sufficiently large (independent of ℓ) so that, for each ℓ , there exists a point $y_\ell \in B$ such that

- (i) $\int_B \text{dist}_{\mathcal{E}}(\varphi_\ell(x), \varphi_\ell(y_\ell)) d\mathcal{H}^k(x) \leq bC\delta_1$;
- (ii) $\varphi_\ell(y_\ell) \in \widetilde{B}_\rho(e_\ell^j)$ for some $j := j(\ell)$.

For each ℓ , let $\tau_\ell \in \text{Aut}(\pi)$ be such that $\tau_\ell(e_\ell^j) = e_\ell^0$. Since deck transformations are isometries, we have

- (i) $\int_B \text{dist}_{\mathcal{E}}(\tau_\ell \circ \varphi_\ell(x), \tau_\ell \circ \varphi_\ell(y_\ell)) d\mathcal{H}^k(x) \leq bC\delta_1$;
- (ii) $\tau_\ell \circ \varphi_\ell(y_\ell) \in \widetilde{B}_\rho(e_\ell^0) \subset \widetilde{B}_{\rho+R_0}(e_0)$.

Thus, by (2.36) and (2.37), we obtain (2.35).

Step 3. Construction of a converging sequence $(\tau_\ell \circ \varphi_\ell)$.

Since $f_\ell \rightarrow f$ in BMO, by [15, Lemma 4] (there the domain is a smooth manifold, but the argument can readily be adapted to Lipschitz manifolds), we have

$$M_r(f_\ell) \rightarrow 0 \text{ uniformly as } r \rightarrow 0, \quad (2.43)$$

which, combined with (2.33), implies that

$$M_r(\tau_\ell \circ \varphi_\ell) \rightarrow 0 \text{ uniformly as } r \rightarrow 0. \quad (2.44)$$

Using [15, Lemma 4] again, (2.44) implies $(\tau_\ell \circ \varphi_\ell)$ is compact. Then, possibly up to a subsequence (still denoted $(\tau_\ell \circ \varphi_\ell)$ for simplicity), we have

$$\tau_\ell \circ \varphi_\ell \rightarrow \tilde{\varphi} \text{ in } \text{BMO}(\mathcal{M}), \quad (2.45)$$

with a map $\tilde{\varphi} \in \text{VMO}(\mathcal{M})$ and $\int_{\mathcal{M}} \tilde{\varphi} = 0$. Here, convergence in BMO means convergence of the BMO seminorms only, which in particular enables us to choose $\tilde{\varphi}$ having zero average.

We now take care of the convergence in $\mathcal{L}^1$. First, since the sequence $(\tau_\ell \circ \varphi_\ell)$ satisfies (2.35) in Step 2, we have, possibly up to a further subsequence,

$$\int_{\mathcal{M}} \tau_\ell \circ \varphi_\ell \rightarrow \eta \quad (2.46)$$

for some constant η . With $\varphi := \tilde{\varphi} + \eta$, we have, by (2.31), (2.46), and (2.45), that $\tau_\ell \circ \varphi_\ell \rightarrow \varphi$ in $(\text{BMO} \cap \mathcal{L}^1)(\mathcal{M}; \mathcal{E})$. Since the $\mathcal{L}^1$ convergence implies, up to a subsequence, a.e. convergence, we have, for a.e. $x \in \mathcal{M}$,

$$f(x) = \lim_{\ell \rightarrow \infty} f_\ell(x) = \lim_{\ell \rightarrow \infty} \pi \circ \varphi_\ell(x) = \pi \circ (\lim_{\ell} \varphi_\ell(x)) = \pi \circ \varphi,$$

and therefore φ is a lifting of f .

So far, (2.45) and (2.46) hold along a subsequence of $(\tau_\ell \circ \varphi_\ell)$. Next, we want to make the whole sequence (φ_ℓ) converge, up to deck transformations, to φ in $\text{BMO} \cap \mathcal{L}^1$. For this purpose, we first notice that, by (2.34), (2.35), and (2.31),

$$\sup_{\ell} \inf_{\tau} \{\|\tau \circ \varphi_\ell - \varphi\|_1 : \tau \in \text{Aut}(\pi)\} < \infty.$$

Let $\tilde{\tau}_\ell$ be such that

$$\|\tilde{\tau}_\ell \circ \varphi_\ell - \varphi\|_1 \leq \inf_{\tau} \{\|\tau \circ \varphi_\ell - \varphi\|_1\} + 1/\ell. \quad (2.47)$$

Thus we have

$$\begin{aligned} \left| \int_{\mathcal{M}} \tilde{\tau}_\ell \circ \varphi_\ell \right| &\lesssim \eta + \|\tau_\ell \circ \varphi_\ell - \varphi\|_1 + 1/\ell \\ &\lesssim \eta + |\tau_\ell \circ \varphi_\ell|_{\text{BMO}} + |\varphi|_{\text{BMO}} + \left| \int_{\mathcal{M}} \tau_\ell \circ \varphi_\ell \right| + 1/\ell. \end{aligned} \quad (2.48)$$

Using (2.35) and (2.34), the right-hand side of (2.48) is bounded, with a bound that does not depend on ℓ .

For any subsequence of $(\tilde{\tau}_\ell \circ \varphi_\ell)$, by (2.48) and the same argument as in (2.43)–(2.45), we can obtain a further subsequence $\tilde{\tau}_{\ell_\beta} \circ \varphi_{\ell_\beta} \rightarrow \varphi'$ in $\text{BMO} \cap \mathcal{L}^1$ and a.e. By Lemma 2.11, there exists $\tau \in \text{Aut}(\pi)$ such that $\tau \circ \varphi' = \varphi$. Therefore, by (2.47) and Fatou's lemma, we have

$$\begin{aligned} \|\tilde{\tau}_{\ell_\beta} \circ \varphi_{\ell_\beta} - \varphi\|_1 &\leq \|(\tau \circ \tilde{\tau}_{\ell_\beta}) \circ \varphi_{\ell_\beta} - \varphi\|_1 + 1/\ell_\beta \\ &= \|\tau \circ (\tilde{\tau}_{\ell_\beta} \circ \varphi_{\ell_\beta}) - \tau \circ \varphi'\|_1 + 1/\ell_\beta \\ &\lesssim \int_{\mathcal{M}} \text{dist}(\tilde{\tau}_{\ell_\beta} \circ \varphi_{\ell_\beta}, \varphi') + 1/\ell_\beta \rightarrow 0, \end{aligned}$$

which means that $\varphi = \varphi'$. Then the whole sequence $(\tilde{\tau}_\ell \circ \varphi_\ell)$ converge to φ in $\text{BMO} \cap \mathcal{L}^1$.

We finish the proof by justifying the claim (2.42) in Step 2.

Step 4. In this step, we prove (2.42).

Since the E_ℓ^j defined in (2.40) are disjoint, by (2.38) and (2.39), we have

$$\sum_j \mathcal{H}^k(E_\ell^j) = \mathcal{H}^k(B \cap f_\ell^{-1}(B_\rho(\mathfrak{n}_\ell))) \geq \mathcal{H}^k(B)/T. \quad (2.49)$$

On the other hand,

$$\begin{aligned}
& \int_B \int_B \text{dist}(\varphi_\ell(x), \varphi_\ell(y)) \, d\mathcal{H}^k(x) \, d\mathcal{H}^k(y) \\
& \geq \sum_j \sum_{j'} \int_{E_\ell^j} \int_{E_\ell^{j'}} \text{dist}(\varphi_\ell(x), \varphi_\ell(y)) \, d\mathcal{H}^k(x) \, d\mathcal{H}^k(y) \\
& \geq \sum_j \sum_{j'} \int_{E_\ell^j} \int_{E_\ell^{j'}} \text{dist}(\widetilde{B}_\rho(e_\ell^j), \widetilde{B}_\rho(e_\ell^{j'})) \, d\mathcal{H}^k(x) \, d\mathcal{H}^k(y) \\
& \geq \sum_j \sum_{j'} \mathcal{H}^k(E_\ell^j) \mathcal{H}^k(E_\ell^{j'}) \text{dist}(\widetilde{B}_\rho(e_\ell^j), \widetilde{B}_\rho(e_\ell^{j'})).
\end{aligned} \tag{2.50}$$

Combining (2.50) and (2.30), we have

$$\sum_j \sum_{j'} \mathcal{H}^k(E_\ell^j) \mathcal{H}^k(E_\ell^{j'}) \text{dist}(\widetilde{B}_\rho(e_\ell^j), \widetilde{B}_\rho(e_\ell^{j'})) \leq C \delta_1 [\mathcal{H}^k(B)]^2. \tag{2.51}$$

Let R be a constant chosen later. For j fixed, define

$$D_\ell^j := \{j' : \text{dist}(\widetilde{B}_\rho(e_\ell^j), \widetilde{B}_\rho(e_\ell^{j'})) \leq R\}. \tag{2.52}$$

Consider $\tau_\ell \in \text{Aut}(\pi)$ such that $\tau_\ell^j(e_\ell^j) = e_\ell^0$. By (2.52) and the fact that $e_\ell^0 \in \widetilde{B}_{R_0}(e_0)$, we have

$$\begin{aligned}
\text{card}(D_\ell^j) & \leq \frac{\mathcal{H}^k(\widetilde{B}_{3\rho+R}(e_\ell^j))}{\inf_{e \in \mathcal{E}} \mathcal{H}^k(\widetilde{B}_\rho(e))} \leq C \mathcal{H}^k(\widetilde{B}_{3\rho+R}(\tau_\ell^j \circ e_\ell^j)) \\
& \leq C \mathcal{H}^k(\widetilde{B}_{R+3\rho+R_0}(e_0)) =: C(R).
\end{aligned} \tag{2.53}$$

Combining (2.51) and (2.53), we have

$$\begin{aligned}
C \delta_1 [\mathcal{H}^k(B)]^2 & \geq R \sum_j \sum_{j' \in (D_\ell^j)^c} \mathcal{H}^k(E_\ell^j) \mathcal{H}^k(E_\ell^{j'}) \\
& \geq R \left[\sum_\ell \mathcal{H}^k(E_\ell^j) \right]^2 - R \sum_j \sum_{j' \in D_\ell^j} \mathcal{H}^k(E_\ell^j) \mathcal{H}^k(E_\ell^{j'}) \\
& \geq R \left[\sum_\ell \mathcal{H}^k(E_\ell^j) \right]^2 - R \left[\sum_j \text{card}(D_\ell^j) \mathcal{H}^k(E_\ell^j) \right] \sup_{j'} \mathcal{H}^k(E_\ell^{j'}).
\end{aligned} \tag{2.54}$$

Combining (2.54) with (2.49), we have

$$\sup_{j'} \mathcal{H}^k(E_\ell^{j'}) \geq \frac{[\sum_\ell \mathcal{H}^k(E_\ell^j)]^2 - C\delta_1[\mathcal{H}^k(B)]^2/R}{C(R) \sum_j \mathcal{H}^k(E_\ell^j)} \geq \frac{[\frac{\mathcal{H}^k(B)}{T}]^2 - C\delta_1[\mathcal{H}^k(B)]^2/R}{C(R)\mathcal{H}^k(B)}. \quad (2.55)$$

Hence, if we let

$$R := R(T) = \frac{2T^2}{C\delta_1}, \quad (2.56)$$

then, combining (2.49), (2.55), and (2.56), we deduce that

$$\sup_\ell \mathcal{H}^k(E_\ell^j) \geq \frac{\mathcal{H}^k(B)}{2T^2 C(R)}. \quad (2.57)$$

Since the constant $C(R)$ in (2.53) does not depend on ℓ , neither does the right-hand side of (2.57), which implies (2.42). $\square$

Remark 2.13. If the covering space $\mathcal{E}$ admits a universal covering $\tilde{\pi}: \tilde{\mathcal{E}} \rightarrow \mathcal{E}$ such that $\tilde{\mathcal{E}} \subset \mathbb{R}^{\tilde{v}}$ has polynomial growth, then the normality assumption in Theorem 2.12 can be removed.

Indeed, it suffices to apply Theorem 2.12 to the normal covering $\tilde{\pi} \circ \pi: \tilde{\mathcal{E}} \rightarrow \mathcal{N}$. This yields a sequence of corresponding liftings $\tilde{\varphi}_\ell, \tilde{\varphi} \in \text{VMO}(\mathcal{M}; \tilde{\mathcal{E}})$ such that $\tilde{\varphi}_\ell \rightarrow \tilde{\varphi}$ in $(\text{BMO} \cap \mathcal{L}^1)(\mathcal{M}; \tilde{\mathcal{E}})$. Then, we obtain the desired sequence by letting $\varphi_\ell := \tilde{\pi} \circ \tilde{\varphi}_\ell$ and $\varphi := \tilde{\pi} \circ \tilde{\varphi}$. $\square$

3 The density of the class $\mathcal{R}_0$

In this section, we assume that $\mathcal{M}$ is a k -dimensional connected compact Lipschitz manifold, and that $\mathcal{F}$ is a smooth manifold embedded into $\mathbb{R}^d$ via a tubed embedding. Let us interestingly mention that, usually, the setting for strong approximation of manifold-valued Sobolev maps is the one of *compact* targets. Here, we are able to work also with some non-compact targets, provided that they can be embedded *via* a tubed embedding — this will always be the case for compact manifolds. To the best of our knowledge, the only other work on the strong approximation problem for non-compact targets is the one of Bousquet, Ponce, and Van Schaftingen [10].

We consider the space

$$W^{s,p}(\mathcal{M}; \mathcal{F}) := \{f \in W^{s,p}(\mathcal{M}; \mathbb{R}^d): f(x) \in \mathcal{F} \text{ for a.e. } x \in \mathcal{M}\}. \quad (3.1)$$

We introduce a class of “almost smooth” maps tailored to the setting of Lipschitz

manifolds, adapting the standard definition of the class $\mathcal{R}_0^\infty$ devised by Bethuel and Zheng [7] as follows:

$$\mathcal{R}_0 := \{f: \mathcal{M} \rightarrow \mathcal{F}: f \in \text{Lip}_{\text{loc}}(\mathcal{M} \setminus \mathcal{S}(f)), \mathcal{S}(f) \text{ is a finite subset of } \mathcal{M}, \\ |f(x) - f(y)| \leq \frac{C(f) \text{dist}(x, y)}{\min\{\text{dist}(x, \mathcal{S}(f)), \text{dist}(y, \mathcal{S}(f))\}}, \forall x, y \in \mathcal{M}\}.$$

In the relevant range $0 < s \leq 1, k-1 \leq sp < k$, we will prove that the class $\mathcal{R}_0$ is dense in $W^{s,p}(\mathcal{M}; \mathcal{F})$. Since the arguments we provide depend on s and sp we divide, for the convenience of the reader, this general statement into a number of separate results.

Theorem 3.1. *Let $k-1 \leq p < k$. Then the set $\mathcal{R}_0$ is dense in $W^{1,p}(\mathcal{M}; \mathcal{F})$.*

Theorem 3.2. *Let $0 < s < 1$ and $k-1 \leq sp < k$. Then the set $\mathcal{R}_0$ is dense in $W^{s,p}(\mathcal{M}; \mathcal{F})$.*

We will also prove the following corollary, which is an application of Theorem 3.1 and 3.2.

Corollary 3.3. *Let $0 < s \leq 1$ and $k-1 \leq sp < k$. Then*

$$\text{Lip}(\mathcal{M}; \mathcal{F}) \text{ is dense in } W^{s,p}(\mathcal{M}; \mathcal{F}) \iff \pi_{k-1}(\mathcal{F}) \simeq \{0\}.$$

Here $\pi_{k-1}(\mathcal{F})$ is the $(k-1)$ -th homotopy group of $\mathcal{F}$.

These results were known when $\mathcal{M}$ is smooth and $\mathcal{F}$ is compact. For such $\mathcal{M}$ and $\mathcal{F}$, Theorem 3.1 and Corollary 3.3 when $s = 1$ were obtained by Bethuel [4], Theorem 3.2 and Corollary 3.3 when $0 < s < 1$ were obtained by Brezis and Mironescu [11]. Their proofs heavily rely on a decomposition of the domain into a suitable grid made of cubes. Our contribution concerns the Lipschitz case. In this more general setting, $\mathcal{M}$ may have no triangulation (see, Kirby and Siebenmann [26, Appendix C, Section 2] and Sullivan [42, Corollary 3]), and it is not clear how to perform such a decomposition on $\mathcal{M}$. For this reason, we instead rely on the *method of good and bad balls*, inspired by Bousquet, Ponce, and Van Schaftingen [9]. As opposed to a covering with a grid of cubes, balls that cover a domain will overlap. One of the key points of the argument is then to separate the covering into finitely many *disjoint* families, in the spirit of the Besicovitch covering theorem, and to perform our constructions inductively on each family. For another use of this method in a slightly different context, let us mention the work of Caniato and Rivière [16] on the approximation of weak Yang–Mills connections, which also relies crucially on a good and bad balls-type method.

Corollary 3.3 is a direct consequence of Theorem 3.1 when $s = 1$, respectively Theorem 3.2 when $0 < s < 1$, relying on the topological extension and shrinking constructions to remove the singularities (see, e.g., Bethuel [4]). For the proof, see Appendix A.

In the following sections, we focus on the proof of Theorem 3.1 and 3.2.

3.1 Proof of Theorem 3.1 when $k - 1 < p < k$

In this section, we consider the supercritical case where $s = 1$ and $k - 1 < p < k$. Although this could be unified with the critical case $p = k - 1$ (see Section 3.3), we provide a simpler, easier to read proof. Indeed, the Morrey embedding $W^{1,p} \hookrightarrow C^0$ in dimension $k - 1$ for $p > k - 1$ allows to simplify some technicalities in the argument.

Proof of Theorem 3.1 when $k - 1 < p < k$. Recall that $\mathcal{M}$ is a Lipschitz manifold and that we assume (2.1). The main idea of the proof is the following. We first separate the covering balls of $\mathcal{M}$ into finitely many families such that, in each family, the balls are disjoint. Next, for each family, we only modify the values of the map inside the balls of the family to obtain an approximating map with the desired regularity inside of the modified region. We repeat this procedure inductively with respect to the families, and show that the map we obtain at the end is an approximation of the original map.

Step 1. Covering of $\mathcal{M}$ with “well-separated” balls.

Without loss of generality, we may assume that $K_1 = 1$ in (2.1). Let $r > 0$ be sufficiently small so that, for any $x \in \mathcal{M}$, $B_{16K_2r}(x) \subset B_{r_0}(x) \subset U_i$ for some $i \in I$, with r_0 as in (2.2).

For such r , we consider a (finite) maximal collection of disjoint balls $B_{r/2}(x_j)$, $j \in J$. Hence, $(B_r(x_j))_j$ is a covering of $\mathcal{M}$. On the other hand, since the k -dimensional Hausdorff measure on $\mathcal{M}$ is a doubling measure and the balls $B_{r/2}(x_j)$ are mutually disjoint, for every j , the balls $B_{3K_2r}(x_j)$ intersect at most θ balls in $(B_{3K_2r}(x_{j'}))_{j'}$, where θ only depends on $\mathcal{M}$. Indeed, for any

$$j' \in J' := \{j' \in J : B_{3K_2r}(x_{j'}) \cap B_{3K_2r}(x_j) \neq \emptyset\},$$

we have $B_{3K_2r}(x_{j'}) \subset B_{9K_2r}(x_j) \subset B_{15K_2r}(x_{j'})$. Since the balls $B_{r/2}(x_j)$ are disjoint, we have

$$\mathcal{H}^k(B_{9K_2r}(x_j)) \geq \sum_{j' \in J'} \mathcal{H}^k(B_{r/2}(x_{j'})) \geq \text{card}(J') \inf_{j' \in J'} \mathcal{H}^k(B_{r/2}(x_{j'})),$$

and therefore

$$\begin{aligned} \text{card}(J') &\leq \sup_{j' \in J'} \frac{\mathcal{H}^k(B_{9K_2r}(x_j))}{\mathcal{H}^k(B_{r/2}(x_{j'}))} \leq \sup_{j' \in J} \frac{\mathcal{H}^k(B_{15K_2r}(x_{j'}))}{\mathcal{H}^k(B_{r/2}(x_{j'}))} \\ &\leq (C_{\mathcal{M}})^{5+\lceil \log_2 K_2 \rceil} \leq \lceil (C_{\mathcal{M}})^{5+\lceil \log_2 K_2 \rceil} \rceil =: \theta, \end{aligned}$$

where $C_{\mathcal{M}}$ is the doubling constant.

Thus, we can divide the set J of indices as $J_1 \cup \dots \cup J_{\theta+1}$ in such a way that, for any $1 \leq \ell \leq \theta + 1$ and any indices $j, j' \in J_\ell$, with $j' \neq j$, we have $B_{3K_2r}(x_{j'}) \cap B_{3K_2r}(x_j) = \emptyset$. Following the approach in [9], we will associate, with each ℓ , a family of ball-like

disjoint sets, slightly larger than the ball $B_r(x_j)$, and approximate f on these sets. The construction is by induction on ℓ .

Set $f_0 := f$. In the following Steps 2 to 5, for each $1 \leq \ell \leq \theta + 1$, we construct inductively f_ℓ such that

- (i) there exists a set D_ℓ such that $\bigcup_{j \in J_\ell} B_{2r}(x_j) \subset D_\ell \subset \bigcup_{j \in J_\ell} B_{3K_2r}(x_j)$,

$$f_\ell = f_{\ell-1} \text{ in } \mathcal{M} \setminus D_\ell, \text{ and } f_\ell \text{ is continuous in } \bigcup_{a \leq \ell} D_a, \quad (3.2)$$

possibly except at the set $\{x_j : j \in \bigcup_{a \leq \ell} J_a\}$;

- (ii) there exists a set $E_\ell \subset D_\ell$ such that

$$\mathcal{H}^k(E_\ell) \lesssim r^p \int_{\mathcal{M}} |\nabla f_{\ell-1}|^p d\mathcal{H}^k, \quad (3.3)$$

and the following estimates hold:

$$\int_{\mathcal{M}} |f_\ell - f_{\ell-1}|^p d\mathcal{H}^k \lesssim r^p \int_{\mathcal{M}} |\nabla f_{\ell-1}|^p d\mathcal{H}^k, \quad (3.4)$$

$$\int_{\mathcal{M}} |\nabla(f_\ell - f_{\ell-1})|^p d\mathcal{H}^k \lesssim \int_{E_\ell} |\nabla f_{\ell-1}|^p d\mathcal{H}^k. \quad (3.5)$$

Step 2. Enlarging the balls.

By our choice of r , we know that $B_{3K_2r}(x_j) \subset U_i$ for some $i \in I$.

Denote $S(0, r)$ the Euclidean sphere of radius r and center 0 in $\mathbb{R}^k$. By Morrey's embedding $W^{1,p}(S(0, r_j)) \hookrightarrow C^0(S(0, r_j))$ for $k - 1 < p < k$ and Tonelli's theorem, there exists $2r \leq r_j \leq 3r$ such that (i) the restriction of $f_{\ell-1} \circ \psi_i$ on the sphere $S(0, r_j)$ coincides with the trace of $f_{\ell-1} \circ \psi_i$ on $S(0, r_j)$ and has a continuous representative; (ii) $\psi_i(S(0, r_j)) \cap \{x_{j'} : j' \in \bigcup_{a \leq \ell-1} J_a\} = \emptyset$; and (iii)

$$\int_{S(0, r_j)} |\nabla(f_{\ell-1} \circ \psi_i)|^p \leq \frac{2}{r} \int_{B_{3r}(0)} |\nabla(f_{\ell-1} \circ \psi_i)|^p. \quad (3.6)$$

Let

$$D_\ell := \bigcup_{j \in J_\ell} \psi_i(B_{r_j}(\psi_i^{-1}(x_j))) \quad (3.7)$$

and set $f_\ell := f_{\ell-1}$ in $\mathcal{M} \setminus D_\ell$. By the assumptions on the constants and the choice of r_j ,

we have

$$B_{2r}(x_j) \subset \psi_i(B_{r_j}(\psi_i^{-1}(x_j))) \subset B_{3K_2r}(x_j).$$

Thus, we have

$$\bigcup_{j \in J_\ell} B_{2r}(x_j) \subset D_\ell \subset \bigcup_{j \in J_\ell} B_{3K_2r}(x_j).$$

Step 3. Construction of f_ℓ in D_ℓ .

Since the sets $\psi_i(B_{r_j}(\psi_i^{-1}(x_j)))$ are disjoint, it suffices to construct f_ℓ in each “ball” $\psi_i(B_{r_j}(\psi_i^{-1}(x_j)))$, $j \in J_\ell$. In the spirit of Bethuel’s approach in [4], we distinguish between *good* and *bad* balls.

Let $\lambda > 0$ to be chosen later. Given $1 \leq \ell \leq \theta + 1$, we say that the ball $B_{3K_2r}(x_j)$ is a *good* ball (for $f_{\ell-1}$) provided that

$$\frac{1}{r^{k-p}} \int_{B_{3K_2r}(x_j)} |\nabla f_{\ell-1}(x)|^p d\mathcal{H}^k(x) \leq \lambda. \quad (3.8)$$

We say that the ball $B_{3K_2r}(x_j)$ is a *bad* ball (for $f_{\ell-1}$) if (3.8) fails.

Step 3.1. Construction of f_ℓ in bad balls.

Assume that $B_{3K_2r}(x_j)$ is a bad ball. Without loss of generality, we assume that $\psi_i^{-1}(x_j) = 0$.

We define f_ℓ in $\psi_i(B_{r_j}(0))$ by

$$f_\ell(x) := f_{\ell-1} \circ \psi_i \left(r_j \frac{\psi_i^{-1}(x)}{|\psi_i^{-1}(x)|} \right). \quad (3.9)$$

By the choice of r_j and (3.9),

$$f_\ell \text{ is continuous in } \overline{\psi_i(B_{r_j}(0))} \setminus \{x_j\}. \quad (3.10)$$

Step 3.2. Construction of f_ℓ in good balls.

Let $B_{3K_2r}(x_j)$ be a good ball. Combining Morrey’s embedding with (3.6), for any w_1 ,

$w_2 \in S(0, r_j)$, we have

$$\begin{aligned}
|f_{\ell-1} \circ \psi_i(w_1) - f_{\ell-1} \circ \psi_i(w_2)| &\lesssim |w_1 - w_2|^{1-\frac{k-1}{p}} \|\nabla(f_{\ell-1} \circ \psi_i)\|_{\mathcal{L}^p(S(0, r_j))} \\
&\lesssim r^{1-\frac{k}{p}} \left(\int_{B_{3r}(0)} |\nabla(f_{\ell-1} \circ \psi_i)|^p \right)^{\frac{1}{p}} \\
&\lesssim r^{1-\frac{k}{p}} \|\nabla f_{\ell-1}\|_{\mathcal{L}^p(B_{3K_2r}(x_j))} \leq \lambda^{\frac{1}{p}},
\end{aligned} \tag{3.11}$$

where in the last inequality, we have used the definition (3.8) of good balls. For λ sufficiently small, (3.11) implies that there exists an $\mathfrak{f} \in \mathcal{F} \subset \mathbb{R}^d$ such that $f_{\ell-1} \circ \psi_i|_{S(0, r_j)} \subset B_{\delta/2}(\mathfrak{f}) \cap \mathcal{F}$, where δ is as in Definition 2.2.

With Π as in Definition 2.2, we define $\Psi_{\mathfrak{f}, \delta}: \mathbb{R}^n \rightarrow \mathcal{F}$ as

$$\Psi_{\mathfrak{f}, \delta}(z) := \begin{cases} \Pi(z), & \text{if } z \in B_\delta(\mathfrak{f}) \\ \Pi\left(\mathfrak{f} + \frac{z-\mathfrak{f}}{|z-\mathfrak{f}|} \delta\right), & \text{otherwise} \end{cases}.$$

Then $\Psi_{\mathfrak{f}, \delta}$ is Lipschitz, and its Lipschitz constant is independent of $\mathfrak{f}$.

Let

$$h: \psi_i(B_{r_j}(0)) \rightarrow \mathcal{F}, \quad h(x) := \Psi_{\mathfrak{f}, \delta} \circ f_{\ell-1}. \tag{3.12}$$

By composition with a Lipschitz function, we have $h \in W^{1,p}(\psi_i(B_{r_j}(0)); \mathcal{F})$.

Next, we approximate h , which is $B_\delta(\mathfrak{f})$ -valued, with a continuous map with values in the same ball. By a standard argument (see, e.g., [4, Lemma 6]), combined with the nearest point projection, for any $c > 0$, we may find $\bar{h} \in W^{1,p}(B_{r_j}(0); B_\delta(\mathfrak{f}) \cap \mathcal{F}) \cap C^0(\overline{B_{r_j}(0)})$ such that: (i) $\bar{h} = h \circ \psi_i$ on $S(0, r_j)$; (ii)

$$\int_{B_{r_j}(0)} |\nabla(\bar{h} - h \circ \psi_i)|^p \, d\mathcal{H}^k \leq c. \tag{3.13}$$

We use below (3.13) with

$$c := \int_{\psi_i(B_{r_j}(0))} |\nabla(h - f_{\ell-1})|^p \, d\mathcal{H}^k. \tag{3.14}$$

We define f_ℓ in $\psi_i(B_{r_j}(0))$ by

$$f_\ell(x) := \bar{h} \circ \psi_i^{-1}(x).$$

Using the fact that $\Psi_{\mathfrak{f}, \delta} \circ f_{\ell-1}(x) = f_{\ell-1}(x)$ on $\psi_i(S(0, r_j))$, we find that f_ℓ belongs to $W^{1,p}$

in $B_{3K_2r}(x_j)$ and (arguing as for (3.10))

f_ℓ is continuous in $\overline{\psi_i(B_{r_j}(0))} \supset B_{2r}(x_j)$.

We conclude (3.2) by noting the fact that, by construction, f_ℓ is continuous in $\overline{D_\ell}$, possibly except at some points x_j with $j \in J_\ell$, and the induction assumption on $f_{\ell-1}$.

Step 4. Proof of (3.3)–(3.5).

This is performed in three sub-steps.

Step 4.1. Estimate of $f_\ell - f_{\ell-1}$ in a bad ball $B_{3K_2r}(x_j)$.

By the definition of f_ℓ , we have

$$\begin{aligned} \int_{\psi_i(B_{r_j}(0))} |\nabla f_\ell|^p \, d\mathcal{H}^k &\lesssim \int_{B_{r_j}(0)} |\nabla(f_\ell \circ \psi_i)|^p \, d\mathcal{H}^k \\ &\lesssim \int_0^{r_j} \left(\frac{t}{r_j}\right)^{k-1-p} \left(\int_{S(0,r_j)} |\nabla(f_{\ell-1} \circ \psi_i)|^p \right) dt. \end{aligned}$$

By (3.6), we obtain

$$\begin{aligned} \int_{\psi_i(B_{r_j}(0))} |\nabla f_\ell|^p \, d\mathcal{H}^k &\lesssim \int_0^{r_j} \frac{t^{k-1-p}}{r_j^{k-p}} \left(\int_{B_{3r}(0)} |\nabla(f_{\ell-1} \circ \psi_i)|^p \right) dt \\ &\lesssim \int_{B_{3r}(0)} |\nabla(f_{\ell-1} \circ \psi_i)|^p \, d\mathcal{H}^k \\ &\lesssim \int_{B_{3K_2r}(x_j)} |\nabla f_{\ell-1}|^p \, d\mathcal{H}^k. \end{aligned} \tag{3.15}$$

Thus, we have

$$\int_{\psi_i(B_{r_j}(0))} |\nabla(f_\ell - f_{\ell-1})|^p \, d\mathcal{H}^k \lesssim \int_{B_{3K_2r}(x_j)} |\nabla f_{\ell-1}|^p \, d\mathcal{H}^k. \tag{3.16}$$

On the other hand, using (3.15) and the Poincaré inequality, we have

$$\begin{aligned} \int_{\psi_i(B_{r_j}(0))} |f_\ell - f_{\ell-1}|^p \, d\mathcal{H}^k &\lesssim \int_{B_{r_j}(0)} |f_\ell \circ \psi_i - f_{\ell-1} \circ \psi_i|^p \, d\mathcal{H}^k \\ &\lesssim r_j^p \int_{B_{r_j}(0)} |\nabla(f_\ell \circ \psi_i) - \nabla(f_{\ell-1} \circ \psi_i)|^p \, d\mathcal{H}^k \\ &\lesssim r_j^p \int_{B_{3K_2r}(x_j)} |\nabla f_{\ell-1}|^p \, d\mathcal{H}^k. \end{aligned} \tag{3.17}$$

Step 4.2. Estimate of $f_\ell - f_{\ell-1}$ in a good ball $B_{3K_2r}(x_j)$.

We first show that, in the good balls, since the energy is small (thanks to (3.8)), the map $f_{\ell-1}$ does not oscillate too much and takes its values in $B_\delta(\mathfrak{f}) \cap \mathcal{F}$ “most of the time”. Let

$$\bar{A}_j := \{u \in B_{r_j}(0) : f_{\ell-1} \circ \psi_i(u) \notin B_\delta(\mathfrak{f})\}, \quad (3.18)$$

and

$$A_j := \psi_i(\bar{A}_j). \quad (3.19)$$

By the construction of h , for $u \in B_{r_j}(0) \setminus \bar{A}_j$, we have

$$h \circ \psi_i(u) = f_{\ell-1} \circ \psi_i(u),$$

and therefore

$$\int_{\psi_i(B_{r_j}(0)) \setminus A_j} |\nabla(h - f_{\ell-1})|^p d\mathcal{H}^k = \int_{B_{r_j}(0) \setminus \bar{A}_j} |\nabla(h \circ \psi_i - f_{\ell-1} \circ \psi_i)|^p d\mathcal{H}^k = 0$$

(see, e.g., [21, Theorem 4.4 (iv)]).

Therefore, using the fact that $h = \Psi_{\mathfrak{f},\delta} \circ f_{\ell-1}$ in A_j (see (3.12)) with $\Psi_{\mathfrak{f},\delta}$ Lipschitz, we obtain

$$\int_{\psi_i(B_{r_j}(0))} |\nabla(h - f_{\ell-1})|^p d\mathcal{H}^k = \int_{A_j} |\nabla(h - f_{\ell-1})|^p d\mathcal{H}^k \lesssim \int_{A_j} |\nabla f_{\ell-1}|^p d\mathcal{H}^k. \quad (3.20)$$

Combining the definition of f_ℓ with (3.20), (3.13), and (3.14), we have

$$\int_{\psi_i(B_{r_j}(0))} |\nabla(f_\ell - f_{\ell-1})|^p d\mathcal{H}^k \lesssim \int_{A_j} |\nabla f_{\ell-1}|^p d\mathcal{H}^k, \quad (3.21)$$

and, by (3.21) and Poincaré’s inequality,

$$\int_{\psi_i(B_{r_j}(0))} |f_\ell - f_{\ell-1}|^p d\mathcal{H}^k \lesssim r^p \int_{B_{3K_2r}(x_j)} |\nabla f_{\ell-1}|^p d\mathcal{H}^k. \quad (3.22)$$

Step 4.3. Summing over $j \in J_\ell$, (3.17) and (3.22) imply (3.4). On the other hand, we obtain

(3.5) by combining (3.16) and (3.21) with $E_\ell := \bigcup_{j \in J_\ell} E_\ell^j$ and

$$E_\ell^j := \begin{cases} A_j, & \text{if } B_{3K_2r}(x_j) \text{ is a good ball for } f_{\ell-1} \\ B_{3K_2r}(x_j), & \text{if } B_{3K_2r}(x_j) \text{ is a bad ball for } f_{\ell-1} \end{cases}. \quad (3.23)$$

We next prove (3.3). For this purpose, we first show that

$$\mathcal{H}^k(A_j) \lesssim r^p \int_{B_{3K_2r}(x_j)} |\nabla f_{\ell-1}|^p d\mathcal{H}^k. \quad (3.24)$$

Indeed, let us consider a smooth function $\Phi: \mathbb{R}_+ \rightarrow [0, 1]$ such that $\Phi = 0$ in $(0, \delta/2]$ and $\Phi = 1$ in $[\delta, +\infty)$. We have $A_j \subset \{x \in \psi_i(B_{r_j}(0)): \Phi(|f_{\ell-1}(x) - \bar{f}|) = 1\}$ by the definition (3.19) of A_j . Then, using the Poincaré inequality again, we have

$$\begin{aligned} \mathcal{H}^k(A_j) &\leq \int_{\psi_i(B_{r_j}(0))} |\Phi(|f_{\ell-1}(x) - \bar{f}|)|^p d\mathcal{H}^k(x) \\ &\lesssim r^p \int_{B_{r_j}(0)} |\nabla(\Phi(|f_{\ell-1} \circ \psi_i - \bar{f}|))|^p d\mathcal{H}^k \lesssim r^p \int_{B_{3K_2r}(x_j)} |\nabla f_{\ell-1}|^p d\mathcal{H}^k, \end{aligned} \quad (3.25)$$

which establishes (3.24).

On the other hand, by (3.8), for $f_{\ell-1}$, there will be at most $\int_{\mathcal{M}} |\nabla f_{\ell-1}|^p d\mathcal{H}^k / (\lambda r^{k-p})$ bad balls. Combining this with (3.24) and the definition (3.23) of E_ℓ , we have

$$\mathcal{H}^k(E_\ell) \lesssim r^p \int_{\mathcal{M}} |\nabla f_{\ell-1}|^p d\mathcal{H}^k,$$

whence (3.3).

Step 5. The induction procedure.

Starting with $\ell = 1$ and $f_0 := f$, we construct as in Steps 2 to 4, a sequence $f_1, f_2, \dots, f_{\theta+1}$ in $W^{1,p}(\mathcal{M}; \mathcal{F})$. Let us note that the construction depends on r , but importantly, θ does not depend on r .

We now turn to the estimate of $\|f_{\theta+1} - f\|_{W^{1,p}}$. First, by (3.5), we have

$$\begin{aligned} \|\nabla f_{\theta+1} - \nabla f\|_{\mathcal{L}^p(\mathcal{M})}^p &\lesssim \|\nabla f_{\theta+1} - \nabla f_\theta\|_{\mathcal{L}^p(\mathcal{M})}^p + \|\nabla f_\theta - \nabla f\|_{\mathcal{L}^p(\mathcal{M})}^p \\ &\lesssim \|\nabla f_\theta\|_{\mathcal{L}^p(E_{\theta+1})}^p + \|\nabla f_\theta - \nabla f\|_{\mathcal{L}^p(\mathcal{M})}^p \\ &= \|\nabla f_\theta - \nabla f + \nabla f\|_{\mathcal{L}^p(E_{\theta+1})}^p + \|\nabla f_\theta - \nabla f\|_{\mathcal{L}^p(\mathcal{M})}^p \\ &\lesssim \|\nabla f\|_{\mathcal{L}^p(E_{\theta+1})}^p + \|\nabla f_\theta - \nabla f\|_{\mathcal{L}^p(\mathcal{M})}^p. \end{aligned}$$

Iterating this estimate, we arrive at

$$\|\nabla f_{\theta+1} - \nabla f\|_{\mathcal{L}^p(\mathcal{M})}^p \lesssim \sum_{\ell=1}^{\theta+1} \|\nabla f\|_{\mathcal{L}^p(E_\ell)}^p \lesssim \|\nabla f\|_{\mathcal{L}^p(\bigcup_{\ell} E_\ell)}^p. \quad (3.26)$$

Next, using (3.26) and (3.4), we have

$$\begin{aligned} \|f_{\theta+1} - f\|_{\mathcal{L}^p(\mathcal{M})} &\leq \|f_{\theta+1} - f_\theta\|_{\mathcal{L}^p(\mathcal{M})} + \|f_\theta - f\|_{\mathcal{L}^p(\mathcal{M})} \\ &\lesssim r \|\nabla f_\theta\|_{\mathcal{L}^p(\mathcal{M})} + \|f_\theta - f\|_{\mathcal{L}^p(\mathcal{M})} \\ &\lesssim r \|\nabla f\|_{\mathcal{L}^p(\mathcal{M})} + r \|\nabla f_\theta - \nabla f\|_{\mathcal{L}^p(\mathcal{M})} + \|f_\theta - f\|_{\mathcal{L}^p(\mathcal{M})} \\ &\lesssim r \|\nabla f\|_{\mathcal{L}^p(\mathcal{M})} + \|f_\theta - f\|_{\mathcal{L}^p(\mathcal{M})} \lesssim r \|\nabla f\|_{\mathcal{L}^p(\mathcal{M})}. \end{aligned} \quad (3.27)$$

Similarly, by (3.3), we have

$$\begin{aligned} \mathcal{H}^k\left(\bigcup_{\ell} E_\ell\right) &\lesssim \sum_{\ell=1}^{\theta+1} r^p \int_{\mathcal{M}} |\nabla f_{\ell-1}|^p \, d\mathcal{H}^k \lesssim \sum_{\ell=1}^{\theta+1} r^p \int_{\mathcal{M}} |\nabla f_{\ell-1} - \nabla f|^p + |\nabla f|^p \, d\mathcal{H}^k \\ &\lesssim \sum_{\ell=1}^{\theta+1} r^p \|\nabla f\|_{\mathcal{L}^p(\mathcal{M})}^p \lesssim r^p \|\nabla f\|_{\mathcal{L}^p(\mathcal{M})}^p. \end{aligned} \quad (3.28)$$

Thus, combining (3.26) and (3.27) with (3.28), we have $\|f_{\theta+1} - f\|_{W^{1,p}} \rightarrow 0$ as $r \rightarrow 0$. It therefore remains to show that $f_{\theta+1}$ can be approximated with maps in $\mathcal{R}_0$.

Step 6. Approximating $f_{\theta+1}$ with maps in $\mathcal{R}_0$.

First of all, performing a standard convolution procedure in local charts and gluing the constructions together, there exists a sequence of Lipschitz maps (not necessarily $\mathcal{F}$ -valued) such that $g_\eta \rightarrow f_{\theta+1}$ in $W^{1,p}(\mathcal{M})$ as $\eta \rightarrow \infty$. Moreover, since $f_{\theta+1}$ is continuous except at $\{x_j; j \in J'\}$ with $J' \subset J$, we can enforce that

$$g_\eta \rightarrow f_{\theta+1} \text{ uniformly in } \mathcal{M} \setminus \left(\bigcup_{j \in J'} \psi_i(B_{\tilde{r}/2}(0)) \right), \quad (3.29)$$

where $\tilde{r} \leq r/2$ is sufficiently small so that, for every $j \in J'$, we have

$$f_{\theta+1} \circ \psi_i(u) = f_{\theta+1} \circ \psi_i\left(\tilde{r} \frac{u}{|u|}\right), \quad \forall u \in B_{\tilde{r}}(0). \quad (3.30)$$

(This is a consequence of our construction of the maps f_ℓ as homogeneous extensions in the bad balls.)

On the other hand, using a mean value argument, for each $j \in J'$, there exists $\tilde{r}/2 \leq$

$\tilde{r}_j \leq \tilde{r}$ such that

$$|g_\eta \circ \psi_i - f_{\theta+1} \circ \psi_i|_{W^{1,p}(S(0,\tilde{r}_j))}^p \leq \frac{2}{\tilde{r}} |g_\eta \circ \psi_i - f_{\theta+1} \circ \psi_i|_{W^{1,p}(B_{\tilde{r}}(0))}^p. \quad (3.31)$$

By (3.29), for sufficiently large η , we can define

$$h_\eta \in W^{1,p}(\mathcal{M}; \mathcal{F}), h_\eta(x) := \begin{cases} \Pi \circ g_\eta \circ \psi_i \left(\tilde{r}_j \frac{\psi_i^{-1}(x)}{|\psi_i^{-1}(x)|} \right), & \text{if } x \in \psi_i(B_{\tilde{r}_j}(0)) \\ \Pi \circ g_\eta(x), & \text{otherwise} \end{cases}.$$

Clearly, the map h_η is in $\mathcal{R}_0$. We now show that $h_\eta \rightarrow f_{\theta+1}$. For this purpose, we first notice that

$$\begin{aligned} \int_{\mathcal{M}} |\nabla(h_\eta - f_{\theta+1})|^p &\lesssim \int_{\mathcal{M} \setminus (\cup_{j \in J'} \psi_i(B_{\tilde{r}/2}(0)))} |\nabla(g_\eta - f_{\theta+1})|^p \\ &\quad + \sum_{j \in J'} \int_{B_{\tilde{r}_j}(0)} |\nabla(h_\eta \circ \psi_i - f_{\theta+1} \circ \psi_i)|^p. \end{aligned} \quad (3.32)$$

The first term on the right-hand side of (3.32) converges to 0 as $\eta \rightarrow \infty$. For the second term, using (3.31), (3.30), and the fact that the homogeneous extension operator is continuous in $\mathcal{L}(W^{1,p}(S(0, r_j)); W^{1,p}(B_{r_j}(0)))$ (see [12, Lemma 3]), we have

$$\begin{aligned} &\sum_{j \in J'} \int_{B_{\tilde{r}_j}(0)} |\nabla(h_\eta \circ \psi_i - f_{\theta+1} \circ \psi_i)|^p \\ &\lesssim \sum_{j \in J'} \int_{B_{\tilde{r}_j}(0)} \left| \nabla \left[\Pi \circ g_\eta \circ \psi_i \left(\tilde{r}_j \frac{u}{|u|} \right) \right] - \nabla \left[\Pi \circ f_{\theta+1} \circ \psi_i \left(\tilde{r}_j \frac{u}{|u|} \right) \right] \right|^p du \\ &\lesssim \sum_{j \in J'} |g_\eta \circ \psi_i - f_{\theta+1} \circ \psi_i|_{W^{1,p}(S(0,\tilde{r}_j))}^p \\ &\lesssim \sum_{j \in J'} |g_\eta \circ \psi_i - f_{\theta+1} \circ \psi_i|_{W^{1,p}(B_{r_j}(0))}^p \\ &\rightarrow 0, \text{ as } \eta \rightarrow \infty \end{aligned}$$

(where the hidden constants may depend on the number $\tilde{r}$ that was fixed above). Therefore, we have

$$\int_{\mathcal{M}} |\nabla(h_\eta - f_{\theta+1})|^p \rightarrow 0, \text{ as } \eta \rightarrow \infty.$$

Similarly, we have $h_\eta \rightarrow f_{\theta+1}$ in $\mathcal{L}^p$, and thus in $W^{1,p}(\mathcal{M}; \mathcal{F})$. This concludes the

proof. □

3.2 Proof of Theorem 3.2 when $k - 1 < sp < k$

In this section, we consider the case where $0 < s < 1$ and $k - 1 < sp < k$, which is the fractional counterpart of the previous section. In this proof, we use exactly the same method as in the case $s = 1$ to construct the approximation. The differences arise when adapting the argument in the previous proof to the fractional setting. The main challenge comes from the *nonlocal* character of the Gagliardo seminorm.

Proof of Theorem 3.2 when $k - 1 < sp < k$. We split the set J of indices as in the proof of Theorem 3.1.

Set $f_0 := f$. We will construct inductively f_ℓ such that

- (i) there exists a set D_ℓ such that $\bigcup_{j \in J_\ell} B_{2r}(x_j) \subset D_\ell \subset \bigcup_{j \in J_\ell} B_{3K_2r}(x_j)$,

$$f_\ell = f_{\ell-1} \text{ in } \mathcal{M} \setminus D_\ell, \text{ and } f_\ell \text{ is continuous in } \bigcup_{a \leq \ell} D_a, \\ \text{possibly except at the set } \left\{ x_j : j \in \bigcup_{a \leq \ell} J_a \right\};$$

- (ii) there exists a set $E_\ell \subset D_\ell$ such that

$$\mathcal{H}^k(E_\ell) \lesssim r^{sp} |f_{\ell-1}|_{W^{s,p}(\mathcal{M})}^p \quad (3.33)$$

and the following estimates hold:

$$\int_{\mathcal{M}} |f_\ell - f_{\ell-1}|^p d\mathcal{H}^k \lesssim r^{sp} |f_{\ell-1}|_{W^{s,p}(\mathcal{M})}^p, \quad (3.34)$$

$$|f_\ell - f_{\ell-1}|_{W^{s,p}(\mathcal{M})}^p \lesssim \int_{E_\ell} \int_{\mathcal{M}} \frac{|f_{\ell-1}(x) - f_{\ell-1}(y)|^p}{|x - y|^{k+sp}} d\mathcal{H}^k(x) d\mathcal{H}^k(y). \quad (3.35)$$

The construction of f_ℓ is essentially the same as in the case where $s = 1$. The main modifications concern (i) the choice of r_j ; (ii) the definition of good balls and bad balls; and (iii) the counterpart of [4, Lemma 6].

The choice of r_j . By Morrey's embedding $W^{s,p}(S(0, r_j); \mathcal{F}) \hookrightarrow C^0(S(0, r_j); \mathcal{F})$ for $k - 1 < sp < k$ and Tonelli's theorem, there exists a $2r \leq r_j \leq 5r/2$ such that (i) the restriction of $f_{\ell-1} \circ \psi_i$ on $S(0, r_j)$ coincides with the trace of $f_{\ell-1} \circ \psi_i$ on $S(0, r_j)$ and has a continuous representative; (ii) $\psi_i(S(0, r_j)) \cap \{x_{j'} : j' \in \bigcup_{a \leq \ell-1} J_a\} = \emptyset$; (iii)

$$|f_{\ell-1} \circ \psi_i|_{W^{s,p}(S(0, r_j))}^p \lesssim \frac{1}{r} |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))}^p \quad (3.36)$$

and (iv)

$$\begin{aligned} \int_{S(0,r_j)} \int_{B_{3r}(0)} \frac{|f_{\ell-1} \circ \psi_i(u) - f_{\ell-1} \circ \psi_i(w)|^p}{|u-w|^{k+sp}} d\mathcal{H}^k(u) d\mathcal{H}^{k-1}(w) \\ \lesssim \frac{1}{r} |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))}^p. \end{aligned} \quad (3.37)$$

The definition of good balls and bad balls. For $j \in J_\ell$, we say that the ball $B_{3K_2r}(x_j)$ is a *good* ball provided that

$$\frac{1}{r^{k-sp}} \int_{B_{3K_2r}(x_j)} \int_{\mathcal{M}} \frac{|f_{\ell-1}(x) - f_{\ell-1}(y)|^p}{|x-y|^{k+sp}} d\mathcal{H}^k(y) d\mathcal{H}^k(x) \leq \lambda, \quad (3.38)$$

with λ to be chosen later. We say that the ball $B_{3K_2r}(x_j)$ is a *bad* ball if (3.38) fails.

The counterpart of [4, Lemma 6].

Lemma 3.4. *Let $0 < s \leq 1$ be such that either $sp > 1$ or $s = 1$ and $p = 1$. (This assumption ensures that $W^{s,p}$ maps have traces.) Assume that $f: B_1(0) \rightarrow \mathcal{F}$ satisfies: (i) $f \in W^{s,p}(B_1(0); B_\delta(\mathfrak{f}) \cap \mathcal{F})$ for some $\mathfrak{f} \in \mathcal{F}$; (ii) the trace $\text{tr}_{S(0,1)} f$ of f on $S(0,1)$ belongs to $W^{s-1/p,p}(S(0,1))$ and has a continuous representative.*

Then, for every $\alpha > 0$, there exists $\bar{f} \in W^{s,p}(B_1(0); B_{4\delta}(\mathfrak{f}) \cap \mathcal{F}) \cap C(\overline{B_1(0)}; B_{4\delta}(\mathfrak{f}) \cap \mathcal{F})$ such that $\text{tr}_{S(0,1)} \bar{f} = \text{tr}_{S(0,1)} f$ and

$$\|f - \bar{f}\|_{W^{s,p}(B_1(0))}^p \leq \alpha.$$

For the proof of Lemma 3.4, see Appendix A. Replacing [4, Lemma 6] with Lemma 3.4, we can choose $\bar{h}$ in the construction of f_ℓ in such a way that

$$|\bar{h} - h \circ \psi_i|_{W^{s,p}(B_{r_j}(0))} \leq |h - f_{\ell-1}|_{W^{s,p}(\psi_i(B_{r_j}(0)))}. \quad (3.39)$$

Regarding the counterpart of (3.11), combining Morrey's embedding with (3.36), for any $w_1, w_2 \in S(0, r_j)$, we have

$$\begin{aligned} |f_{\ell-1} \circ \psi_i(w_1) - f_{\ell-1} \circ \psi_i(w_2)| &\lesssim r^{s-\frac{k-1}{p}} r_j^{-\frac{1}{p}} |f_{\ell-1} \circ \psi_i|_{W^{s,p}(S(0,r_j))} \\ &\lesssim r^{s-\frac{k}{p}} |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))} \leq \lambda^{1/p}, \end{aligned}$$

where in the last inequality, we use the definition of good balls (3.38). We can now construct f_ℓ as in the previous section thanks to the above modifications.

For further use, we note that f_ℓ is in $W^{s,p}(\mathcal{M}; \mathcal{F})$, using that we do not change the values of $f_{\ell-1}$ on $\psi_i(S(0, r_j))$.

We next estimate the difference $f_\ell - f_{\ell-1}$.

Step 1. In this step, we prove the following inequality:

$$|f_\ell - f_{\ell-1}|_{W^{s,p}(\mathcal{M})}^p \lesssim \sum_{j \in J_\ell} |f_\ell - f_{\ell-1}|_{W^{s,p}(\psi_i(B_{r_j}(0)))}^p. \quad (3.40)$$

Indeed, setting, for simplicity, $f' := f_\ell - f_{\ell-1}$, and using the fact that f_ℓ coincides with $f_{\ell-1}$ in $\mathcal{M} \setminus D_\ell$, we have

$$\begin{aligned} |f'|_{W^{s,p}(\mathcal{M})}^p &= \int_{D_\ell} \int_{D_\ell} \frac{|f'(x) - f'(y)|^p}{|x - y|^{k+sp}} d\mathcal{H}^k(x) d\mathcal{H}^k(y) \\ &\quad + 2 \int_{D_\ell} \int_{\mathcal{M} \setminus D_\ell} \frac{|f'(y)|^p}{|x - y|^{k+sp}} d\mathcal{H}^k(x) d\mathcal{H}^k(y) \\ &=: \text{I} + \text{II}. \end{aligned} \quad (3.41)$$

For II, let r_0 be as in (2.2). Using Hardy's inequality (see, e.g., [27, Exercise 9.17]), which is scaling invariant, and Poincaré's inequality, we have

$$\begin{aligned} \text{II} &\leq \sum_{j \in J_\ell} \int_{\psi_i(B_{r_j}(0))} \int_{\mathcal{M} \setminus \psi_i(B_{r_j}(0))} \frac{|f'(y)|^p}{|x - y|^{k+sp}} d\mathcal{H}^k(x) d\mathcal{H}^k(y) \\ &\leq \sum_{j \in J_\ell} \int_{\psi_i(B_{r_j}(0))} \int_{B_{r_0}(x_j) \setminus \psi_i(B_{r_j}(0))} \frac{|f'(y)|^p}{|x - y|^{k+sp}} d\mathcal{H}^k(x) d\mathcal{H}^k(y) \\ &\quad + \sum_{j \in J_\ell} \int_{\psi_i(B_{r_j}(0))} \int_{\mathcal{M} \setminus B_{r_0}(x_j)} \frac{|f'(y)|^p}{|x - y|^{k+sp}} d\mathcal{H}^k(x) d\mathcal{H}^k(y) \\ &\lesssim \sum_{j \in J_\ell} \int_{B_{r_j}(0)} \int_{(B_{r_j}(0))^c} \frac{|f' \circ \psi_i(v)|^p}{|u - v|^{k+sp}} du dv + \sum_{j \in J_\ell} \int_{\psi_i(B_{r_j}(0))} |f'(y)|^p d\mathcal{H}^k(y) \\ &\lesssim \sum_{j \in J_\ell} \int_{B_{r_j}(0)} \frac{|f' \circ \psi_i(v)|^p}{[\text{dist}(v, S(0, r_j))]^{sp}} dv + \sum_{j \in J_\ell} \int_{\psi_i(B_{r_j}(0))} |f'(y)|^p d\mathcal{H}^k(y) \\ &\lesssim \sum_{j \in J_\ell} |f'|_{W^{s,p}(\psi_i(B_{r_j}(0)))}^p. \end{aligned} \quad (3.42)$$

Let us note that, here, the hidden constants depend on r_0 .

On the other hand, by (3.42), we have

$$\begin{aligned}
& \sum_{j \in J_\ell} \sum_{j' \neq j} \int_{\psi_i(B_{r_j}(0))} \int_{\psi_i(B_{r_{j'}}(0))} \frac{|f'(x) - f'(y)|^p}{|x - y|^{k+sp}} d\mathcal{H}^k(x) d\mathcal{H}^k(y) \\
& \lesssim \sum_{j \in J_\ell} \sum_{j' \neq j} \int_{\psi_i(B_{r_j}(0))} \int_{\psi_i(B_{r_{j'}}(0))} \frac{|f'(y)|^p}{|x - y|^{k+sp}} d\mathcal{H}^k(x) d\mathcal{H}^k(y) \\
& \lesssim \sum_{j \in J_\ell} \int_{\psi_i(B_{r_j}(0))} \int_{\mathcal{M} \setminus \psi_i(B_{r_j}(0))} \frac{|f'(y)|^p}{|x - y|^{k+sp}} d\mathcal{H}^k(x) d\mathcal{H}^k(y) \\
& \lesssim \sum_{j \in J_\ell} |f'|_{W^{s,p}(\psi_i(B_{r_j}(0)))}^p
\end{aligned} \tag{3.43}$$

and thus,

$$\begin{aligned}
\text{I} &= \sum_{j \in J_\ell} \sum_{j' \in J_\ell} \int_{\psi_i(B_{r_j}(0))} \int_{\psi_i(B_{r_{j'}}(0))} \frac{|f'(x) - f'(y)|^p}{|x - y|^{k+sp}} d\mathcal{H}^k(x) d\mathcal{H}^k(y) \\
&\lesssim \sum_{j \in J_\ell} |f'|_{W^{s,p}(\psi_i(B_{r_j}(0)))}^p.
\end{aligned} \tag{3.44}$$

Combining (3.41), (3.42), and (3.44), we obtain (3.40).

Step 2. Estimate of $|f_\ell - f_{\ell-1}|_{W^{s,p}(\psi_i(B_{r_j}(0)))}^p$ in a bad ball $B_{3K_2r}(x_j)$.

Using the definition of f_ℓ , combined with (3.36) and the $W^{s,p}$ estimate of homogeneous extensions (this follows, by scaling, from [12, Lemma 3]), we have

$$\begin{aligned}
& \int_{\psi_i(B_{r_j}(0))} \int_{\psi_i(B_{r_j}(0))} \frac{|f_\ell(x) - f_\ell(y)|^p}{|x - y|^{k+sp}} d\mathcal{H}^k(y) d\mathcal{H}^k(x) \\
& \lesssim r_j |f_{\ell-1} \circ \psi_i|_{W^{s,p}(S(0,r_j))}^p \lesssim |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))}^p
\end{aligned} \tag{3.45}$$

and thus

$$|f_\ell - f_{\ell-1}|_{W^{s,p}(\psi_i(B_{r_j}(0)))}^p \lesssim |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))}^p. \tag{3.46}$$

Step 3. Estimate of $|f_\ell - f_{\ell-1}|_{W^{s,p}(\psi_i(B_{r_j}(0)))}^p$ in a good ball $B_{3K_2r}(x_j)$.

Let $\bar{A}_j$, A_j , and h be as in (3.18), (3.19) and (3.12). We first notice that, in $\psi_i(B_{r_j}(0))$, we have $h = \Psi_{\bar{f},\delta} \circ f_{\ell-1}$, and therefore

$$|h \circ \psi_i(u) - h \circ \psi_i(v)| \lesssim |f_{\ell-1} \circ \psi_i(u) - f_{\ell-1} \circ \psi_i(v)|, \forall u, v \in B_{r_j}(0).$$

Consequently,

$$\begin{aligned}
& |h - f_{\ell-1}|_{W^{s,p}(\psi_i(B_{r_j}(0)))}^p \\
& \leq 2 \int_{\bar{A}_j} \int_{B_{r_j}(0)} \frac{|(h - f_{\ell-1}) \circ \psi_i(u) - (h - f_{\ell-1}) \circ \psi_i(v)|^p}{|u - v|^{k+sp}} dv du \\
& \lesssim \int_{\bar{A}_j} \int_{B_{r_j}(0)} \frac{|f_{\ell-1} \circ \psi_i(u) - f_{\ell-1} \circ \psi_i(v)|^p}{|u - v|^{k+sp}} dv du \\
& \lesssim \int_{A_j} \int_{B_{3K_2r}(x_j)} \frac{|f_{\ell-1}(x) - f_{\ell-1}(y)|^p}{|x - y|^{k+sp}} d\mathcal{H}^k(y) d\mathcal{H}^k(x).
\end{aligned} \tag{3.47}$$

Combining the definition of f_ℓ with (3.47) and (3.39), we have

$$|f_\ell - f_{\ell-1}|_{W^{s,p}(\psi_i(B_{r_j}(0)))}^p \lesssim \int_{A_j} \int_{B_{3K_2r}(x_j)} \frac{|f_{\ell-1}(x) - f_{\ell-1}(y)|^p}{|x - y|^{k+sp}} d\mathcal{H}^k(y) d\mathcal{H}^k(x). \tag{3.48}$$

Step 4. Repeating the induction procedure as in Section 3.1, we obtain a map $f_{\theta+1} \in W^{s,p}(\mathcal{M}; \mathcal{F})$.

For $1 \leq \ell \leq \theta + 1$, using (3.40), (3.46), (3.48), and the Poincaré inequality, we have

$$\begin{aligned}
\int_{\mathcal{M}} |f_\ell - f_{\ell-1}|^p d\mathcal{H}^k & \lesssim \sum_{j \in I_\ell} r^{sp} |f_\ell - f_{\ell-1}|_{W^{s,p}(\psi_i(B_{r_j}(0)))}^p \\
& \lesssim r^{sp} \int_{E_\ell} \int_{\mathcal{M}} \frac{|f_{\ell-1}(x) - f_{\ell-1}(y)|^p}{|x - y|^{k+sp}} d\mathcal{H}^k(x) d\mathcal{H}^k(y),
\end{aligned} \tag{3.49}$$

and

$$|f_\ell - f_{\ell-1}|_{W^{s,p}(\mathcal{M})}^p \lesssim \int_{E_\ell} \int_{\mathcal{M}} \frac{|f_{\ell-1}(x) - f_{\ell-1}(y)|^p}{|x - y|^{k+sp}} d\mathcal{H}^k(x) d\mathcal{H}^k(y), \tag{3.50}$$

with E_ℓ defined as (3.23). This conclude the proof of (3.34) and (3.35).

We now prove (3.33). On the one hand, the set A_j satisfies (similarly to (3.24))

$$\mathcal{H}^k(A_j) \lesssim r^{sp} |f_{\ell-1}|_{W^{s,p}(B_{3K_2r}(x_j))}^p. \tag{3.51}$$

On the other hand, by (3.38), for $f_{\ell-1}$, there will be at most $|f_{\ell-1}|_{W^{s,p}}^p / (\lambda r^{k-sp})$ bad balls. Combining this fact with (3.51), we obtain

$$\mathcal{H}^k(E_\ell) \lesssim r^{sp} |f_{\ell-1}|_{W^{s,p}(\mathcal{M})}^p. \tag{3.52}$$

We now turn to the estimate of $\|f_{\theta+1} - f\|_{W^{s,p}}$. By (3.50), we have

$$\begin{aligned}
& |f_{\theta+1} - f|_{W^{s,p}(\mathcal{M})}^p \\
& \lesssim \int_{E_{\theta+1}} \int_{\mathcal{M}} \frac{|f_{\theta}(x) - f_{\theta}(y)|^p}{|x - y|^{k+sp}} d\mathcal{H}^k(x) d\mathcal{H}^k(y) + |f_{\theta} - f|_{W^{s,p}(\mathcal{M})}^p \\
& \lesssim \int_{E_{\theta+1}} \int_{\mathcal{M}} \frac{|f(x) - f(y)|^p}{|x - y|^{k+sp}} d\mathcal{H}^k(x) d\mathcal{H}^k(y) + |f_{\theta} - f|_{W^{s,p}(\mathcal{M})}^p \\
& \lesssim \int_{\bigcup_{\ell} E_{\ell}} \int_{\mathcal{M}} \frac{|f(x) - f(y)|^p}{|x - y|^{k+sp}} d\mathcal{H}^k(x) d\mathcal{H}^k(y).
\end{aligned} \tag{3.53}$$

Combining (3.49) and (3.53), we find that

$$\begin{aligned}
& \|f_{\theta+1} - f\|_{\mathcal{L}^p(\mathcal{M})}^p \\
& \lesssim r^{sp} \int_{E_{\theta+1}} \int_{\mathcal{M}} \frac{|f_{\theta}(x) - f_{\theta}(y)|^p}{|x - y|^{k+sp}} d\mathcal{H}^k(x) d\mathcal{H}^k(y) + \|f_{\theta} - f\|_{\mathcal{L}^p(\mathcal{M})}^p \\
& \lesssim r^{sp} \int_{E_{\theta+1}} \int_{\mathcal{M}} \frac{|f(x) - f(y)|^p}{|x - y|^{k+sp}} d\mathcal{H}^k(x) d\mathcal{H}^k(y) + \|f_{\theta} - f\|_{\mathcal{L}^p(\mathcal{M})}^p \\
& \lesssim r^{sp} |f|_{W^{s,p}(\mathcal{M})}^p.
\end{aligned} \tag{3.54}$$

Similarly, combining (3.52) with (3.53), we have

$$\mathcal{H}^k\left(\bigcup_{\ell} E_{\ell}\right) \lesssim r^{sp} \sum_{\ell=1}^{\theta+1} |f_{\ell-1} - f|_{W^{s,p}(\mathcal{M})}^p + r^{sp} |f|_{W^{s,p}(\mathcal{M})}^p \lesssim r^{sp} |f|_{W^{s,p}(\mathcal{M})}^p. \tag{3.55}$$

As a direct consequence of (3.53), (3.54), and (3.55), we have $f_{\theta+1} \rightarrow f$ in $W^{s,p}(\mathcal{M}; \mathcal{F})$ as $r \rightarrow 0$.

We complete the proof of the theorem by repeating Step 6 in the previous proof to approximate $f_{\theta+1}$ with maps in $\mathcal{R}_0$. All the arguments in that step have straightforward fractional counterparts. $\square$

3.3 Proof of Theorem 3.1 when $p = k - 1$

We now turn to the critical case where $s = 1$ and $p = k - 1$. The case where $k = 2$, $s = 1$, and $p = 1$ can be treated essentially exactly as in Section 3.1 while, instead of Morrey's embedding, we use the fact that $W^{1,1}(S(0, r_j))$ maps have absolutely continuous representatives and, in a good ball, the following inequality:

$$|f_{\ell-1} \circ \psi_i(w_1) - f_{\ell-1} \circ \psi_i(w_2)| \leq \int_{S(0, r_j)} |\nabla(f_{\ell-1} \circ \psi_i)| d\mathcal{H}^1 \lesssim \lambda, \forall w_1, w_2 \in S(0, r_j).$$

We omit the proof of Theorem 3.1 in this case and focus on the more delicate case where $k \geq 3$. While the strategy of the proof is similar to the one in the supercritical case $s = 1$, $k - 1 < p < k$, we have to cope with an additional difficulty: When $k \geq 3$, $W^{1,k-1}$ maps on $(k - 1)$ -dimensional spheres need not be Hölder continuous, or even continuous. A way to handle this consists in smoothing the restriction of f to “generic” spheres. A second difference with respect to the supercritical case occurs in the definition of the good balls, which has to be adapted.

Proof of Theorem 3.1 when $p = k - 1$. Let $k \geq 3$. Starting from the “well-separated” balls that cover $\mathcal{M}$ and the *enlarged* balls constructed as in the supercritical case, we will construct inductively, for sufficiently small r , f_ℓ such that

- (i) there exists a set D_ℓ such that $\bigcup_{j \in J_\ell} B_{2r}(x_j) \subset D_\ell \subset \bigcup_{j \in J_\ell} B_{3K_2r}(x_j)$,

$$f_\ell = f_{\ell-1} \text{ in } \mathcal{M} \setminus D_\ell, \text{ and } f_\ell \text{ is continuous in } \bigcup_{a \leq \ell} D_a, \quad (3.56)$$

possibly except at the set $\left\{x_j : j \in \bigcup_{a \leq \ell} J_a\right\}$;

- (ii) there exists a set $E_\ell \subset D_\ell$ such that

$$\mathcal{H}^k(E_\ell) \lesssim r^{k-2} \int_{\mathcal{M}} |\nabla f_{\ell-1}|^{k-1} d\mathcal{H}^k + r^{1/(k-1)}, \quad (3.57)$$

and the following estimates hold:

$$\int_{\mathcal{M}} |f_\ell - f_{\ell-1}|^{k-1} d\mathcal{H}^k \lesssim r^{k-1} \int_{\mathcal{M}} |\nabla f_{\ell-1}|^{k-1} d\mathcal{H}^k, \quad (3.58)$$

$$\begin{aligned} \int_{\mathcal{M}} |\nabla(f_\ell - f_{\ell-1})|^{k-1} d\mathcal{H}^k &\lesssim \int_{E_\ell} |\nabla f_{\ell-1}|^{k-1} d\mathcal{H}^k \\ &\quad + r^{\frac{1}{k-1}} \int_{\mathcal{M}} |\nabla f_{\ell-1}|^{k-1} d\mathcal{H}^k. \end{aligned} \quad (3.59)$$

Let $\lambda > 0$ to be chosen later. Given a map $g \in W^{1,k-1}(\mathcal{M}; \mathcal{F})$ and $1 \leq \ell \leq \theta + 1$, for $j \in J_\ell$, we say that the ball $B_{3K_2r}(x_j)$ is a *good* ball provided

$$\frac{1}{r^2} \int_{B_{3K_2r}(x_j)} |\nabla f_{\ell-1}(x)|^{k-1} d\mathcal{H}^k(x) \leq \lambda. \quad (3.60)$$

We say that the ball $B_{3K_2r}(x_j)$ is a *bad* ball if (3.60) fails.

Let us note that (3.57) – (3.60) are quantitatively different from their counterparts in Section 3.1. The important point in our approach is to use a different power of r for the

prefactor in the definition of good balls. This results in good balls being defined *via* constraining a bigger, not scale invariant, quantity. As a result, we get a better control on our map in good balls, in exchange of having more bad balls. However, we are taking profit of the fact that working with the scale invariant energy gives some margin of security when estimating the number of bad balls, some of which we are therefore allowed to give up here. To the best of our knowledge, it is the first time that such an idea is used in the context of approximation problems for manifold-valued Sobolev mappings.

Let D_ℓ be as in (3.7). We let $f_\ell = f_{\ell-1}$ in $\mathcal{M} \setminus D_\ell$ and construct f_ℓ in D_ℓ in Steps 1-2.

Step 1. Construction of f_ℓ in bad balls.

We use the same notation as in Step 3 of the proof in the supercritical case. Let

$$\bar{g}_1: B_{r_j}(0) \setminus \{0\} \rightarrow \mathcal{F}, \bar{g}_1(u) := f_{\ell-1} \circ \psi_i \left(r_j \frac{u}{|u|} \right). \quad (3.61)$$

We obtain an approximation $\bar{g}_2$ of $\bar{g}_1$ *via* the following lemma.

Lemma 3.5. *Assume that either (i) $k \geq 3$, $0 < s \leq 1$, and $k-1 \leq sp < k$; or (ii) $k = 2$, $s = 1$, and $p = 1$. Consider a map $g \in W^{s,p}(S(0,1); \mathcal{F})$, and denote by $H(g)$ the homogeneous extension of g inside $B_1(0)$. Then, for every $\alpha > 0$, there exists $h \in W^{s,p}(B_1(0); \mathcal{F}) \cap C(B_1(0) \setminus \{0\})$ such that $\text{tr}_{S(0,1)} h = g$ and*

$$\|H(g) - h\|_{W^{s,p}(B_1(0))} \leq \alpha.$$

Moreover, we may choose h such that if g is continuous at some point $\omega \in S(0,1)$, then h is continuous up to ω .

For the proof of Lemma 3.5, see Appendix A. Using Lemma 3.5 for $\bar{g}_1$ in $B_{r_j}(0)$ instead of $B_1(0)$, we obtain a map $\bar{g}_2$ such that

- (i) $\bar{g}_2$ is continuous in $B_{r_j}(0) \setminus \{0\}$;
- (ii) $\bar{g}_2 = \bar{g}_1$ on $S(0, r_j)$;
- (iii) we have

$$\int_{B_{r_j}(0)} |\nabla(\bar{g}_1 - \bar{g}_2)|^{k-1} d\mathcal{H}^k \leq \int_{B_{3K_2r}(x_j)} |\nabla f_{\ell-1}|^{k-1} d\mathcal{H}^k.$$

We define f_ℓ in $\psi_i(B_{r_j}(0))$ by

$$f_\ell(x) := \bar{g}_2 \circ \psi_i^{-1}(x). \quad (3.62)$$

Step 2. Construction of f_ℓ in good balls.

We use the same notation as in the previous step, but this time, $B_{3K_2r}(x_j)$ is a good ball.

Let

$$\tilde{h}: S(0, 1) \rightarrow \mathcal{F}, \tilde{h}(\omega) := f_{\ell-1} \circ \psi_i(r_j \omega). \quad (3.63)$$

Since $\tilde{h}$ need not have bounded oscillation anymore, we first suitably extend $\tilde{h}$ in a small annulus near $S(0, 1)$. Then, we prove that the extension has small oscillation when restricted to a suitably chosen sphere of radius smaller than 1, and finally use Lemma 3.4 in the ball enclosed by such a sphere.

For $0 < \varepsilon < r_j/4$, which will later on be chosen *depending on* r , define

$$F: S(0, r_j) \times (0, 2\varepsilon) \rightarrow \mathcal{F}, F(w, t) := \begin{cases} \tilde{h} * \rho_{t/r_j}(w/r_j), & \text{if } t \leq \varepsilon \\ F(w, 2\varepsilon - t), & \text{if } \varepsilon \leq t \leq 2\varepsilon \end{cases}. \quad (3.64)$$

Here, the convolution is defined in the following sense:

$$h * \rho_t(\omega) := \int_{B_1(0)} h \circ R^\omega \circ \psi(u) \rho_t(u) du, \quad (3.65)$$

where ρ is a standard radial mollifier in the Euclidean $\mathbb{R}^{k-1}$, R^ω is any rotation of $\mathbb{R}^k$ around the origin that sends the north pole $\mathcal{O}$ of $S(0, 1)$ to ω , and $\psi: B_1(0) \rightarrow \tilde{B}_1(\mathcal{O})$ is a parametrization of the geodesic ball $\tilde{B}_1(\mathcal{O})$ centered at $\mathcal{O}$ of radius 1 on $S(0, 1)$ that satisfies: (i) ψ is a diffeomorphism; (ii) $\text{dist}(\psi(u), \mathcal{O}) = |u|$ for any $u \in B_1(0)$; (iii) for any rotation R of $\mathbb{R}^k$ around the axis $0x_k$, we have $R \circ \psi(u) = \psi(R(u, 0))$; and (iv) ψ maps any segment $[0, x)$, for $x \in \partial B_1(0)$, onto a great circle (which will therefore be a longitude).

We will discuss some properties of the convolution and their proofs in Appendix B.

We first study the oscillation of $F(\cdot, \varepsilon)$. For $w_1, w_2 \in S(0, r_j)$, by (B.1) (with $t_0 = \varepsilon/r_j$), we have

$$\begin{aligned} & |\tilde{h} * \rho_{\varepsilon/r_j}(w_1/r_j) - \tilde{h} * \rho_{\varepsilon/r_j}(w_2/r_j)| \\ & \lesssim \frac{r_j}{\varepsilon} \|\nabla \tilde{h}\|_{\mathcal{L}^{k-1}(S(0, 1))} \leq \frac{r_j}{\varepsilon} \|\nabla(f_{\ell-1} \circ \psi_i)\|_{\mathcal{L}^{k-1}(S(0, r_j))}. \end{aligned} \quad (3.66)$$

Combining (3.66), (3.6) with $p = k - 1$, and (3.60), we have

$$|\tilde{h} * \rho_{\varepsilon/r_j}(w_1/r_j) - \tilde{h} * \rho_{\varepsilon/r_j}(w_2/r_j)| \lesssim \frac{r^{\frac{k}{k-1}}}{\varepsilon} \lambda^{\frac{1}{k-1}}. \quad (3.67)$$

Similarly, using (B.2), we have for $t \leq \varepsilon$,

$$\text{dist}(\tilde{h} * \rho_{t/r_j}(w_1/r_j), \mathcal{F}) \lesssim \|\nabla \tilde{h}\|_{\mathcal{L}^{k-1}(S(0,1))} \lesssim (r\lambda)^{\frac{1}{k-1}}. \quad (3.68)$$

From now on, we let

$$\varepsilon := r^{k/(k-1)}.$$

Let us note that this is indeed compatible with the constraint $\varepsilon < r_j/4$, provided that we choose r sufficiently small. By (3.67) and (3.68), we can choose λ and r_1 sufficiently small universal constants such that, for $r < r_1$: (i) the map $\tilde{F} := \Pi \circ F$ is well-defined and $\tilde{F} \in W^{1,k-1}(S(0, r_j) \times (0, 2\varepsilon); \mathcal{F})$; and (ii) $\tilde{F}(\cdot, \varepsilon) \subset B_{\delta/2}(\mathfrak{f})$ for some $\mathfrak{f} \in \mathcal{F}$.

We next construct the approximation of $f_{\ell-1}$. For $0 < \varepsilon < r_j/4$, we consider the bi-Lipschitz map $\Psi: B_{r_j}(0) \setminus B_{r_j/2}(0) \rightarrow S(0, r_j) \times (0, r_j/2)$ defined by

$$\Psi(u) = \left(r_j \frac{u}{|u|}, r_j - |u| \right). \quad (3.69)$$

Clearly, for sufficiently small r_j , Ψ is bi-Lipschitz and its bi-Lipschitz constants do not depend on r_j . We define $\bar{h}_1: B_{r_j}(0) \rightarrow \mathcal{F}$ by

$$\bar{h}_1(u) := \begin{cases} \tilde{F} \circ \Psi(u), & \text{if } r_j - 2\varepsilon \leq |u| \leq r_j \\ f_{\ell-1} \circ \psi_i(u), & \text{if } |u| \leq r_j - 3\varepsilon \\ f_{\ell-1} \circ \psi_i(3u - 2\frac{r_j-3\varepsilon}{|u|}u), & \text{if } r_j - 3\varepsilon \leq |u| \leq r_j - 2\varepsilon \end{cases}. \quad (3.70)$$

We project $\bar{h}_1$ as in the supercritical case, by setting

$$\bar{h}_2(u) := \begin{cases} \Psi_{\mathfrak{f},\delta} \circ \bar{h}_1(u), & \text{if } u \in B_{r_j-\varepsilon}(0) \\ \bar{h}_1(u), & \text{if } u \in B_{r_j}(0) \setminus B_{r_j-\varepsilon}(0) \end{cases}. \quad (3.71)$$

Using Lemma 3.4 for $\bar{h}_2$ in $B_{r_j-\varepsilon}(0)$, there exists a map $\bar{h}_3 \in W^{1,k-1}(B_{r_j}(0); B_\delta(\mathfrak{f}))$ such that

- (i) $\bar{h}_3$ is continuous in $B_{r_j-\varepsilon}(0)$;
- (ii) $\bar{h}_3 = \bar{h}_2 = \bar{h}_1$ in $B_{r_j}(0) \setminus B_{r_j-\varepsilon}(0)$;
- (iii) we have

$$\int_{B_{r_j}(0)} |\nabla(\bar{h}_3 - \bar{h}_2)|^{k-1} \leq \int_{B_{r_j}(0)} |\nabla(\bar{h}_1 - \bar{h}_2)|^{k-1}. \quad (3.72)$$

We now define f_ℓ in $\psi_i(B_{r_j}(0))$ by $f_\ell := \bar{h}_3 \circ \psi_i^{-1}$.

By construction of f_ℓ and Lemma 3.5, the following holds: If $f_{\ell-1}$ is continuous at $x \in \psi_i(S(0, r_j))$, then f_ℓ is also continuous at x . This implies (3.56).

Step 3. Proof of (3.57)–(3.59).

As in the supercritical case, we divide the proof into three sub-steps.

Step 3.1. Estimate of $f_\ell - f_{\ell-1}$ in a bad ball $B_{3K_2r}(x_j)$.

Consider $\bar{g}_1$ as in (3.61). As in the supercritical case, we have

$$\int_{B_{r_j}(0)} |\nabla(\bar{g}_1 - f_{\ell-1} \circ \psi_i)|^{k-1} d\mathcal{H}^k \lesssim \int_{B_{3K_2r}(x_j)} |\nabla f_{\ell-1}|^{k-1} d\mathcal{H}^k. \quad (3.73)$$

Combining (3.73) and the definition (3.62) of f_ℓ , we have

$$\int_{\psi_i(B_{r_j}(0))} |\nabla(f_\ell - f_{\ell-1})|^{k-1} d\mathcal{H}^k \lesssim \int_{B_{3K_2r}(x_j)} |\nabla f_{\ell-1}|^{k-1} d\mathcal{H}^k. \quad (3.74)$$

Step 3.2. Estimate of $f_\ell - f_{\ell-1}$ in a good ball $B_{3K_2r}(x_j)$.

Since

$$\begin{aligned} \int_{\psi_i(B_{r_j}(0))} |\nabla(f_\ell - f_{\ell-1})|^{k-1} d\mathcal{H}^k &\lesssim \int_{B_{r_j}(0)} |\nabla(\bar{h}_3 - f_{\ell-1} \circ \psi_i)|^{k-1} \\ &\lesssim \int_{B_{r_j}(0)} |\nabla(\bar{h}_3 - \bar{h}_2)|^{k-1} + \int_{B_{r_j}(0)} |\nabla(\bar{h}_2 - \bar{h}_1)|^{k-1} \\ &\quad + \int_{B_{r_j}(0)} |\nabla(\bar{h}_1 - f_{\ell-1} \circ \psi_i)|^{k-1}, \end{aligned} \quad (3.75)$$

it suffices to estimate the last three terms in (3.75).

As in (3.20), we first have

$$\begin{aligned} &\int_{B_{r_j}(0)} |\nabla(\bar{h}_2 - \bar{h}_1)|^{k-1} d\mathcal{H}^{k-1} \\ &= \int_{\bar{A}_j} |\nabla \bar{h}_2 - \nabla \bar{h}_1|^{k-1} d\mathcal{H}^{k-1} \\ &\lesssim \int_{\bar{A}_j} |\nabla \bar{h}_1|^{k-1} d\mathcal{H}^{k-1} \\ &\lesssim \int_{\bar{A}_j} |\nabla f_{\ell-1} \circ \psi_i|^{k-1} d\mathcal{H}^{k-1} + \int_{B_{r_j}(0)} |\nabla(\bar{h}_1 - f_{\ell-1} \circ \psi_i)|^{k-1} d\mathcal{H}^{k-1}, \end{aligned} \quad (3.76)$$

where

$$\bar{A}_j^1 := \{u \in B_{r_j-\varepsilon}(0) : \bar{h}_1(u) \notin B_\delta(\mathfrak{f})\}. \quad (3.77)$$

Combining (3.72) with (3.76), in order to estimate the last three terms in (3.75), it remains to estimate $\|\nabla(\bar{h}_1 - f_{\ell-1} \circ \psi_i)\|_{\mathcal{L}^{k-1}}$. By the definition of $\bar{h}_1$, we have

$$\begin{aligned} \int_{B_{r_j}(0)} |\nabla(\bar{h}_1 - f_{\ell-1} \circ \psi_i)|^{k-1} &= \int_{B_{r_j}(0) \setminus B_{r_j-3\varepsilon}(0)} |\nabla(\bar{h}_1 - f_{\ell-1} \circ \psi_i)|^{k-1} \\ &\lesssim \int_{B_{r_j}(0) \setminus B_{r_j-3\varepsilon}(0)} |\nabla(f_{\ell-1} \circ \psi_i)|^{k-1} \\ &\quad + \int_{S(0, r_j) \times (0, \varepsilon)} |\nabla F(w, t)|^{k-1} d\mathcal{H}^{k-1}(w) dt. \end{aligned} \quad (3.78)$$

To estimate the last integral of (3.78), let $\tilde{h}$ be as in (3.63). By Lemma B.2 and the choice of r_j (see (3.6) with $p = k - 1$), we have

$$\begin{aligned} &\int_{S(0, r_j) \times (0, \varepsilon)} |\nabla F(w, t)|^{k-1} d\mathcal{H}^{k-1}(w) dt \\ &\lesssim r_j \int_{S(0, 1) \times (0, \varepsilon/r_j)} |\nabla[\tilde{h} * \rho_{t'}(\omega)]|^{k-1} d\mathcal{H}^{k-1}(\omega) dt' \\ &\lesssim \varepsilon \int_{S(0, 1)} |\nabla(f_{\ell-1} \circ \psi_i(r_j \cdot))|^{k-1} d\mathcal{H}^{k-1} \lesssim r^{\frac{1}{k-1}} \int_{B_{3K_2r}(x_j)} |\nabla f_{\ell-1}|^{k-1} d\mathcal{H}^k. \end{aligned} \quad (3.79)$$

Combining (3.72), (3.76), (3.78), and (3.79), we have

$$\int_{\psi_i(B_{r_j}(0))} |\nabla(f_\ell - f_{\ell-1})|^{k-1} \lesssim \int_{A_j^1 \cup A_j^2} |\nabla f_{\ell-1}|^{k-1} + r^{\frac{1}{k-1}} \int_{B_{3K_2r}(x_j)} |\nabla f_{\ell-1}|^{k-1}. \quad (3.80)$$

Here, $A_j^1 := \psi_i(\bar{A}_j^1)$, with $\bar{A}_j^1$ as in (3.77), and $A_j^2 := \psi_i(B_{r_j}(0) \setminus B_{r_j-3\varepsilon}(0))$.

Step 3.3. As in the supercritical case, summing over $j \in J_\ell$, we obtain maps $f_\ell \in W^{1, k-1}(\mathcal{M}; \mathcal{F})$, $1 \leq \ell \leq \theta + 1$, such that, by (3.74) and (3.80), (3.58) and (3.59) hold, with $E_\ell := \bigcup_{j \in J_\ell} E_\ell^j$ and

$$E_\ell^j := \begin{cases} A_j^1 \cup A_j^2, & \text{if } B_{3K_2r}(x_j) \text{ is a good ball for } f_{\ell-1} \\ B_{3K_2r}(x_j), & \text{if } B_{3K_2r}(x_j) \text{ is a bad ball for } f_{\ell-1} \end{cases}. \quad (3.81)$$

We next prove (3.57). Using the same calculation as in (3.24), we have

$$\mathcal{H}^k(\bar{A}_j^1) \lesssim r^{k-1} \int_{B_{r_j}(0)} |\nabla \bar{h}_1|^{k-1}, \quad (3.82)$$

and thus, combining (3.82) and (3.79) with (3.78), we obtain

$$\begin{aligned} \mathcal{H}^k(A_j^1) &\lesssim r^{k-1} \left[\int_{B_{3K_2r}(x_j)} |\nabla f_{\ell-1}|^{k-1} d\mathcal{H}^k + r^{\frac{1}{k-1}} \int_{B_{3K_2r}(x_j)} |\nabla f_{\ell-1}|^{k-1} d\mathcal{H}^k \right] \\ &\lesssim r^{k-1} \int_{B_{3K_2r}(x_j)} |\nabla f_{\ell-1}|^{k-1} d\mathcal{H}^k, \end{aligned} \quad (3.83)$$

and (recall that $\varepsilon = r^{k/(k-1)}$)

$$\mathcal{H}^k(A_j^2) \lesssim |B_{r_j}(0) \setminus B_{r_j-3\varepsilon}(0)| \lesssim [r^k - (r-3\varepsilon)^k] \lesssim r^{k+1/(k-1)}. \quad (3.84)$$

By (3.60), for $f_{\ell-1}$, there will be at most $\int_{\mathcal{M}} |\nabla f_{\ell-1}|^{k-1} d\mathcal{H}^k / \lambda r^2$ bad balls, and the total number of balls (good and bad) is of order at most r^{-k} . Combining this with (3.83), (3.84), and (3.81), we have

$$\mathcal{H}^k(E_\ell) \lesssim (r^{k-1} + r^{k-2}) \int_{\mathcal{M}} |\nabla f_{\ell-1}|^{k-1} d\mathcal{H}^k + r^{1/(k-1)}, \quad (3.85)$$

which implies (3.57).

Now we turn to the estimate of $\|f_{\theta+1} - f\|_{W^{1,k-1}}$. First, by (3.59) and an iteration procedure, we have

$$\begin{aligned} &\|\nabla f_{\theta+1} - \nabla f\|_{\mathcal{L}^{k-1}(\mathcal{M})}^{k-1} \\ &\lesssim \|\nabla f_\theta\|_{\mathcal{L}^{k-1}(E_{\theta+1})}^{k-1} + r^{\frac{1}{k-1}} \|\nabla f_\theta\|_{\mathcal{L}^{k-1}(\mathcal{M})}^{k-1} + \|\nabla f_\theta - \nabla f\|_{\mathcal{L}^{k-1}(\mathcal{M})}^{k-1} \\ &\lesssim \|\nabla f\|_{\mathcal{L}^{k-1}(E_{\theta+1})}^{k-1} + r^{\frac{1}{k-1}} \|\nabla f\|_{\mathcal{L}^{k-1}(\mathcal{M})}^{k-1} + \|\nabla f_\theta - \nabla f\|_{\mathcal{L}^{k-1}(\mathcal{M})}^{k-1} \\ &\lesssim \sum_{\ell=1}^{\theta+1} \|\nabla f\|_{\mathcal{L}^{k-1}(E_\ell)}^{k-1} + (\theta+1) r^{\frac{1}{k-1}} \|\nabla f\|_{\mathcal{L}^{k-1}(\mathcal{M})}^{k-1} \\ &\lesssim \|\nabla f\|_{\mathcal{L}^{k-1}(\cup_{\ell} E_\ell)}^{k-1} + r^{\frac{1}{k-1}} \|\nabla f\|_{\mathcal{L}^{k-1}(\mathcal{M})}^{k-1}. \end{aligned} \quad (3.86)$$

Then, using (3.58) and (3.86) again combined with an iteration procedure, we have

$$\begin{aligned}
& \|f_{\theta+1} - f\|_{\mathcal{L}^{k-1}(\mathcal{M})} \\
& \lesssim r\|\nabla f_{\theta}\|_{\mathcal{L}^{k-1}(\mathcal{M})} + \|f_{\theta} - f\|_{\mathcal{L}^{k-1}(\mathcal{M})} \\
& \lesssim r\|\nabla f\|_{\mathcal{L}^{k-1}(\mathcal{M})} + r\|\nabla f_{\theta} - \nabla f\|_{\mathcal{L}^{k-1}(\mathcal{M})} + \|f_{\theta} - f\|_{\mathcal{L}^{k-1}(\mathcal{M})} \\
& \lesssim r\|\nabla f\|_{\mathcal{L}^{k-1}(\mathcal{M})} + \|f_{\theta} - f\|_{\mathcal{L}^{k-1}(\mathcal{M})} \\
& \lesssim r\|\nabla f\|_{\mathcal{L}^{k-1}(\mathcal{M})}.
\end{aligned} \tag{3.87}$$

On the other hand, combining the same argument with (3.85), we have

$$\mathcal{H}^k\left(\bigcup_{\ell} E_{\ell}\right) \lesssim (r^{k-1} + r^{k-2})\|\nabla f\|_{\mathcal{L}^{k-1}(\mathcal{M})}^{k-1} + r^{\frac{1}{k-1}}. \tag{3.88}$$

Thus, combining (3.86), (3.87), and (3.88), we have $\|f_{\theta+1} - f\|_{W^{1,k-1}} \rightarrow 0$ as $r \rightarrow 0$. Repeating Step 6 as in the supercritical case, the map $f_{\theta+1}$ can be approximated with maps in $\mathcal{R}_0$. $\square$

3.4 Proof of Theorem 3.2 when $sp = k - 1$ and $k \geq 3$

When $0 < s < 1$ and $sp = k - 1$, the construction of approximation map is again the same as the case where $s = 1$, except when $k = 2$. In this section, we focus on the fractional counterpart of the proof in Section 3.4, and then deal with the case where $k = 2$ in Section 3.5.

Proof of Theorem 3.2 when $sp = k - 1$ when $k \geq 3$. The proof combines arguments from Sections 3.2 and 3.3. We use the same notation as in Section 3.2. We will construct inductively f_{ℓ} such that

- (i) there exists a set D_{ℓ} such that $\bigcup_{j \in J_{\ell}} B_{2r}(x_j) \subset D_{\ell} \subset \bigcup_{j \in J_{\ell}} B_{3K_2 r}(x_j)$,

$$\begin{aligned}
& f_{\ell} = f_{\ell-1} \text{ in } \mathcal{M} \setminus D_{\ell}, \text{ and } f_{\ell} \text{ is continuous in } \bigcup_{a \leq \ell} D_a, \\
& \text{possibly except at the set } \left\{x_j : j \in \bigcup_{a \leq \ell} J_a\right\};
\end{aligned}$$

- (ii) there exists a set $E_{\ell} \subset D_{\ell}$ such that

$$\mathcal{H}^k(E_{\ell}) \lesssim r^{k-2} |f_{\ell-1}|_{W^{s,p}(\mathcal{M})}^p + r^{\frac{1}{k-1}}, \tag{3.89}$$

and the following estimates hold:

$$\int_{\mathcal{M}} |f_\ell - f_{\ell-1}|^p d\mathcal{H}^k \lesssim r^{k-1} |f_{\ell-1}|_{W^{s,p}(\mathcal{M})}^p \quad (3.90)$$

$$\begin{aligned} |f_\ell - f_{\ell-1}|_{W^{s,p}(\mathcal{M})}^p &\lesssim \int_{E_\ell} \int_{\mathcal{M}} \frac{|f_{\ell-1}(x) - f_{\ell-1}(y)|^p}{|x - y|^{2k-1}} d\mathcal{H}^k(x) d\mathcal{H}^k(y) \\ &\quad + r^{\frac{1}{k-1}} |f_{\ell-1}|_{W^{s,p}(\mathcal{M})}^p. \end{aligned} \quad (3.91)$$

(The exponent $2k - 1$ in the middle line is the same as the familiar exponent $k + sp$ arising in the definition of $W^{s,p}$, since, in this section, $sp = k - 1$.)

Given $f_{\ell-1} \in W^{s,p}(\mathcal{M}; \mathcal{F})$, the construction of f_ℓ is almost the same as the case where $s = 1$, with three slight modifications. First, the r_j we choose is the same as the one in the supercritical case (Section 3.2). Second, we adapt the definition of good balls and bad balls. More specifically, for $j \in J_\ell$, we say that the ball $B_{3K_2r}(x_j)$ is a *good* ball provided

$$\frac{1}{r^2} \int_{B_{3K_2r}(x_j)} \int_{\mathcal{M}} \frac{|f_{\ell-1}(x) - f_{\ell-1}(y)|^p}{|x - y|^{2k-1}} d\mathcal{H}^k(y) d\mathcal{H}^k(x) \leq \lambda, \quad (3.92)$$

with λ to be chosen later. We say that the ball $B_{3K_2r}(x_j)$ is a *bad* ball if (3.92) fails. Third, for $w_1, w_2 \in S(0, r_j)$, we replace the oscillation computation (3.66) with (using (B.1))

$$\begin{aligned} &|\tilde{h} * \rho_{\varepsilon/r_j}(w_1/r_j) - \tilde{h} * \rho_{\varepsilon/r_j}(w_2/r_j)| \\ &\lesssim \left(\frac{r_j}{\varepsilon}\right)^s |\tilde{h}|_{W^{s,p}(S(0,1))} \leq \left(\frac{r_j}{\varepsilon}\right)^s \|f_{\ell-1} \circ \psi_i\|_{W^{s,p}(S(0,r_j))}. \end{aligned} \quad (3.93)$$

Step 1. Estimate of $f_\ell - f_{\ell-1}$ in a bad ball $B_{3K_2r}(x_j)$.

Similarly to the case where $s = 1$, let $\bar{g}_1$ be as in (3.61), and let $\bar{g}_2$ be an approximation of $\bar{g}_1$ obtained *via* Lemma 3.5. We have

$$|\bar{g}_1 - f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{r_j}(0))}^p \lesssim |f_{\ell-1}|_{W^{s,p}(B_{3K_2r}(x_j))}^p \quad (3.94)$$

and, suitably choosing α when applying Lemma 3.5,

$$|\bar{g}_2 - \bar{g}_1|_{W^{s,p}(B_{r_j}(0))}^p \leq |f_{\ell-1}|_{W^{s,p}(B_{3K_2r}(x_j))}^p. \quad (3.95)$$

Combining (3.94) and (3.95) with the definition of f_ℓ in (3.62), we find that

$$|f_\ell - f_{\ell-1}|_{W^{s,p}(\psi_i(B_{r_j}(0)))}^p \lesssim |f_{\ell-1}|_{W^{s,p}(B_{3K_2r}(x_j))}^p. \quad (3.96)$$

Step 2. Estimate of $f_\ell - f_{\ell-1}$ in a good ball $B_{3K_2r}(x_j)$.

We use the same notation as in Section 3.3. Let $\bar{h}_1$ be as in (3.70), let $\bar{h}_2$ be as in (3.71), and let $\bar{h}_3$ be an approximation of $\bar{h}_2$ obtained *via* Lemma 3.4. In this step, we will obtain the fractional counterpart of (3.80); see (3.115) below. For this purpose, we will estimate, as in Section 3.3, the last three terms in (3.75) (with the $W^{1,k}$ seminorm replaced with the $W^{s,p}$ seminorm).

First, since $\bar{h}_2 = \bar{h}_1$ in $B_{r_j}(0) \setminus B_{r_j-\varepsilon}(0)$, using Hardy's inequality, we have

$$\begin{aligned} & |\bar{h}_2 - \bar{h}_1|_{W^{s,p}(B_{r_j}(0))}^p \\ & \leq |\bar{h}_2 - \bar{h}_1|_{W^{s,p}(B_{r_j-\varepsilon}(0))}^p + 2 \int_{B_{r_j-\varepsilon}(0)} \int_{B_{r_j}(0) \setminus B_{r_j-\varepsilon}(0)} \frac{|\bar{h}_2(v) - \bar{h}_1(v)|^p}{|u - v|^{2k-1}} du dv \\ & \lesssim |\bar{h}_2 - \bar{h}_1|_{W^{s,p}(B_{r_j-\varepsilon}(0))}^p. \end{aligned}$$

The same calculation as in (3.47) (with $B_{r_j}(0)$ replaced with $B_{r_j-\varepsilon}(0)$) then implies that

$$\begin{aligned} |\bar{h}_2 - \bar{h}_1|_{W^{s,p}(B_{r_j}(0))}^p & \lesssim \int_{\bar{A}_j^{-1}} \int_{B_{r_j}(0)} \frac{|\bar{h}_1(u) - \bar{h}_1(v)|^p}{|u - v|^{2k-1}} du dv \\ & \lesssim \int_{\bar{A}_j^{-1}} \int_{B_{r_j}(0)} \frac{|f_{\ell-1} \circ \psi_i(u) - f_{\ell-1} \circ \psi_i(v)|^p}{|u - v|^{2k-1}} du dv \\ & \quad + |\bar{h}_1 - f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{r_j}(0))}^p, \end{aligned} \tag{3.97}$$

where $\bar{A}_j^{-1}$ is as in (3.77).

In view of (3.97) and of the definition of $\bar{h}_3$, in order to obtain (3.115) below, it remains to estimate $|\bar{H}|_{W^{s,p}(B_{r_j}(0))}$, with $\bar{H} := \bar{h}_1 - f_{\ell-1} \circ \psi_i$. In the following three sub-steps, we are going to prove that

$$\begin{aligned} |\bar{H}|_{W^{s,p}(B_{r_j}(0))}^p & \lesssim \int_{B_{r_j}(0) \setminus B_{r_j-3\varepsilon}(0)} \int_{B_{3r}(0)} \frac{|f_{\ell-1} \circ \psi_i(u) - f_{\ell-1} \circ \psi_i(v)|^p}{|u - v|^{2k-1}} dv du \\ & \quad + r^{\frac{1}{k-1}} |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))}^p. \end{aligned} \tag{3.98}$$

More precisely, in the first two sub-steps, we will prove that

$$\begin{aligned} |\bar{H}|_{W^{s,p}(B_{r_j}(0))}^p & \lesssim |F|_{W^{s,p}(S(0,r_j) \times (0,\varepsilon))}^p + r^{\frac{1}{k-1}} |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))}^p \\ & \quad + \int_{B_{r_j}(0) \setminus B_{r_j-3\varepsilon}(0)} \int_{B_{3r}(0)} \frac{|f_{\ell-1} \circ \psi_i(u) - f_{\ell-1} \circ \psi_i(v)|^p}{|u - v|^{2k-1}} dv du, \end{aligned} \tag{3.99}$$

where F is as in (3.64), and in the last sub-step, we prove that

$$|F|_{W^{s,p}(S(0,r_j) \times (0,\varepsilon))}^p \lesssim r^{\frac{1}{k-1}} |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))}^p. \quad (3.100)$$

For this purpose, we first note that, by the definition of $\bar{h}_1$, we have

$$\begin{aligned} & |\bar{H}|_{W^{s,p}(B_{r_j}(0))}^p \\ & \leq 2 \int_{B_{r_j}(0) \setminus B_{r_j-3\varepsilon}(0)} \int_{B_{r_j-3\varepsilon}(0)} \frac{|\bar{H}(u)|^p}{|u-v|^{2k-1}} dv du + |\bar{H}|_{W^{s,p}(B_{r_j}(0) \setminus B_{r_j-3\varepsilon}(0))}^p. \end{aligned} \quad (3.101)$$

Step 2.1. Estimate of the first term on the right-hand side of (3.101).

We decompose

$$B_{r_j}(0) \setminus B_{r_j-3\varepsilon}(0) = (B_{r_j-2\varepsilon}(0) \setminus B_{r_j-3\varepsilon}(0)) \bigcup (B_{r_j}(0) \setminus B_{r_j-2\varepsilon}(0)).$$

For $r_j - 3\varepsilon \leq |u| \leq r_j - 2\varepsilon$, by the definition (3.70) of $\bar{h}_1$ in $B_{r_j-2\varepsilon}(0)$, we have

$$\begin{aligned} & \int_{B_{r_j-2\varepsilon}(0) \setminus B_{r_j-3\varepsilon}(0)} \int_{B_{r_j-3\varepsilon}(0)} \frac{|\bar{H}(u)|^p}{|u-v|^{2k-1}} dv du \\ & = \int_{B_{r_j-2\varepsilon}(0) \setminus B_{r_j-3\varepsilon}(0)} \int_{B_{r_j-3\varepsilon}(0)} \frac{|\bar{H}(u) - \bar{H}(v)|^p}{|u-v|^{2k-1}} dv du \\ & \lesssim \int_{B_{r_j}(0) \setminus B_{r_j-3\varepsilon}(0)} \int_{B_{r_j-3\varepsilon}(0)} \frac{|f_{\ell-1} \circ \psi_i(u) - f_{\ell-1} \circ \psi_i(v)|^p}{|u-v|^{2k-1}} dv du \\ & \leq \int_{B_{r_j}(0) \setminus B_{r_j-3\varepsilon}(0)} \int_{B_{3r}(0)} \frac{|f_{\ell-1} \circ \psi_i(u) - f_{\ell-1} \circ \psi_i(v)|^p}{|u-v|^{2k-1}} dv du. \end{aligned} \quad (3.102)$$

On the other hand, for $r_j - 2\varepsilon \leq |u| \leq r_j$, we have (with $\tilde{h}$ as in (3.63))

$$\begin{aligned} & \int_{B_{r_j}(0) \setminus B_{r_j-2\varepsilon}(0)} \int_{B_{r_j-3\varepsilon}(0)} \frac{|\bar{H}(u)|^p}{|u-v|^{2k-1}} dv du \\ & \lesssim \int_{B_{r_j}(0) \setminus B_{r_j-2\varepsilon}(0)} \int_{B_{r_j-3\varepsilon}(0)} \frac{|\bar{h}_1(u) - \tilde{h}(u/|u|)|^p}{|u-v|^{2k-1}} dv du \\ & \quad + \int_{B_{r_j}(0) \setminus B_{r_j-2\varepsilon}(0)} \int_{B_{r_j-3\varepsilon}(0)} \frac{|\tilde{h}(u/|u|) - f_{\ell-1} \circ \psi_i(v)|^p}{|u-v|^{2k-1}} dv du \\ & \quad + \int_{B_{r_j}(0) \setminus B_{r_j-3\varepsilon}(0)} \int_{B_{3r}(0)} \frac{|f_{\ell-1} \circ \psi_i(v) - f_{\ell-1} \circ \psi_i(u)|^p}{|u-v|^{2k-1}} dv du. \end{aligned} \quad (3.103)$$

We next estimate the third integral in (3.103). We first note that, when $r_j - 2\varepsilon \leq |u| \leq r_j$ and $|v| \leq r_j - 3\varepsilon$, we have $|r_j u / |u| - u| = r_j - |u| \leq 2\varepsilon$ and $|u - v| \geq |u| - |v| \geq \varepsilon$. Therefore,

$$|r_j u / |u| - v| \leq |r_j u / |u| - u| + |u - v| \leq 2\varepsilon + |u - v| \leq 3|u - v|. \quad (3.104)$$

Thus,

$$\begin{aligned} & \int_{B_{r_j}(0) \setminus B_{r_j-2\varepsilon}(0)} \int_{B_{r_j-3\varepsilon}(0)} \frac{|\tilde{h}(u/|u|) - f_{\ell-1} \circ \psi_i(v)|^p}{|u - v|^{2k-1}} dv du \\ & \lesssim \int_{B_{r_j}(0) \setminus B_{r_j-2\varepsilon}(0)} \int_{B_{r_j-3\varepsilon}(0)} \frac{|f_{\ell-1} \circ \psi_i(r_j u / |u|) - f_{\ell-1} \circ \psi_i(v)|^p}{|r_j u / |u| - v|^{2k-1}} dv du \\ & = \int_{r_j-2\varepsilon}^{r_j} \int_{S(0, r_j)} \int_{B_{r_j-3\varepsilon}(0)} \left(\frac{t}{r_j} \right)^{k-1} \frac{|f_{\ell-1} \circ \psi_i(w) - f_{\ell-1} \circ \psi_i(v)|^p}{|w - v|^{2k-1}} dv d\mathcal{H}^{k-1}(w) dt \\ & \lesssim \int_{r_j-2\varepsilon}^{r_j} \frac{t^{k-1}}{r_j^k} |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))}^p dt \lesssim r^{\frac{1}{k-1}} |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))}^p, \end{aligned} \quad (3.105)$$

where the last line follows from the choice of r_j as in (3.37).

We now turn to the second integral in (3.103). By the definition of $\bar{h}_1$ in $B_{r_j}(0) \setminus B_{r_j-2\varepsilon}(0)$, we have

$$\begin{aligned} & \int_{B_{r_j}(0) \setminus B_{r_j-2\varepsilon}(0)} \int_{B_{r_j-3\varepsilon}(0)} \frac{|\bar{h}_1(u) - \tilde{h}(u/|u|)|^p}{|u - v|^{2k-1}} dv du \\ & \lesssim \frac{1}{\varepsilon^{k-1}} \int_{B_{r_j}(0) \setminus B_{r_j-2\varepsilon}(0)} |\bar{h}_1(u) - \tilde{h}(u/|u|)|^p du \\ & \lesssim \frac{1}{\varepsilon^{k-1}} \int_{S(0, r_j)} \int_0^{2\varepsilon} |F(w, t) - \tilde{h}(w/r_j)|^p dt d\mathcal{H}^{k-1}(w) \\ & \lesssim \frac{1}{\varepsilon^{k-1}} \int_{S(0, r_j)} \int_0^\varepsilon |\tilde{h} * \rho_{t/r_j}(w/r_j) - \tilde{h}(w/r_j)|^p dt d\mathcal{H}^{k-1}(w) \\ & \quad + \frac{1}{\varepsilon^{k-1}} \int_{S(0, r_j)} \int_\varepsilon^{2\varepsilon} |\tilde{h} * \rho_{(2\varepsilon-t)/r_j}(w/r_j) - \tilde{h}(w/r_j)|^p dt d\mathcal{H}^{k-1}(w) \\ & \lesssim \frac{r_j^k}{\varepsilon^{k-1}} \int_{S(0,1)} \int_0^{\varepsilon/r_j} |\tilde{h} * \rho_t(\omega) - \tilde{h}(\omega)|^p dt d\mathcal{H}^{k-1}(\omega). \end{aligned}$$

Therefore, using: (j) Hölder's inequality; (jj) Tonelli's theorem; and (jjj) the choice of r_j

as in (3.36), we have

$$\begin{aligned}
& \int_{B_{r_j}(0) \setminus B_{r_j-2\varepsilon}(0)} \int_{B_{r_j-3\varepsilon}(0)} \frac{|\bar{h}_1(u) - \tilde{h}(u/|u|)|^p}{|u-v|^{2k-1}} dv du \\
& \lesssim \frac{r_j^k}{\varepsilon^{k-1}} \int_{S(0,1)} \int_0^{\varepsilon/r_j} \int_{\tilde{B}_t(\omega)} \frac{|\tilde{h}(\omega') - \tilde{h}(\omega)|^p}{t^{k-1}} d\mathcal{H}^{k-1}(\omega') dt d\mathcal{H}^{k-1}(\omega) \\
& \lesssim \varepsilon \int_{S(0,1)} \int_0^{\varepsilon/r_j} \int_{\tilde{B}_t(\omega)} \frac{|\tilde{h}(\omega') - \tilde{h}(\omega)|^p}{t^{2k-1}} d\mathcal{H}^{k-1}(\omega') dt d\mathcal{H}^{k-1}(\omega) \\
& \lesssim \varepsilon \int_{S(0,1)} \int_{S(0,1)} \int_{|\omega-\omega'|}^{\infty} \frac{|\tilde{h}(\omega') - \tilde{h}(\omega)|^p}{t^{2k-1}} dt d\mathcal{H}^{k-1}(\omega') d\mathcal{H}^{k-1}(\omega) \\
& \lesssim \varepsilon \int_{S(0,1)} \int_{S(0,1)} \frac{|\tilde{h}(\omega') - \tilde{h}(\omega)|^p}{|\omega-\omega'|^{2k-2}} d\mathcal{H}^{k-1}(\omega') d\mathcal{H}^{k-1}(\omega) \\
& \lesssim \varepsilon |\tilde{h}|_{W^{s,p}(S(0,1))}^p \lesssim \varepsilon |f_\ell \circ \psi_i|_{W^{s,p}(S(0,r_j))}^p \lesssim r^{\frac{1}{k-1}} |f_\ell \circ \psi_i|_{W^{s,p}(B_{3r}(0))}^p.
\end{aligned} \tag{3.106}$$

Combining (3.102)–(3.106), we obtain

$$\begin{aligned}
& \int_{B_{r_j}(0) \setminus B_{r_j-3\varepsilon}(0)} \int_{B_{r_j-3\varepsilon}(0)} \frac{|\bar{H}(u)|^p}{|u-v|^{2k-1}} dv du \\
& \lesssim \int_{B_{r_j}(0) \setminus B_{r_j-3\varepsilon}(0)} \int_{B_{3r}(0)} \frac{|f_{\ell-1} \circ \psi_i(u) - f_{\ell-1} \circ \psi_i(v)|^p}{|u-v|^{2k-1}} dv du \\
& \quad + r^{\frac{1}{k-1}} |f_\ell \circ \psi_i|_{W^{s,p}(B_{3r}(0))}^p.
\end{aligned} \tag{3.107}$$

Step 2.2. Estimate of the second term on the right-hand side of (3.101).

For this purpose, we start by noticing that

$$|\bar{H}|_{W^{s,p}(B_{r_j}(0) \setminus B_{r_j-3\varepsilon}(0))}^p \lesssim |\bar{h}_1|_{W^{s,p}(B_{r_j}(0) \setminus B_{r_j-3\varepsilon}(0))}^p + |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{r_j}(0) \setminus B_{r_j-3\varepsilon}(0))}^p. \tag{3.108}$$

By the definition of $\bar{h}_1$, we have

$$\begin{aligned}
& |\bar{h}_1|_{W^{s,p}(B_{r_j}(0) \setminus B_{r_j-3\varepsilon}(0))}^p \\
& \lesssim |\bar{h}_1|_{W^{s,p}(B_{r_j-2\varepsilon}(0) \setminus B_{r_j-3\varepsilon}(0))}^p + |\bar{h}_1|_{W^{s,p}(B_{r_j}(0) \setminus B_{r_j-2\varepsilon}(0))}^p \\
& \quad + \int_{B_{r_j-2\varepsilon}(0) \setminus B_{r_j-3\varepsilon}(0)} \int_{B_{r_j}(0) \setminus B_{r_j-2\varepsilon}(0)} \frac{|\bar{h}_1(u) - \bar{h}_1(v)|^p}{|u - v|^{2k-1}} dv du \\
& \lesssim |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{r_j}(0) \setminus B_{r_j-3\varepsilon}(0))}^p + |F|_{W^{s,p}(S(0,r_j) \times (0,\varepsilon))}^p \\
& \quad + \int_{B_{r_j-2\varepsilon}(0) \setminus B_{r_j-3\varepsilon}(0)} \int_{B_{r_j}(0) \setminus B_{r_j-2\varepsilon}(0)} \frac{|\bar{h}_1(u) - \tilde{h}(v/|v|)|^p}{|u - v|^{2k-1}} dv du \\
& \quad + \int_{B_{r_j-2\varepsilon}(0) \setminus B_{r_j-3\varepsilon}(0)} \int_{B_{r_j}(0) \setminus B_{r_j-\varepsilon}(0)} \frac{|\bar{h}_1(v) - \tilde{h}(v/|v|)|^p}{|u - v|^{2k-1}} dv du \\
& \quad + \int_{B_{r_j-2\varepsilon}(0) \setminus B_{r_j-3\varepsilon}(0)} \int_{B_{r_j-\varepsilon}(0) \setminus B_{r_j-2\varepsilon}(0)} \frac{|\bar{h}_1(v) - \tilde{h}(v/|v|)|^p}{|u - v|^{2k-1}} dv du.
\end{aligned} \tag{3.109}$$

We next estimate the last three integrals in (3.109). In order to estimate the first of the three integrals, we first note that, when $|u| \leq r_j - 2\varepsilon$ and $|v| \geq r_j - 2\varepsilon$, we have $|(r_j - 2\varepsilon)v/|v| - v| = \text{dist}(v, B_{r_j-2\varepsilon}(0)) \leq |u - v|$. Hence,

$$\begin{aligned}
|u - v| & \geq |u - (r_j - 2\varepsilon)v/|v|| - |(r_j - 2\varepsilon)v/|v| - v| \\
& \geq |u - (r_j - 2\varepsilon)v/|v|| - |u - v|,
\end{aligned}$$

so that

$$|u - v| \geq \frac{1}{2}|u - (r_j - 2\varepsilon)v/|v||.$$

Therefore,

$$\begin{aligned}
& \int_{B_{r_j-2\varepsilon}(0) \setminus B_{r_j-3\varepsilon}(0)} \int_{B_{r_j}(0) \setminus B_{r_j-2\varepsilon}(0)} \frac{|\bar{h}_1(u) - \tilde{h}(v/|v|)|^p}{|u-v|^{2k-1}} dv du \\
&= \int_{B_{r_j-2\varepsilon}(0) \setminus B_{r_j-3\varepsilon}(0)} \int_{B_{r_j}(0) \setminus B_{r_j-2\varepsilon}(0)} \frac{|\bar{h}_1(u) - \bar{h}_1((r_j-2\varepsilon)v/|v|)|^p}{|u-v|^{2k-1}} dv du \\
&\lesssim \int_{B_{r_j-2\varepsilon}(0) \setminus B_{r_j-3\varepsilon}(0)} \int_{B_{r_j}(0) \setminus B_{r_j-2\varepsilon}(0)} \frac{|\bar{h}_1(u) - \bar{h}_1((r_j-2\varepsilon)v/|v|)|^p}{|u-(r_j-2\varepsilon)v/|v||^{2k-1}} dv du \\
&= \int_{B_{r_j-2\varepsilon}(0) \setminus B_{r_j-3\varepsilon}(0)} \int_{r_j-2\varepsilon}^{r_j} \int_{S(0,t)} \frac{|\bar{h}_1(u) - \bar{h}_1((r_j-2\varepsilon)w'/t)|^p}{|u-(r_j-2\varepsilon)w'/t|^{2k-1}} d\mathcal{H}^{k-1}(w') dt du \\
&= \int_{B_{r_j-2\varepsilon}(0) \setminus B_{r_j-3\varepsilon}(0)} \int_{r_j-2\varepsilon}^{r_j} \int_{S(0,r_j-2\varepsilon)} \left(\frac{t}{r_j-2\varepsilon} \right)^{k-1} \frac{|\bar{h}_1(u) - \bar{h}_1(w)|^p}{|u-w|^{2k-1}} d\mathcal{H}^{k-1}(w) dt du \\
&\lesssim \varepsilon \int_{B_{r_j-2\varepsilon}(0) \setminus B_{r_j-3\varepsilon}(0)} \int_{S(0,r_j-2\varepsilon)} \frac{|\bar{h}_1(u) - \bar{h}_1(w)|^p}{|u-w|^{2k-1}} d\mathcal{H}^{k-1}(w) du.
\end{aligned}$$

Then, using a change of variables, the definition (3.70) for $\bar{h}_1$, and the choice of r_j from (3.37), we obtain

$$\begin{aligned}
& \int_{B_{r_j-2\varepsilon}(0) \setminus B_{r_j-3\varepsilon}(0)} \int_{B_{r_j}(0) \setminus B_{r_j-2\varepsilon}(0)} \frac{|\bar{h}_1(u) - \tilde{h}(v/|v|)|^p}{|u-v|^{2k-1}} dv du \\
&\lesssim \varepsilon \int_{B_{r_j-3\varepsilon}(0)} \int_{S(0,r_j)} \frac{|f_{\ell-1} \circ \psi_i(u) - f_{\ell-1} \circ \psi_i(w)|^p}{|u-w|^{2k-1}} d\mathcal{H}^{k-1}(w) du \\
&\lesssim \varepsilon \int_{B_{r_j}(0)} \int_{S(0,r_j)} \frac{|f_{\ell-1} \circ \psi_i(u) - f_{\ell-1} \circ \psi_i(w)|^p}{|u-w|^{2k-1}} d\mathcal{H}^{k-1}(w) du \\
&\lesssim r^{\frac{1}{k-1}} |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))}.
\end{aligned} \tag{3.110}$$

We now turn to the second of the three last integrals in (3.109). By the definition of

$\bar{h}_1$ in $B_{r_j-2\varepsilon}(0) \setminus B_{r_j-3\varepsilon}(0)$ and a change of variables *via* Ψ (as in (3.69)), we have

$$\begin{aligned}
& \int_{B_{r_j-2\varepsilon}(0) \setminus B_{r_j-3\varepsilon}(0)} \int_{B_{r_j-\varepsilon}(0) \setminus B_{r_j-2\varepsilon}(0)} \frac{|\bar{h}_1(v) - \tilde{h}(v/|v|)|^p}{|u-v|^{2k-1}} dv du \\
& \lesssim \int_{B_{r_j-\varepsilon}(0) \setminus B_{r_j-2\varepsilon}(0)} \frac{|\bar{h}_1(v) - \tilde{h}(v/|v|)|^p}{[\text{dist}(v, S(0, r_j - 2\varepsilon))]^{k-1}} dv \\
& = \int_{B_{r_j-\varepsilon}(0) \setminus B_{r_j-2\varepsilon}(0)} \frac{|\tilde{F} \circ \Psi(v) - \tilde{F} \circ \Psi(r_j v/|v|)|^p}{[\text{dist}(v, S(0, r_j - 2\varepsilon))]^{k-1}} dv \\
& \lesssim \int_{S(0, r_j) \times (\varepsilon, 2\varepsilon)} \frac{|F(w, t) - F(w, 0)|^p}{(2\varepsilon - t)^{k-1}} d\mathcal{H}^{k-1}(w) dt \\
& = \int_{S(0, r_j) \times (\varepsilon, 2\varepsilon)} \frac{|\tilde{h} * \rho_{(2\varepsilon-t)/r_j}(w/r_j) - \tilde{h}(w/r_j)|^p}{(2\varepsilon - t)^{k-1}} d\mathcal{H}^{k-1}(w) dt.
\end{aligned}$$

Using a change of variables and the choice of r_j (3.36), we now obtain

$$\begin{aligned}
& \int_{B_{r_j-2\varepsilon}(0) \setminus B_{r_j-3\varepsilon}(0)} \int_{B_{r_j-\varepsilon}(0) \setminus B_{r_j-2\varepsilon}(0)} \frac{|\bar{h}_1(v) - \tilde{h}(v/|v|)|^p}{|u-v|^{2k-1}} dv du \\
& \lesssim \int_{S(0, r_j) \times (0, \varepsilon)} \frac{|\tilde{h} * \rho_{t/r_j}(w/r_j) - \tilde{h}(w/r_j)|^p}{t^{k-1}} d\mathcal{H}^{k-1}(w) dt \\
& \lesssim r_j \int_{S(0,1)} \int_0^{\varepsilon/r_j} \frac{|\tilde{h} * \rho_t(\omega) - \tilde{h}(\omega)|^p}{t^{k-1}} dt d\mathcal{H}^{k-1}(\omega) \\
& \lesssim r_j \int_{S(0,1)} \int_0^{\varepsilon/r_j} \int_{\tilde{B}_t(\omega)} \frac{|\tilde{h}(\omega') - \tilde{h}(\omega)|^p}{t^{2k-2}} d\mathcal{H}^{k-1}(\omega') dt d\mathcal{H}^{k-1}(\omega) \\
& \lesssim \varepsilon \int_{S(0,1)} \int_0^{\varepsilon/r_j} \int_{\tilde{B}_t(\omega)} \frac{|\tilde{h}(\omega') - \tilde{h}(\omega)|^p}{t^{2k-1}} d\mathcal{H}^{k-1}(\omega') dt d\mathcal{H}^{k-1}(\omega) \\
& \lesssim \varepsilon |\tilde{h}|_{W^{s,p}(S(0,1))}^p \lesssim r^{\frac{1}{k-1}} |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))},
\end{aligned} \tag{3.111}$$

where the last line is the same as the one in (3.106).

For the last integral in (3.109), a calculation in the spirit of (3.106) implies that

$$\begin{aligned}
& \int_{B_{r_j-2\varepsilon}(0) \setminus B_{r_j-3\varepsilon}(0)} \int_{B_{r_j}(0) \setminus B_{r_j-\varepsilon}(0)} \frac{|\bar{h}_1(v) - \tilde{h}(v/|v|)|^p}{|u-v|^{2k-1}} dv du \\
& \lesssim r^{\frac{1}{k-1}} |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))}.
\end{aligned} \tag{3.112}$$

Combining (3.108), (3.109), (3.110), (3.111), and (3.112), we obtain

$$\begin{aligned} |\overline{H}|_{W^{s,p}(B_{r_j}(0) \setminus B_{r_j-3\varepsilon}(0))}^p &\lesssim |F|_{W^{s,p}(S(0,r_j) \times (0,\varepsilon))}^p + r^{\frac{1}{k-1}} |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))}^p \\ &\quad + |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{r_j}(0) \setminus B_{r_j-3\varepsilon}(0))}^p \end{aligned} \quad (3.113)$$

Therefore, (3.101), (3.107), and (3.113) imply (3.99).

Step 2.3. Estimate of F .

We will prove (3.100). By the definition of F as in (3.64), Lemma B.2, and the choice of r_j (see (3.36)), we have

$$\begin{aligned} |F|_{W^{s,p}(S(0,r_j) \times (0,\varepsilon))}^p &\lesssim r_j |\tilde{h} * \rho(\cdot)|_{W^{s,p}(S(0,1) \times (0,\varepsilon/r_j))}^p \lesssim \varepsilon |\tilde{h}|_{W^{s,p}(S(0,1))}^p \\ &\lesssim \varepsilon |f_{\ell-1} \circ \psi_i|_{W^{s,p}(S(0,r_j))}^p \lesssim r^{\frac{1}{k-1}} |f_{\ell-1}|_{W^{s,p}(B_{3K_2r}(x_j))}^p. \end{aligned} \quad (3.114)$$

We obtain (3.100) from (3.114).

Estimate (3.98) follows from (3.99) and (3.100).

In view of (3.75), we obtain the following estimate of $f_\ell - f_{\ell-1}$ in a good ball by combining Lemma 3.4, (3.97), and (3.98):

$$\begin{aligned} |f_\ell - f_{\ell-1}|_{W^{s,p}(\psi_i(B_{r_j}(0)))}^p &\lesssim \int_{A_j^1 \cup A_j^2} \int_{B_{3K_2r}(x_j)} \frac{|f_{\ell-1}(x) - f_{\ell-1}(y)|^p}{|x - y|^{2k-1}} d\mathcal{H}^k(x) d\mathcal{H}^k(y) \\ &\quad + r^{\frac{1}{k-1}} |f_{\ell-1}|_{W^{s,p}(B_{3K_2r}(x_j))}^p. \end{aligned} \quad (3.115)$$

Here, $A_j^1 := \psi_i(\overline{A_j}^1)$ and $A_j^2 := \psi_i(B_{r_j}(0) \setminus B_{r_j-3\varepsilon}(0))$ are the same as in the previous section.

Combining (3.96) and (3.115) with the Poincaré inequality and summing over j , we obtain (3.90) and (3.91).

We next prove (3.89). For this purpose, we first note that, as in (3.51), we have

$$\mathcal{H}^k(\overline{A_j}^1) \lesssim r^{k-1} |\overline{h}_1|_{W^{s,p}(B_{r_j}(0))}^p \quad (3.116)$$

and then, by (3.116) and (3.98), we have

$$\mathcal{H}^k(A_j^1) \lesssim r^{k-1} |f_{\ell-1}|_{W^{s,p}(B_{3K_2r}(x_j))}^p \quad (3.117)$$

and

$$\mathcal{H}^k(A_j^2) \lesssim |B_{r_j}(0) \setminus B_{r_j-3\varepsilon}(0)| \lesssim r_j^k - (r_j - 3\varepsilon)^k \lesssim r^{k+1/(k-1)}. \quad (3.118)$$

Summing over $j \in J_\ell$ in (3.117) and (3.118) (similarly to (3.85)), we derive (3.89).

In view of Section 3.3, in order to complete the proof of Theorem 3.2 when $sp = k - 1$ and $k \geq 3$, it suffices to repeat the induction procedure presented in Section 3.3 to obtain $f_{\theta+1}$, and then to approximate $f_{\theta+1}$ with maps in $\mathcal{R}_0$ by repeating Step 6 in Section 3.1. $\square$

3.5 Proof of Theorem 3.2 when $sp = 1$ and $k = 2$

The case where $sp = 1$ and $0 < s < 1$ requires a separate argument since, in contrast with the $W^{1,1}$ case, both trace theory and the Hardy inequality fail. The idea of constructing the approximation is essentially the same as in the case $k \geq 3$, with careful calculations and an adaptation from “traces” to “good restrictions”.

Proof of Theorem 3.2 when $sp = 1$ when $k = 2$. We use the same notation as in Section 3.2. We will construct inductively f_ℓ such that

- (i) there exists a set D_ℓ such that $\bigcup_{j \in J_\ell} B_{2r}(x_j) \subset D_\ell \subset \bigcup_{j \in J_\ell} B_{3K_2 r}(x_j)$,

$$f_\ell = f_{\ell-1} \text{ in } \mathcal{M} \setminus D_\ell, \text{ and } f_\ell \text{ is continuous in } \bigcup_{a \leq \ell} D_a, \\ \text{possibly except at the set } \left\{ x_j : j \in \bigcup_{a \leq \ell} J_a \right\};$$

- (ii) there exists a set $E_\ell \subset D_\ell$ such that

$$\mathcal{H}^2(E_\ell) \lesssim r^{1/2} |f_{\ell-1}|_{W^{s,p}(\mathcal{M})}^p + r^{1/4}, \quad (3.119)$$

and the following estimates hold:

$$\int_{\mathcal{M}} |f_\ell - f_{\ell-1}|^p d\mathcal{H}^2 \lesssim r |f_{\ell-1}|_{W^{s,p}(\mathcal{M})}^p, \quad (3.120)$$

$$|f_\ell - f_{\ell-1}|_{W^{s,p}(\mathcal{M})}^p \lesssim \int_{E_\ell} \int_{\mathcal{M}} \frac{|f_{\ell-1}(x) - f_{\ell-1}(y)|^p}{|x - y|^3} d\mathcal{H}^2(x) d\mathcal{H}^2(y) \\ + r^{1/4} |f_{\ell-1}|_{W^{s,p}(\mathcal{M})}^p. \quad (3.121)$$

(The exponent 3 in the middle line is the same as the familiar exponent $k + sp$ arising in the definition of $W^{s,p}$, since, in this section, $sp = k - 1 = 1$.)

Given $f_{\ell-1} \in W^{s,p}(\mathcal{M}; \mathcal{F})$, the construction of f_ℓ is almost the same as in the case where $k \geq 3$, with four slight modifications: (i) the definition of good balls and bad balls; (ii) the counterpart of Lemma 3.5; (iii) the counterpart of Lemma 3.4; and (iv) the choice of ε .

The definition of good balls and bad balls. For $j \in J_\ell$, we say that the ball $B_{3K_2r}(x_j)$ is a *good* ball provided that

$$\frac{1}{r^{3/2}} \int_{B_{3K_2r}(x_j)} \int_{\mathcal{M}} \frac{|f_{\ell-1}(x) - f_{\ell-1}(y)|^p}{|x - y|^3} d\mathcal{H}^2(y) d\mathcal{H}^2(x) \leq \lambda, \quad (3.122)$$

with λ to be chosen later. We say that the ball $B_{3K_2r}(x_j)$ is a *bad* ball if (3.122) fails.

The counterpart of Lemma 3.5. In a bad ball, to get the approximation $\bar{g}_2$ of the map $\bar{g}_1$ defined as in (3.61), instead of using Lemma 3.5, we use the following lemma.

Lemma 3.6. *Let $k = 2$, and assume that $0 < s < 1$ and $sp = 1$. Consider a map $g \in W^{s,p}(S(0, 1); \mathcal{F})$, and let $H(g)$ be the homogeneous extension of g . Then, for every $\alpha > 0$, there exists $\bar{g}: \overline{B_1(0)} \rightarrow \mathcal{F}$ such that: (i) $\bar{g} \in W^{s,p}(B_1(0); \mathcal{F}) \cap C(B_1(0) \setminus \{0\})$; (ii) if g is continuous at some point $\omega \in S(0, 1)$, then $\bar{g}$ is continuous up to ω and $\bar{g}(\omega) = g(\omega)$; (iii) we have*

$$\|H(g) - \bar{g}\|_{W^{s,p}(B_1(0))} \leq \alpha;$$

and (iv) we have

$$\int_{S(0,1)} \int_{B_1(0)} \frac{|H(g)(u) - \bar{g}(u)|^p}{|u - \omega|^3} du d\mathcal{H}^1(\omega) \leq \alpha. \quad (3.123)$$

For the proof of Lemma 3.6, see Appendix A.

The counterpart of Lemma 3.4. In a good ball, to obtain an approximation $\bar{h}_3$ of the map $\bar{h}_2$ defined as in (3.71), instead of using Lemma 3.4, we use Lemma 3.7 below.

Lemma 3.7. *Let $k \geq 2$. Let $0 < s < 1$ be such that $sp = 1$. Assume that $f: \overline{B_1(0)} \rightarrow \mathcal{F}$ satisfies: (i) $f \in W^{s,p}(B_1(0); B_\delta(\mathfrak{f}) \cap \mathcal{F})$ for some $\mathfrak{f} \in \mathcal{F}$; (ii) the restriction $f|_{S(0,1)}$ of f on $S(0, 1)$ is in $W^{s,p}(S(0, 1))$ and has a continuous representative; (iii) it holds*

$$\int_{S(0,1)} \int_{B_1(0)} \frac{|f(u) - f(\omega)|^p}{|u - \omega|^{k+1}} du d\mathcal{H}^{k-1}(\omega) < \infty. \quad (3.124)$$

Then, for every $\alpha > 0$, there exists $\bar{f} \in W^{s,p}(B_1(0); B_{4\delta}(\mathfrak{f}) \cap \mathcal{F}) \cap C(\overline{B_1(0)}; B_{4\delta}(\mathfrak{f}) \cap \mathcal{F})$ such that (i) $\bar{f}|_{S(0,1)} = f|_{S(0,1)}$; (ii) we have

$$\|f - \bar{f}\|_{W^{s,p}(B_1(0))} \leq \alpha;$$

and (iii) we have

$$\int_{S(0,1)} \int_{B_1(0)} \frac{|\bar{f}(u) - f(u)|^p}{|u - \omega|^{k+1}} du d\mathcal{H}^{k-1}(\omega) \leq \alpha. \quad (3.125)$$

For the proof of Lemma 3.7, see Appendix A.

Remark 3.8. Lemma 3.7, with $k + 1$ replaced with $k + sp$, still holds when $0 < s < 1$ and $sp > 1$. This is a direct consequence of Lemma 3.4. $\square$

The choice of ε . In this proof, we let $\varepsilon := r^{5/4}$.

By (3.93), (3.122) and (3.36), in a good ball, for $w_1, w_2 \in S(0, r_j)$, we have

$$|\tilde{h} * \rho_{\varepsilon/r_j}(w_1/r_j) - \tilde{h} * \rho_{\varepsilon/r_j}(w_2/r_j)| \lesssim \left(\frac{r^{3/2}}{\varepsilon} \lambda \right)^s.$$

Therefore, we can choose λ and r_1 to be sufficiently small universal constants so that, for $r < r_1$: (i) the map $\tilde{F} := \Pi \circ F$ is well-defined and $\tilde{F} \in W^{1,k-1}(S(0, r_j) \times (0, 2\varepsilon); \mathcal{F})$; and (ii) $\tilde{F}(\cdot, \varepsilon) \subset B_{\delta/2}(\mathfrak{f})$ for some $\mathfrak{f} \in \mathcal{F}$.

Step 1. A preliminary estimate for $\tilde{f} := f_\ell - f_{\ell-1}$.

We first note that, by repeating the calculations in (3.42) and (3.43), we have

$$\begin{aligned} |\tilde{f}|_{W^{s,p}(\mathcal{M})}^p &\lesssim \sum_{j \in J_\ell} \int_{\psi_i(B_{r_j}(0))} \int_{B_{r_0}(x_j) \setminus \psi_i(B_{r_j}(0))} \frac{|\tilde{f}(y)|^p}{|x - y|^3} d\mathcal{H}^2(x) d\mathcal{H}^2(y) \\ &\quad + \sum_{j \in J_\ell} \|\tilde{f}\|_{W^{s,p}(\psi_i(B_{r_j}(0)))}^p. \end{aligned} \quad (3.126)$$

Changing variables, we find that

$$\begin{aligned} &\int_{\psi_i(B_{r_j}(0))} \int_{B_{r_0}(x_j) \setminus \psi_i(B_{r_j}(0))} \frac{|\tilde{f}(y)|^p}{|x - y|^3} d\mathcal{H}^2(x) d\mathcal{H}^2(y) \\ &\lesssim \int_{B_{r_j}(0)} \int_{\mathbb{R}^k \setminus B_{r_j}(0)} \frac{|\tilde{f} \circ \psi_i(v)|^p}{|u - v|^3} du dv \lesssim \int_{B_{r_j}(0)} \frac{|\tilde{f} \circ \psi_i(v)|^p}{r_j - |v|} dv. \end{aligned} \quad (3.127)$$

On the other hand, we have

$$\int_{B_{r_j}(0)} |\tilde{f} \circ \psi_i(v)|^p dv \leq r \int_{B_{r_j}(0)} \frac{|\tilde{f} \circ \psi_i(v)|^p}{r_j - |v|} dv. \quad (3.128)$$

Combining (3.126), (3.127), and (3.128), we have

$$|\tilde{f}|_{W^{s,p}(\mathcal{M})}^p \lesssim \sum_{j \in \mathcal{I}_\ell} \left[|\tilde{f} \circ \psi_i|_{W^{s,p}(B_{r_j}(0))}^p + \int_{B_{r_j}(0)} \frac{|\tilde{f} \circ \psi_i(v)|^p}{r_j - |v|} dv \right] =: \mathcal{L}. \quad (3.129)$$

Step 2. Estimate of $\mathcal{L}$ in a bad ball $B_{3K_2r}(x_j)$.

We first estimate the first term on the right-hand side of (3.129). Let $\bar{g}_1$ be as in (3.61). In the spirit of (3.45) and (3.46), we have

$$|\bar{g}_1 - f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{r_j}(0))} \lesssim |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))}. \quad (3.130)$$

On the other hand, by Lemma 3.6, we have

$$|\bar{g}_2 - \bar{g}_1|_{W^{s,p}(B_{r_j}(0))} \leq |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))}. \quad (3.131)$$

Combining (3.130), (3.131), and the definition of f_ℓ , we have

$$\begin{aligned} |\tilde{f} \circ \psi_i|_{W^{s,p}(B_{r_j}(0))} &= |\bar{g}_2 - f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))} \\ &\leq |\bar{g}_2 - \bar{g}_1|_{W^{s,p}(B_{3r}(0))} + |\bar{g}_1 - f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))} \\ &\lesssim |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))}. \end{aligned} \quad (3.132)$$

We now turn to the last term in (3.129). Using (3.37), and the fact that $|w - v| \lesssim r_j - |v|$ provided that $w \in S(0, r_j) \cap B_{r_j-|v|}(r_j v/|v|)$, we have

$$\begin{aligned} &\int_{B_{r_j}(0)} \frac{|\bar{g}_1(v) - f_{\ell-1} \circ \psi_i(v)|^p}{r_j - |v|} dv \\ &\lesssim \int_{B_{r_j}(0)} \int_{S(0, r_j) \cap B_{r_j-|v|}(r_j v/|v|)} \frac{|\bar{g}_1(v) - f_{\ell-1} \circ \psi_i(v)|^p}{|w - v|} d\mathcal{H}^1(w) dv \\ &\lesssim r_j \int_{S(0, r_j)} \int_{B_{r_j}(0)} \frac{|\bar{g}_1(v) - f_{\ell-1} \circ \psi_i(v)|^p}{|w - v|^3} dv d\mathcal{H}^1(w) \\ &\lesssim r_j \int_{S(0, r_j)} \int_{B_{r_j}(0)} \frac{|f_{\ell-1} \circ \psi_i(r_j v/|v|) - f_{\ell-1} \circ \psi_i(w)|^p}{|w - v|^3} dv d\mathcal{H}^1(w) \\ &\quad + r_j \int_{S(0, r_j)} \int_{B_{r_j}(0)} \frac{|f_{\ell-1} \circ \psi_i(w) - f_{\ell-1} \circ \psi_i(v)|^p}{|w - v|^3} dv d\mathcal{H}^1(w). \end{aligned} \quad (3.133)$$

We estimate the first integral on the right-hand side of (3.133). Using [11, (4.14)], we

have

$$\begin{aligned}
& \int_{S(0,r_j)} \int_{B_{r_j}(0)} \frac{|f_{\ell-1} \circ \psi_i(r_j v/|v|) - f_{\ell-1} \circ \psi_i(w)|^p}{|w-v|^3} dv d\mathcal{H}^1(w) \\
& \lesssim \int_{S(0,r_j)} \int_{S(0,r_j)} \int_0^{r_j} \frac{t}{r_j} \frac{|f_{\ell-1} \circ \psi_i(\tilde{w}) - f_{\ell-1} \circ \psi_i(w)|^p}{|w - t\tilde{w}/r_j|^3} dt d\mathcal{H}^1(\tilde{w}) d\mathcal{H}^1(w) \\
& = \int_{S(0,1)} \int_{S(0,1)} \int_0^1 t \frac{|f_{\ell-1} \circ \psi_i(r_j \tilde{\omega}) - f_{\ell-1} \circ \psi_i(r_j \omega)|^p}{|\omega - t\tilde{\omega}|^3} dt d\mathcal{H}^1(\tilde{\omega}) d\mathcal{H}^1(\omega) \\
& \leq |f_{\ell-1} \circ \psi_i(r_j \cdot)|_{W^{s,p}(S(0,1))}^p = |f_{\ell-1} \circ \psi_i|_{W^{s,p}(S(0,r_j))}^p.
\end{aligned} \tag{3.134}$$

Combining (3.133) and (3.134) with the choice of r_j (see (3.36) and (3.37)), we have

$$\int_{B_{r_j}(0)} \frac{|\bar{g}_1(v) - f_{\ell-1} \circ \psi_i(v)|^p}{r_j - |v|} dv \lesssim |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))}^p. \tag{3.135}$$

Using (3.135) and the choice of $\bar{g}_2$ (Lemma 3.6), we obtain

$$\int_{B_{r_j}(0)} \frac{|\tilde{f} \circ \psi_i(v)|^p}{r_j - |v|} dv \lesssim |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))}^p.$$

Therefore, by (3.132), we have

$$|\tilde{f} \circ \psi_i|_{W^{s,p}(B_{r_j}(0))}^p + \int_{B_{r_j}(0)} \frac{|\tilde{f} \circ \psi_i(v)|^p}{r_j - |v|} dv \lesssim |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))}^p, \tag{3.136}$$

and, using (3.128), we conclude that

$$\|\tilde{f} \circ \psi_i\|_{\mathcal{L}^p(B_{r_j}(0))}^p \lesssim r |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))}^p.$$

Step 3. Estimate of $\mathcal{L}$ in a good ball $B_{3K_2r}(x_j)$.

Let $\bar{h}_1$ be as in (3.70), $\bar{h}_2$ be as in (3.71), and let $\bar{h}_3$ be an approximation of $\bar{h}_2$ obtained via Lemma 3.7.

In order to estimate $\mathcal{L}$, we decompose $\tilde{f} \circ \psi_i$ as

$$\tilde{f} \circ \psi_i = (\bar{h}_3 - \bar{h}_2) + (\bar{h}_2 - \bar{h}_1) + (\bar{h}_1 - f_{\ell-1} \circ \psi_i),$$

and estimate the contribution of each of the three above terms in $\mathcal{L}$.

Step 3.1. In this sub-step, we prove that

$$|\bar{h}_3 - \bar{h}_2|_{W^{s,p}(B_{r_j}(0))}^p + \int_{B_{r_j}(0)} \frac{|\bar{h}_3(v) - \bar{h}_2(v)|^p}{r_j - |v|} dv \lesssim |\bar{h}_2 - \bar{h}_1|_{W^{s,p}(B_{r_j-\varepsilon}(0))}^p. \quad (3.137)$$

By Lemma 3.7, we can choose the map $\bar{h}_3$ such that

$$|\bar{h}_3 - \bar{h}_2|_{W^{s,p}(B_{r_j-\varepsilon}(0))}^p \leq |\bar{h}_2 - \bar{h}_1|_{W^{s,p}(B_{r_j-\varepsilon}(0))}^p,$$

and

$$\int_{S(0,r_j-\varepsilon)} \int_{B_{r_j-\varepsilon}(0)} \frac{|\bar{h}_3(v) - \bar{h}_2(v)|^p}{|v-w|^3} dv d\mathcal{H}^{k-1}(w) \leq |\bar{h}_2 - \bar{h}_1|_{W^{s,p}(B_{r_j-\varepsilon}(0))}^p. \quad (3.138)$$

Therefore, using an inequality in the spirit of (3.104),

$$\begin{aligned} & |\bar{h}_3 - \bar{h}_2|_{W^{s,p}(B_{r_j}(0))}^p \\ & \leq |\bar{h}_3 - \bar{h}_2|_{W^{s,p}(B_{r_j-\varepsilon}(0))}^p + 2 \int_{B_{r_j}(0) \setminus B_{r_j-\varepsilon}(0)} \int_{B_{r_j-\varepsilon}(0)} \frac{|\bar{h}_3(v) - \bar{h}_2(v)|^p}{|v-u|^3} dv du \\ & \lesssim |\bar{h}_3 - \bar{h}_2|_{W^{s,p}(B_{r_j-\varepsilon}(0))}^p + \int_{B_{r_j}(0) \setminus B_{r_j-\varepsilon}(0)} \int_{B_{r_j-\varepsilon}(0)} \frac{|\bar{h}_3(v) - \bar{h}_2(v)|^p}{|v - (r_j - \varepsilon)u/|u||^3} dv du \quad (3.139) \\ & \lesssim |\bar{h}_3 - \bar{h}_2|_{W^{s,p}(B_{r_j-\varepsilon}(0))}^p + \int_{r_j-\varepsilon}^{r_j} \frac{t}{r_j - \varepsilon} |\bar{h}_2 - \bar{h}_1|_{W^{s,p}(B_{r_j-\varepsilon}(0))}^p dt \\ & \lesssim |\bar{h}_2 - \bar{h}_1|_{W^{s,p}(B_{r_j-\varepsilon}(0))}^p. \end{aligned}$$

Now, observe that, if $w \in S(0, r_j - \varepsilon) \cap B_{r_j-|v|}((r_j - \varepsilon)v/|v|)$, then $|w - v| \lesssim r_j - |v|$. Hence, relying on (3.138) and on the fact that $\varepsilon = r^{5/4}$, so that $r_j - \varepsilon \geq r_j/2$ whenever r is

sufficiently small, we estimate

$$\begin{aligned}
& \int_{B_{r_j}(0)} \frac{|\bar{h}_3(v) - \bar{h}_2(v)|^p}{r_j - |v|} dv \\
&= \int_{B_{r_j-\varepsilon}(0)} \frac{|\bar{h}_3(v) - \bar{h}_2(v)|^p}{r_j - |v|} dv \\
&\lesssim \int_{B_{r_j-\varepsilon}(0)} \int_{S(0, r_j-\varepsilon) \cap B_{r_j-|v|}((r_j-\varepsilon)v/|v|)} \frac{|\bar{h}_3(v) - \bar{h}_2(v)|^p}{|w - v|} d\mathcal{H}^1(w) dv \\
&\lesssim \int_{B_{r_j-\varepsilon}(0)} \int_{S(0, r_j-\varepsilon) \cap B_{r_j-|v|}((r_j-\varepsilon)v/|v|)} \frac{|\bar{h}_3(v) - \bar{h}_2(v)|^p}{|w - v|(r_j - |v|)} d\mathcal{H}^1(w) dv \\
&\lesssim r_j \int_{S(0, r_j-\varepsilon)} \int_{B_{r_j-\varepsilon}(0)} \frac{|\bar{h}_3(v) - \bar{h}_2(v)|^p}{|w - v|^3} dv d\mathcal{H}^1(w) \\
&\lesssim r_j |\bar{h}_2 - \bar{h}_1|_{W^{s,p}(B_{r_j-\varepsilon}(0))}^p.
\end{aligned} \tag{3.140}$$

Together, (3.139) and (3.140) imply (3.137).

Step 3.2. In this sub-step, we prove that

$$\begin{aligned}
& |\bar{h}_2 - \bar{h}_1|_{W^{s,p}(B_{r_j}(0))}^p + \int_{B_{r_j}(0)} \frac{|\bar{h}_2(v) - \bar{h}_1(v)|^p}{r_j - |v|} dv \\
&\lesssim \int_{\bar{A}_j^{-1}} \int_{B_{r_j}(0)} \frac{|f_{\ell-1} \circ \psi_i(u) - f_{\ell-1} \circ \psi_i(v)|^p}{|u - v|^3} du dv \\
&\quad + |\bar{h}_1 - f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{r_j}(0))}^p + r^{1/4} |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))}^p.
\end{aligned} \tag{3.141}$$

Step 3.2.1. We first prove that (with $\bar{A}_j^1$ as in (3.77))

$$\begin{aligned}
& |\bar{h}_2 - \bar{h}_1|_{W^{s,p}(B_{r_j}(0))}^p \\
&\lesssim \int_{\bar{A}_j^{-1}} \int_{B_{r_j}(0)} \frac{|f_{\ell-1} \circ \psi_i(u) - f_{\ell-1} \circ \psi_i(v)|^p}{|u - v|^3} du dv \\
&\quad + |\bar{h}_1 - f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{r_j}(0))}^p + r^{1/4} |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))}^p.
\end{aligned} \tag{3.142}$$

By the definition (3.71) of $\bar{h}_2$, we have

$$\begin{aligned} & |\bar{h}_2 - \bar{h}_1|_{W^{s,p}(B_{r_j}(0))}^p \\ & \leq |\bar{h}_2 - \bar{h}_1|_{W^{s,p}(B_{r_j-\varepsilon}(0))}^p + 2 \int_{B_{r_j-\varepsilon}(0)} \int_{B_{r_j}(0) \setminus B_{r_j-\varepsilon}(0)} \frac{|\bar{h}_2(v) - \bar{h}_1(v)|^p}{|u-v|^3} du dv. \end{aligned} \quad (3.143)$$

Repeating the calculation in (3.97), we find that, with $\bar{A}_j^1$ as in (3.77), we have

$$\begin{aligned} |\bar{h}_2 - \bar{h}_1|_{W^{s,p}(B_{r_j-\varepsilon}(0))}^p & \lesssim \int_{\bar{A}_j^1} \int_{B_{r_j}(0)} \frac{|f_{\ell-1} \circ \psi_i(u) - f_{\ell-1} \circ \psi_i(v)|^p}{|u-v|^3} du dv \\ & \quad + |\bar{h}_1 - f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{r_j}(0))}^p. \end{aligned} \quad (3.144)$$

We next estimate the crossed integral in (3.143). For this purpose, we first note that, since $\bar{h}_1(w) \in B_{\delta/2}(\mathfrak{f})$, $\forall w \in S(0, r_j - \varepsilon)$, we have $\Psi_{\mathfrak{f},\delta} \circ \bar{h}_1 = \bar{h}_1$ on $S(0, r_j - \varepsilon)$, and thus

$$\begin{aligned} & \int_{B_{r_j-\varepsilon}(0)} \int_{B_{r_j}(0) \setminus B_{r_j-\varepsilon}(0)} \frac{|\bar{h}_2(v) - \bar{h}_1(v)|^p}{|u-v|^3} du dv \\ & = \int_{B_{r_j-\varepsilon}(0)} \int_{B_{r_j}(0) \setminus B_{r_j-\varepsilon}(0)} \frac{|\Psi_{\mathfrak{f},\delta} \circ \bar{h}_1(v) - \bar{h}_1(v)|^p}{|u-v|^3} du dv \\ & \lesssim \int_{B_{r_j-\varepsilon}(0)} \int_{B_{r_j}(0) \setminus B_{r_j-\varepsilon}(0)} \frac{|\Psi_{\mathfrak{f},\delta} \circ \bar{h}_1(v) - \Psi_{\mathfrak{f},\delta} \circ \bar{h}_1((r_j - \varepsilon)u/|u|)|^p}{|u-v|^3} du dv \\ & \quad + \int_{B_{r_j-\varepsilon}(0)} \int_{B_{r_j}(0) \setminus B_{r_j-\varepsilon}(0)} \frac{|\bar{h}_1(v) - \bar{h}_1((r_j - \varepsilon)u/|u|)|^p}{|u-v|^3} du dv \\ & \lesssim \int_{B_{r_j-\varepsilon}(0)} \int_{B_{r_j}(0) \setminus B_{r_j-\varepsilon}(0)} \frac{|\bar{h}_1(v) - \bar{h}_1((r_j - \varepsilon)u/|u|)|^p}{|u-v|^3} du dv = \text{I} + \text{II}. \end{aligned} \quad (3.145)$$

Here, in order to estimate the last integral of (3.145), we decompose

$$B_{r_j-\varepsilon}(0) = (B_{r_j-\varepsilon}(0) \setminus B_{r_j-2\varepsilon}(0)) \bigcup B_{r_j-2\varepsilon}(0).$$

For $v \in B_{r_j-\varepsilon}(0) \setminus B_{r_j-2\varepsilon}(0)$, we have

$$\begin{aligned}
\mathbf{I} &:= \int_{B_{r_j-\varepsilon}(0) \setminus B_{r_j-2\varepsilon}(0)} \int_{B_{r_j}(0) \setminus B_{r_j-\varepsilon}(0)} \frac{|\bar{h}_1(v) - \bar{h}_1((r_j - \varepsilon)u/|u|)|^p}{|u - v|^3} du dv \\
&\lesssim \int_{\varepsilon}^{2\varepsilon} \int_{S(0, r_j)} \int_0^{\varepsilon} \int_{S(0, r_j)} \frac{|\tilde{h} * \rho_{(2\varepsilon-t)/r_j}(w_1/r_j) - \tilde{h} * \rho_{\varepsilon/r_j}(w_2/r_j)|^p}{[|w_2 - w_1| + (t - t')]^3} \\
&\quad \times d\mathcal{H}^1(w_2) dt' d\mathcal{H}^1(w_1) dt \\
&\lesssim \int_{\varepsilon}^{2\varepsilon} \int_{S(0, r_j)} \int_{S(0, r_j)} \frac{|\tilde{h} * \rho_{(2\varepsilon-t)/r_j}(w_1/r_j) - \tilde{h} * \rho_{\varepsilon/r_j}(w_2/r_j)|^p}{[|w_2 - w_1| + (t - \varepsilon)]^2} d\mathcal{H}^1(w_2) d\mathcal{H}^1(w_1) dt \\
&\lesssim r_j \int_0^{\varepsilon/r_j} \int_{S(0,1)} \int_{S(0,1)} \frac{|\tilde{h} * \rho_t(\omega_1) - \tilde{h} * \rho_{\varepsilon/r_j}(\omega_2)|^p}{[|\omega_2 - \omega_1| + (\varepsilon/r_j - t)]^2} d\mathcal{H}^1(\omega_2) d\mathcal{H}^1(\omega_1) dt.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\mathbf{I} &\lesssim r_j \int_0^{\varepsilon/r_j} \int_{S(0,1)} \int_{S(0,1)} \frac{|\tilde{h} * \rho_{\varepsilon/r_j}(\omega_1) - \tilde{h} * \rho_{\varepsilon/r_j}(\omega_2)|^p}{[|\omega_2 - \omega_1| + (\varepsilon/r_j - t)]^2} d\mathcal{H}^1(\omega_2) d\mathcal{H}^1(\omega_1) dt \\
&\quad + r_j \int_0^{\varepsilon/r_j} \int_{S(0,1)} \int_{S(0,1)} \frac{|\tilde{h} * \rho_t(\omega_1) - \tilde{h} * \rho_{\varepsilon/r_j}(\omega_1)|^p}{[|\omega_2 - \omega_1| + (\varepsilon/r_j - t)]^2} d\mathcal{H}^1(\omega_2) d\mathcal{H}^1(\omega_1) dt.
\end{aligned} \tag{3.146}$$

By Lemma B.1, we have

$$\begin{aligned}
&\int_0^{\varepsilon/r_j} \int_{S(0,1)} \int_{S(0,1)} \frac{|\tilde{h} * \rho_{\varepsilon/r_j}(\omega_1) - \tilde{h} * \rho_{\varepsilon/r_j}(\omega_2)|^p}{[|\omega_2 - \omega_1| + (\varepsilon/r_j - t)]^2} d\mathcal{H}^1(\omega_2) d\mathcal{H}^1(\omega_1) dt \\
&\lesssim \frac{\varepsilon}{r_j} \int_{S(0,1)} \int_{S(0,1)} \frac{|\tilde{h} * \rho_{\varepsilon/r_j}(\omega_1) - \tilde{h} * \rho_{\varepsilon/r_j}(\omega_2)|^p}{|\omega_2 - \omega_1|^2} d\mathcal{H}^1(\omega_2) d\mathcal{H}^1(\omega_1) \\
&\lesssim \frac{\varepsilon}{r_j} |\tilde{h}|_{W^{s,p}(S(0,1))}^p.
\end{aligned} \tag{3.147}$$

For the second integral on the right-hand side of (3.146) we find, similarly to the calcu-

lation in Lemma B.2, that

$$\begin{aligned}
& \int_0^{\varepsilon/r_j} \int_{S(0,1)} \int_{S(0,1)} \frac{|\tilde{h} * \rho_t(\omega_1) - \tilde{h} * \rho_{\varepsilon/r_j}(\omega_1)|^p}{[|\omega_2 - \omega_1| + (\varepsilon/r_j - t)]^2} d\mathcal{H}^1(\omega_2) d\mathcal{H}^1(\omega_1) dt \\
& \lesssim \int_0^{\varepsilon/r_j} \int_{S(0,1)} \frac{|\tilde{h} * \rho_t(\omega_1) - \tilde{h} * \rho_{\varepsilon/r_j}(\omega_1)|^p}{\varepsilon/r_j - t} d\mathcal{H}^1(\omega_1) dt \\
& \lesssim \int_0^{\varepsilon/r_j} \int_{S(0,1)} \int_{\tilde{B}_a(\omega_1)} \frac{|\tilde{h}(\omega) - \tilde{h}(\omega_1)|^p}{a^2} d\mathcal{H}^1(\omega) d\mathcal{H}^1(\omega_1) da \\
& \lesssim \int_0^{\varepsilon/r_j} \int_{S(0,1)} \int_{\tilde{B}_a(\omega)} \frac{|\tilde{h}(\omega) - \tilde{h}(\omega_1)|^p}{|\omega - \omega_1|^2} d\mathcal{H}^1(\omega) d\mathcal{H}^1(\omega_1) da \\
& = \frac{\varepsilon}{r_j} |\tilde{h}|_{W^{s,p}(S(0,1))}^p.
\end{aligned} \tag{3.148}$$

Combining (3.146) with (3.147), (3.148), and the choice of r_j in (3.36), we obtain

$$I \lesssim \varepsilon |\tilde{h}|_{W^{s,p}(S(0,1))}^p \lesssim \varepsilon |f_{\ell-1} \circ \psi_i|_{W^{s,p}(S(0,r_j))}^p \lesssim r^{1/4} |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))}^p. \tag{3.149}$$

To complete the estimate of the last integral of (3.145), we now consider $v \in B_{r_j-2\varepsilon}(0)$ and note that

$$\begin{aligned}
\Pi &:= \int_{B_{r_j-2\varepsilon}(0)} \int_{B_{r_j}(0) \setminus B_{r_j-\varepsilon}(0)} \frac{|\bar{h}_1(v) - \bar{h}_1((r_j - \varepsilon)u/|u|)|^p}{|u - v|^3} du dv \\
&\lesssim \int_{B_{r_j-2\varepsilon}(0)} \int_{B_{r_j}(0) \setminus B_{r_j-\varepsilon}(0)} \frac{|\bar{h}_1(v) - \bar{h}_1((r_j - \varepsilon)u/|u|)|^p}{|(r_j - \varepsilon)u/|u| - v|^3} du dv \\
&\lesssim \int_{r_j-\varepsilon}^{r_j} \int_{S(0,r_j-\varepsilon)} \int_{B_{r_j-2\varepsilon}(0)} \frac{|\bar{h}_1(v) - \bar{h}_1(w)|^p}{|w - v|^3} dv d\mathcal{H}^1(w) dt \\
&\lesssim \varepsilon \int_{S(0,r_j-\varepsilon)} \int_{B_{r_j-2\varepsilon}(0)} \frac{|\bar{h}_1(v) - \bar{h}_1(w)|^p}{|w - v|^3} dv d\mathcal{H}^1(w),
\end{aligned}$$

which implies that

$$\begin{aligned}
\Pi &\lesssim \varepsilon \int_{S(0,r_j-\varepsilon)} \int_{B_{r_j-2\varepsilon}(0)} \frac{|\bar{h}_1(v) - \bar{h}_1(r_j w/(r_j - \varepsilon))|^p}{|\omega - v|^3} dv d\mathcal{H}^1(w) \\
&\quad + \varepsilon \int_{S(0,r_j-\varepsilon)} \int_{B_{r_j-2\varepsilon}(0)} \frac{|\bar{h}_1(w) - \bar{h}_1(r_j w/(r_j - \varepsilon))|^p}{|w - v|^3} dv d\mathcal{H}^1(w).
\end{aligned} \tag{3.150}$$

For the first integral on the right-hand side of (3.150), using the definition (3.70) of $\bar{h}_1$

and the choice of r_j in (3.37), we have

$$\begin{aligned}
& \int_{S(0, r_j - \varepsilon)} \int_{B_{r_j - 2\varepsilon}(0)} \frac{|\bar{h}_1(v) - \bar{h}_1(r_j w / (r_j - \varepsilon))|^p}{|w - v|^3} dv d\mathcal{H}^1(w) \\
& \lesssim \int_{S(0, r_j)} \int_{B_{r_j - 2\varepsilon}(0)} \frac{|\bar{h}_1(v) - \bar{h}_1(w')|^p}{|w' - v|^3} dv d\mathcal{H}^1(w') \\
& \leq \int_{S(0, r_j)} \int_{B_{r_j}(0)} \frac{|f_{\ell-1} \circ \psi_i(u) - f_{\ell-1} \circ \psi_i(w')|^p}{|w' - u|^3} du d\mathcal{H}^1(w') \\
& \lesssim \frac{1}{r_j} |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))}^p.
\end{aligned} \tag{3.151}$$

We next consider the last integral in (3.150). Using again the definitions (3.70) of $\bar{h}_1$ and (3.63) of $\tilde{h}$, we have

$$\begin{aligned}
& \int_{S(0, r_j - \varepsilon)} \int_{B_{r_j - 2\varepsilon}(0)} \frac{|\bar{h}_1(r_j w / (r_j - \varepsilon)) - \bar{h}_1(w)|^p}{|w - v|^3} dv d\mathcal{H}^1(w) \\
& \lesssim \frac{1}{\varepsilon} \int_{S(0, r_j - \varepsilon)} |\bar{h}_1(r_j w / (r_j - \varepsilon)) - \bar{h}_1(w)|^p d\mathcal{H}^1(w) \\
& \lesssim \frac{1}{\varepsilon} \int_{S(0, r_j)} |\tilde{h}(w' / r_j) - \tilde{h} * \rho_{\varepsilon/r_j}(w' / r_j)|^p d\mathcal{H}^1(w') \\
& \lesssim \frac{r_j}{\varepsilon} \int_{S(0, 1)} |\tilde{h}(\omega) - \tilde{h} * \rho_{\varepsilon/r_j}(\omega)|^p d\mathcal{H}^1(\omega) \\
& \lesssim \left(\frac{r_j}{\varepsilon}\right)^2 \int_{S(0, 1)} \int_{\tilde{B}_{\varepsilon/r_j}(\omega)} |\tilde{h}(\omega) - \tilde{h}(\omega')|^p d\mathcal{H}^1(\omega') d\mathcal{H}^1(\omega) \\
& \lesssim \int_{S(0, 1)} \int_{\tilde{B}_{\varepsilon/r_j}(\omega)} \frac{|\tilde{h}(\omega) - \tilde{h}(\omega')|^p}{|\omega' - \omega|^2} d\mathcal{H}^1(\omega') d\mathcal{H}^1(\omega) \lesssim |\tilde{h}|_{W^{s,p}(S(0, 1))}^p.
\end{aligned} \tag{3.152}$$

Combining (3.150) with (3.151), (3.152), and the choice of r_j in (3.36), we obtain

$$\Pi \lesssim r^{1/4} |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))}^p. \tag{3.153}$$

Consequently, (3.145), (3.149), and (3.153) imply that

$$\int_{B_{r_j - \varepsilon}(0)} \int_{B_{r_j}(0) \setminus B_{r_j - \varepsilon}(0)} \frac{|\bar{h}_2(v) - \bar{h}_1(v)|^p}{|u - v|^{2k-1}} du dv \lesssim r^{1/4} |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))}^p. \tag{3.154}$$

Therefore, using (3.143), (3.144), and (3.154), we find (3.142).

Step 3.2.2. Estimate of $\bar{A}_j^1$.

Recall that $\bar{A}_j^1$ is as in (3.77). Since Φ we use for (3.25) is Lipschitz, we have

$$\begin{aligned}\Phi(|\bar{h}_1(v) - \mathfrak{f}|) &= \Phi(|\bar{h}_1(v) - \mathfrak{f}|) - \Phi(|\bar{h}_1((r_j - \varepsilon)u/|u|) - \mathfrak{f}|) \\ &\lesssim |\bar{h}_1(v) - \bar{h}_1((r_j - \varepsilon)u/|u|)|, \quad \forall u \in \mathbb{R}^2.\end{aligned}$$

Averaging this estimate over $B_{r_j}(0) \setminus B_{r_j-\varepsilon}(0)$ with respect to u and using the estimate obtained for the last term in (3.145) (coming from (3.149) and (3.153)), we have

$$\begin{aligned}\mathcal{H}^2(\bar{A}_j^1) &\leq \int_{B_{r_j-\varepsilon}(0)} |\Phi(|\bar{h}_1(v) - \mathfrak{f}|)|^p \, dv \\ &\lesssim \int_{B_{r_j-\varepsilon}(0)} \int_{B_{r_j}(0) \setminus B_{r_j-\varepsilon}(0)} |\bar{h}_1(v) - \bar{h}_1((r_j - \varepsilon)u/|u|)|^p \, du \, dv \\ &\lesssim \frac{(r_j)^2}{\varepsilon} \int_{B_{r_j-\varepsilon}(0)} \int_{B_{r_j}(0) \setminus B_{r_j-\varepsilon}(0)} \frac{|\bar{h}_1(v) - \bar{h}_1((r_j - \varepsilon)u/|u|)|^p}{|u - v|^3} \, du \, dv \\ &\lesssim r |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))}^p.\end{aligned}\tag{3.155}$$

Step 3.2.3. We prove that

$$\begin{aligned}\int_{B_{r_j}(0)} \frac{|\bar{h}_2(v) - \bar{h}_1(v)|^p}{r_j - |v|} \, dv &\lesssim \int_{\bar{A}_j^1} \int_{B_{r_j}(0)} \frac{|f_{\ell-1} \circ \psi_i(u) - f_{\ell-1} \circ \psi_i(v)|^p}{|u - v|^3} \, du \, dv \\ &\quad + |\bar{h}_1 - f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{r_j}(0))}^p.\end{aligned}\tag{3.156}$$

For this purpose, we define

$$\bar{A}_j^1 := \{u \in B_{r_j}(0) : \bar{h}_1(u) \notin B_\delta(\mathfrak{f})\}.$$

By the definition (3.70) of $\bar{h}_1$, (3.155), and (3.122), we have

$$\mathcal{H}^2(\bar{A}_j^1) \lesssim \mathcal{H}^2(\bar{A}_j^1) \lesssim r |f_\ell \circ \psi_i|_{W^{s,p}(B_{3r}(0))}^p \lesssim r^{5/2} \lambda.\tag{3.157}$$

We first note that, for $u \in B_{r_j-\varepsilon}(0) \setminus \bar{A}_j^1$, $\Psi_{\mathfrak{f},\delta} \circ \bar{h}_1(u) = \bar{h}_1(u)$.

Using this and the fact that $\bar{A}_j^1 \subset B_{r_j-\varepsilon}(0)$, we obtain

$$\begin{aligned}
& \int_{B_{r_j}(0)} \frac{|\bar{h}_2(v) - \bar{h}_1(v)|^p}{r_j - |v|} dv \\
&= \int_{\bar{A}_j^1} \frac{|\Psi_{\bar{f},\delta} \circ \bar{h}_1(v) - \bar{h}_1(v)|^p}{r_j - |v|} dv \\
&= \int_{\bar{A}_j^1} \int_{B_\varepsilon(v) \setminus \bar{A}_j^1} \frac{|\Psi_{\bar{f},\delta} \circ \bar{h}_1(v) - \bar{h}_1(v)|^p}{r_j - |v|} du dv \\
&\lesssim \int_{\bar{A}_j^1} \int_{B_\varepsilon(v) \setminus \bar{A}_j^1} \frac{|\bar{h}_1(v) - \bar{h}_1(u)|^p}{|u - v|} du dv \\
&\lesssim \int_{\bar{A}_j^1} \int_{B_\varepsilon(v) \setminus \bar{A}_j^1} \frac{|\bar{h}_1(v) - \bar{h}_1(u)|^p}{|u - v|^3} \frac{\mathcal{H}^2(B_\varepsilon(v))}{\mathcal{H}^2(B_\varepsilon(v)) - \mathcal{H}^2(\bar{A}_j^1)} du dv.
\end{aligned} \tag{3.158}$$

By (3.157), we have

$$\frac{\mathcal{H}^2(\bar{A}_j^1)}{\mathcal{H}^2(B_\varepsilon(v))} \lesssim \frac{r^{5/2}\lambda}{\varepsilon^2} = \lambda.$$

Therefore, we can choose λ sufficiently small such that, by (3.158),

$$\begin{aligned}
& \int_{B_{r_j}(0)} \frac{|\bar{h}_2(v) - \bar{h}_1(v)|^p}{r_j - |v|} dv \lesssim \int_{\bar{A}_j^1} \int_{B_{r_j}(0)} \frac{|\bar{h}_1(v) - \bar{h}_1(u)|^p}{|u - v|^3} du dv \\
&\lesssim \int_{\bar{A}_j^1} \int_{B_{r_j}(0)} \frac{|f_{\ell-1} \circ \psi_i(u) - f_{\ell-1} \circ \psi_i(v)|^p}{|u - v|^3} du dv \\
&\quad + |\bar{h}_1 - f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{r_j}(0))}^p,
\end{aligned}$$

which implies (3.156).

Combining (3.142) with (3.156), we obtain (3.141).

Step 3.3. In this sub-step, we prove that

$$\begin{aligned}
& |\bar{h}_1 - f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{r_j}(0))}^p + \int_{B_{r_j}(0)} \frac{|\bar{h}_1(v) - f_{\ell-1} \circ \psi_i(v)|^p}{r_j - |v|} dv \\
&\lesssim \int_{B_{r_j}(0) \setminus B_{r_j-3\varepsilon}(0)} \int_{B_{3r}(0)} \frac{|f_{\ell-1} \circ \psi_i(v) - f_{\ell-1} \circ \psi_i(u)|^p}{|u - v|^3} du dv \\
&\quad + r^{1/4} |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))}^p.
\end{aligned} \tag{3.159}$$

Let $G := \bar{h}_1 - f_{\ell-1} \circ \psi_i$. Proceeding exactly as in the proof of (3.98), with $\varepsilon = r^{k/(k-1)}$ replaced with $\varepsilon = r^{5/4}$, we find that

$$\begin{aligned} |G|_{W^{s,p}(B_{r_j}(0))}^p &\lesssim \int_{B_{r_j}(0) \setminus B_{r_j-3\varepsilon}(0)} \int_{B_{3r}(0)} \frac{|f_{\ell-1} \circ \psi_i(u) - f_{\ell-1} \circ \psi_i(v)|^p}{|u-v|^3} dv du \\ &\quad + r^{1/4} |f_{\ell-1}|_{W^{s,p}(B_{3K_2r}(x_j))}^p. \end{aligned} \quad (3.160)$$

On the other hand, we note that,

$$\begin{aligned} &\int_{B_{r_j}(0)} \frac{|G(v)|^p}{r_j - |v|} dv \\ &= \int_{B_{r_j}(0) \setminus B_{r_j-3\varepsilon}(0)} \frac{|G(v)|^p}{r_j - |v|} dv \\ &\lesssim \int_{B_{r_j}(0) \setminus B_{r_j-3\varepsilon}(0)} \int_{(B_{3r_j}(0) \setminus B_{r_j}(0)) \cap B_{r_j-|v|}(r_j v/|v|)} \frac{|G(v)|^p}{|u-v|} du dv \\ &\lesssim \int_{B_{3r}(0) \setminus B_{r_j}(0)} \int_{B_{r_j}(0) \setminus B_{r_j-3\varepsilon}(0)} \frac{|f_{\ell-1} \circ \psi_i(v) - f_{\ell-1} \circ \psi_i(u)|^p}{|v-u|^3} dv du \\ &\quad + \int_{B_{3r}(0) \setminus B_{r_j}(0)} \int_{B_{r_j}(0) \setminus B_{r_j-3\varepsilon}(0)} \frac{|\bar{h}_1(v) - f_{\ell-1} \circ \psi_i(u)|^p}{|v-u|^3} dv du. \end{aligned} \quad (3.161)$$

To estimate the last integral of (3.161), we decompose

$$B_{r_j}(0) \setminus B_{r_j-3\varepsilon}(0) = (B_{r_j-2\varepsilon}(0) \setminus B_{r_j-3\varepsilon}(0)) \bigcup (B_{r_j}(0) \setminus B_{r_j-2\varepsilon}(0)).$$

For $v \in B_{r_j-2\varepsilon}(0) \setminus B_{r_j-3\varepsilon}(0)$, we have

$$\begin{aligned} &\int_{B_{3r}(0) \setminus B_{r_j}(0)} \int_{B_{r_j-2\varepsilon}(0) \setminus B_{r_j-3\varepsilon}(0)} \frac{|\bar{h}_1(v) - f_{\ell-1} \circ \psi_i(u)|^p}{|u-v|^3} dv du \\ &\lesssim \int_{B_{3r}(0) \setminus B_{r_j}(0)} \int_{B_{r_j-2\varepsilon}(0) \setminus B_{r_j-3\varepsilon}(0)} \frac{|\bar{h}_1(v) - f_{\ell-1} \circ \psi_i(u)|^p}{|3v - 2(r_j - 3\varepsilon)v/|v| - u|^3} dv du \\ &\lesssim \int_{B_{3r}(0) \setminus B_{r_j}(0)} \int_{B_{r_j}(0) \setminus B_{r_j-3\varepsilon}(0)} \frac{|f_{\ell-1} \circ \psi_i(v') - f_{\ell-1} \circ \psi_i(u)|^p}{|v' - u|^3} dv' du. \end{aligned} \quad (3.162)$$

Second, by the definition (3.70) of $\bar{h}_1$, we have

$$\begin{aligned}
& \int_{B_{3r}(0) \setminus B_{r_j}(0)} \int_{B_{r_j}(0) \setminus B_{r_j-2\varepsilon}(0)} \frac{|\bar{h}_1(v) - f_{\ell-1} \circ \psi_i(u)|^p}{|u-v|^3} dv du \\
& \lesssim \int_{B_{3r}(0) \setminus B_{r_j}(0)} \int_{B_{r_j}(0) \setminus B_{r_j-\varepsilon}(0)} \frac{|\bar{h}_1(v) - f_{\ell-1} \circ \psi_i(u)|^p}{|u-v|^3} dv du \\
& \lesssim \int_{B_{3r}(0) \setminus B_{r_j}(0)} \int_{B_{r_j}(0) \setminus B_{r_j-\varepsilon}(0)} \frac{|\bar{h}_1(v) - f_{\ell-1} \circ \psi_i(r_j v/|v|)|^p}{|u-v|^3} dv du \\
& \quad + \int_{B_{3r}(0) \setminus B_{r_j}(0)} \int_{B_{r_j}(0) \setminus B_{r_j-\varepsilon}(0)} \frac{|f_{\ell-1} \circ \psi_i(u) - f_{\ell-1} \circ \psi_i(r_j v/|v|)|^p}{|u-v|^3} dv du.
\end{aligned} \tag{3.163}$$

We consider the first of the last two integrals in (3.163). By the definition of $\bar{h}_1$, the choice of r_j in (3.36), and a calculation in spirit of (3.106), we obtain

$$\begin{aligned}
& \int_{B_{3r}(0) \setminus B_{r_j}(0)} \int_{B_{r_j}(0) \setminus B_{r_j-\varepsilon}(0)} \frac{|\bar{h}_1(v) - f_{\ell-1} \circ \psi_i(r_j v/|v|)|^p}{|u-v|^3} dv du \\
& \lesssim \int_{B_{r_j}(0) \setminus B_{r_j-\varepsilon}(0)} \frac{|\bar{h}_1(v) - f_{\ell-1} \circ \psi_i(r_j v/|v|)|^p}{\text{dist}(v, S(0, r_j))} dv \\
& \lesssim \int_0^\varepsilon \int_{S(0, r_j)} \frac{|\tilde{h} * \rho_{t/r_j}(w/r_j) - \tilde{h}(w/r_j)|^p}{t} d\mathcal{H}^1(w) dt \\
& \lesssim r_j \int_0^{\varepsilon/r_j} \int_{S(0,1)} \frac{|\tilde{h} * \rho_t(\omega) - \tilde{h}(\omega)|^p}{t} d\mathcal{H}^1(\omega) dt \\
& \lesssim \varepsilon \int_0^{\varepsilon/r_j} \int_{S(0,1)} \frac{|\tilde{h} * \rho_t(\omega) - \tilde{h}(\omega)|^p}{t^2} d\mathcal{H}^1(\omega) dt \\
& \lesssim \varepsilon |\tilde{h}|_{W^{s,p}(S(0,1))}^p = \varepsilon |f_{\ell-1} \circ \psi_i|_{W^{s,p}(S(0, r_j))}^p \lesssim r^{1/4} |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))}^p.
\end{aligned} \tag{3.164}$$

As for the second of the last two integrals in (3.163), by the choice of r_j in (3.36), we

have

$$\begin{aligned}
& \int_{B_{3r}(0) \setminus B_{r_j}(0)} \int_{B_{r_j}(0) \setminus B_{r_j-\varepsilon}(0)} \frac{|f_{\ell-1} \circ \psi_i(u) - f_{\ell-1} \circ \psi_i(r_j v / |v|)|^p}{|u - v|^3} dv du \\
& \lesssim \int_{B_{3r}(0) \setminus B_{r_j}(0)} \int_{B_{r_j}(0) \setminus B_{r_j-\varepsilon}(0)} \frac{|f_{\ell-1} \circ \psi_i(u) - f_{\ell-1} \circ \psi_i(r_j v / |v|)|^p}{|u - r_j v / |v||^3} dv du \\
& \lesssim \int_{B_{3r}(0) \setminus B_{r_j}(0)} \int_{S(0, r_j)} \int_{r_j-\varepsilon}^{r_j} \frac{t}{r_j} \frac{|f_{\ell-1} \circ \psi_i(u) - f_{\ell-1} \circ \psi_i(w)|^p}{|u - w|^3} dt d\mathcal{H}^1(w) du \quad (3.165) \\
& \lesssim \varepsilon \int_{B_{3r}(0)} \int_{S(0, r_j)} \frac{|f_{\ell-1} \circ \psi_i(u) - f_{\ell-1} \circ \psi_i(w)|^p}{|u - w|^3} d\mathcal{H}^1(w) du \\
& \lesssim r^{1/4} |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))}^p.
\end{aligned}$$

Combining, (3.161), (3.162), (3.163), (3.164), and (3.165), we obtain

$$\begin{aligned}
& \int_{B_{r_j}(0)} \frac{|G(v)|^p}{r_j - |v|} dv \\
& \lesssim \int_{B_{r_j}(0) \setminus B_{r_j-3\varepsilon}(0)} \int_{B_{3r}(0)} \frac{|f_{\ell-1} \circ \psi_i(v) - f_{\ell-1} \circ \psi_i(u)|^p}{|u - v|^3} du dv \quad (3.166) \\
& \quad + r^{1/4} |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))}^p.
\end{aligned}$$

Therefore, by (3.160) and (3.166), we obtain (3.159).

To conclude Step 3, we combine (3.137), (3.141), and (3.159), and find that

$$\begin{aligned}
& |\tilde{f} \circ \psi_i|_{W^{s,p}(B_{r_j}(0))}^p + \int_{B_{r_j}(0)} \frac{|\tilde{f} \circ \psi_i(v)|^p}{r_j - |v|} dv \\
& \lesssim \int_{\bar{A}_j^{-1} \cup \bar{A}_j^{-2}} \int_{B_{3r}(0)} \frac{|f_{\ell-1} \circ \psi_i(v) - f_{\ell-1} \circ \psi_i(u)|^p}{|u - v|^3} du dv \quad (3.167) \\
& \quad + r^{1/4} |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))}^p.
\end{aligned}$$

In turn, (3.128) and (3.167) yield

$$\begin{aligned}
\|\tilde{f} \circ \psi_i\|_{\mathcal{L}^p(B_{r_j}(0))}^p & \lesssim r \int_{\bar{A}_j^{-1} \cup \bar{A}_j^{-2}} \int_{B_{3r}(0)} \frac{|f_{\ell-1} \circ \psi_i(v) - f_{\ell-1} \circ \psi_i(u)|^p}{|u - v|^3} du dv \quad (3.168) \\
& \quad + r^{5/4} |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))}^p.
\end{aligned}$$

Summing up over j , by (3.129), (3.136), and (3.167), we obtain

$$|\tilde{f}|_{W^{s,p}(\mathcal{M})}^p \lesssim \int_{E_\ell} \int_{\mathcal{M}} \frac{|f_{\ell-1}(x) - f_{\ell-1}(y)|^p}{|x - y|^3} d\mathcal{H}^2(x) d\mathcal{H}^2(y) + r^{1/4} |f_{\ell-1}|_{W^{s,p}(\mathcal{M})}^p$$

(where E_ℓ is as in (3.81)), whence (3.121).

On the other hand, by (3.168),

$$\begin{aligned} \|\tilde{f}\|_{\mathcal{L}^p(\mathcal{M})}^p &\lesssim r \int_{E_\ell} \int_{\mathcal{M}} \frac{|f_{\ell-1}(x) - f_{\ell-1}(y)|^p}{|x - y|^3} d\mathcal{H}^2(x) d\mathcal{H}^2(y) + r^{5/4} |f_{\ell-1} \circ \psi_i|_{W^{s,p}(B_{3r}(0))}^p \\ &\lesssim r |f_{\ell-1}|_{W^{s,p}(\mathcal{M})}^p, \end{aligned}$$

whence (3.120).

We next prove (3.119). In the spirit of (3.85) and using (3.155), we have

$$\begin{aligned} \mathcal{H}^2(E_\ell) &\lesssim \sum_{\text{good balls}} r |f_{\ell-1}|_{W^{s,p}(B_{3K_2r}(x_j))}^p + \frac{r_j^2 - (r_j - \varepsilon)^2}{r^2} + \frac{|f_{\ell-1}|_{W^{s,p}(\mathcal{M})}^p}{\lambda r^{3/2}} r^2 \\ &\lesssim r^{1/2} |f_{\ell-1}|_{W^{s,p}(\mathcal{M})}^p + r^{1/4}. \end{aligned}$$

To complete the proof of Theorem 3.2 when $sp = 1$ and $k = 2$, it suffices to repeat the induction procedure presented in Section 3.3 to obtain $f_{\theta+1}$, and then to approximate $f_{\theta+1}$ with maps in $\mathcal{R}_0$ by repeating Step 6 in Section 3.1. $\square$

4 Defining the pullback *via* lifting

In this section, we move to our main goal, namely, taking profit of a lifting procedure to extend the range where a robust definition can be given to the pullback $f^*\omega$ in the sense of distributions. This makes use in a crucial way of the techniques we developed in the previous sections on liftings as well as on strong approximation of Sobolev maps with almost smooth maps.

4.1 A natural extension of $d[\varphi^*\gamma]$

In this section, $\mathcal{M}$ is a simply-connected, compact k -dimensional Lipschitz manifold, and $\mathcal{F}$ is a complete Riemannian manifold, embedded into $\mathbb{R}^d$ *via* a tubed embedding. The main result of this section explains how to give a robust meaning to the pullback of a $(k-1)$ -form by a $W^{1,k-1}(\mathcal{M}; \mathcal{F})$ map. Later on, we will impose more stringent restrictions on our target and assume it to be a covering space $\mathcal{E}$, but as far as we are

concerned with pullbacks by first order Sobolev maps, we can work in this very general setting.

More specifically, the main result of this section is the following.

Theorem 4.1. *Assume that $k \geq 2$. Let γ be a smooth $(k-1)$ -form with bounded coefficients on $\mathcal{F}$. Then, the map*

$$W^{1,k-1}(\mathcal{M}; \mathcal{F}) \ni \varphi \mapsto T\varphi = T_\gamma \varphi,$$

where

$$\langle T\varphi, \xi \rangle := \int_{\mathcal{M}} \varphi^* \gamma \wedge d\xi \quad \text{for every } \xi \in \text{Lip}(\mathcal{M}),$$

can be extended by continuity to $W^{1-1/k,k}(\mathcal{M}; \mathcal{F})$.

The extension, still denoted $\varphi \mapsto T\varphi$, satisfies

$$|\langle T\varphi, \xi \rangle| \leq C |\varphi|_{W^{1-1/k,k}}^k |\xi|_{\text{Lip}}, \quad (4.1)$$

for C a finite constant independent of f and ξ .

Proof of Theorem 4.1. Let us start with a formal calculation. Let $\Phi: \mathcal{M} \times (0, \infty) \rightarrow \mathbb{R}^d$ be an extension of φ to be defined later. Set $\tilde{\xi}(x, \varepsilon) := \xi(x)$ and let $\tilde{\gamma}$ be an extension of γ to $\mathbb{R}^d$. Formally, we have

$$\int_{\mathcal{M}} \varphi^* \gamma \wedge d\xi = - \int_{\mathcal{M} \times (0, \infty)} d[\Phi^* \tilde{\gamma} \wedge d\tilde{\xi}] = - \int_{\mathcal{M} \times (0, \infty)} \Phi^*(d\tilde{\gamma}) \wedge d\tilde{\xi}. \quad (4.2)$$

We next justify (4.2) for an appropriate choice of Φ . The argument is quite similar to the proof of Theorem 5.7 in [19].

Step 1. Definition of an appropriate extension operator $\varphi \mapsto \Phi$.

Consider $\{(U_i, V_i, \psi_i)\}$ as in Section 2.1. Let $\varepsilon_1 = \varepsilon_1(\mathcal{M}) > 0$ be such that the open sets

$$U'_i := \{x \in U_i : B_{\varepsilon_1}((\psi_i)^{-1}(x)) \subset V_i\} = \psi_i(\{v \in V_i : \text{dist}(v, (V_i)^c) > \varepsilon_1\})$$

cover $\mathcal{M}$.

Consider a Lipschitz partition of unity $(\xi_i)_{i \in I}$ subordinated to the cover $(U'_i)_{i \in I}$ of $\mathcal{M}$. Let $\rho \in C_c^\infty(B_1(0))$ be a mollifier. For $\varphi \in \mathcal{L}^1(\mathcal{M}; \mathbb{R}^d)$ and $v \in V_i$, set $\bar{\varphi}_i(v) := \varphi \circ \psi_i(v)$. For $0 < \varepsilon < \varepsilon_1$, set

$$\Phi(x, \varepsilon) := \bar{\varphi}_\varepsilon(x) = \sum_i \xi_i(x) [(\bar{\varphi}_i * \rho_\varepsilon)((\psi_i)^{-1}(x))].$$

For $x \in \mathcal{M}$ and the other non-negative values of ε , we set

$$\Phi(x, \varepsilon) := \begin{cases} \varphi(x), & \text{if } \varepsilon = 0 \\ \oint_{\mathcal{M}} \varphi, & \text{if } \varepsilon \geq 2\varepsilon_1 \\ \left(2 - \frac{\varepsilon}{\varepsilon_1}\right)\Phi(x, \varepsilon_1) + \left(\frac{\varepsilon}{\varepsilon_1} - 1\right) \oint_{\mathcal{M}} \varphi, & \text{if } \varepsilon_1 < \varepsilon < 2\varepsilon_1 \end{cases}.$$

The linear operator $\mathcal{L}^1(\mathcal{M}; \mathbb{R}^d) \ni \varphi \mapsto \Phi$ has the following properties (with finite constants independent of φ).

(1) We have $\Phi \in \text{Lip}(\mathcal{M} \times [a, b])$ for any $0 < a < b < \infty$.

(2) For $\varepsilon_1 \leq \varepsilon \leq 2\varepsilon_1$, we have

$$\begin{aligned} \partial_\varepsilon \Phi(x, \varepsilon) &= \frac{1}{\varepsilon_1} \left(\oint_{\mathcal{M}} \varphi - \Phi(x, \varepsilon_1) \right) \\ &= -\frac{1}{\varepsilon_1} \sum_i \xi_i(x) \left[\left(\varphi - \oint_{\mathcal{M}} \varphi \right) \circ \psi_i \right] * \rho_{\varepsilon_1}((\psi_i)^{-1}(x)) \end{aligned}$$

and therefore

$$\begin{aligned} |\partial_\varepsilon \Phi(x, \varepsilon)| &\lesssim \int_{\mathcal{M}} \int_{\mathcal{M}} |\varphi(y) - \varphi(z)| d\mathcal{H}^k(y) d\mathcal{H}^k(z), \\ &\quad \forall x \in \mathcal{M}, \forall \varepsilon_1 \leq \varepsilon \leq 2\varepsilon_1. \end{aligned} \tag{4.3}$$

(3) For $0 < \varepsilon \leq \varepsilon_1$ and $\mathcal{H}^k$ -a.e. $x \in \mathcal{M}$, we have

$$|\nabla \Phi(x, \varepsilon)| \lesssim \frac{1}{\varepsilon} \oint_{B_\varepsilon(x)} |\varphi(y) - \varphi(x)| d\mathcal{H}^k(y). \tag{4.4}$$

For the above properties, see [19, Lemma 5.8].

Step 2. Justification of (4.2) when $\varphi \in W^{1,k-1}(\mathcal{M}; \mathcal{F})$.

Let $0 < a < 2\varepsilon_1 < b < \infty$. By (1) and the Stokes like formula for Lipschitz maps (see, e.g., [19, Proposition 5.9]), we have

$$\int_{\mathcal{M} \times (a,b)} \Phi^*(d\tilde{\gamma}) \wedge d\tilde{\xi} = \int_{\mathcal{M}} (\Phi_b)^* \tilde{\gamma} \wedge d\xi - \int_{\mathcal{M}} (\Phi_a)^* \tilde{\gamma} \wedge d\xi,$$

where we use the notation $\Phi_\varepsilon(x) := \Phi(x, \varepsilon)$. Using the fact that, by definition, $\Phi(x, \varepsilon)$

is constant for $x \in \mathcal{M}$ and $\varepsilon \geq 2\varepsilon_1$, we find that

$$\int_{\mathcal{M}} (\Phi_a)^* \tilde{\gamma} \wedge d\xi = - \int_{\mathcal{M} \times (a,b)} \Phi^*(d\tilde{\gamma}) \wedge d\tilde{\xi} = - \int_{\mathcal{M} \times (a,\infty)} \Phi^*(d\tilde{\gamma}) \wedge d\tilde{\xi}. \quad (4.5)$$

We next note that

$$\Phi_\varepsilon \rightarrow \varphi \text{ in } W^{1,k-1}(\mathcal{M}) \text{ as } \varepsilon \rightarrow 0. \quad (4.6)$$

In order to prove this, it suffices to prove that $\bar{\varphi}_{i,\varepsilon} := \xi_i [(\bar{\varphi}_i * \rho_\varepsilon)((\psi_i)^{-1}(\cdot))] \rightarrow \xi_i \varphi$ in $W^{1,k-1}(\mathcal{M})$. Clearly, we have

$$\bar{\varphi}_{i,\varepsilon} \circ \psi_i = (\xi_i \circ \psi_i) (\bar{\varphi}_i * \rho_\varepsilon) \rightarrow (\xi_i \circ \psi_i) \bar{\varphi}_i = (\xi_i \circ \psi_i) (\varphi \circ \psi_i) \text{ in } W^{1,k-1}(V_i). \quad (4.7)$$

Combining (4.7) with the fact that $\psi_i^{-1} \circ \psi_j$ is bi-Lipschitz, we obtain that $\bar{\varphi}_{i,\varepsilon} \circ \psi_j \rightarrow (\xi_i \circ \psi_j) (\varphi \circ \psi_j)$ in $W^{1,k-1}(V_j)$, and therefore $\bar{\varphi}_{i,\varepsilon} \rightarrow \xi_i \varphi$ in $W^{1,k-1}(\mathcal{M})$.

From (4.6) and the $\mathcal{L}^{k-1}$ -continuity of the superposition with Carathéodory functions (see, e.g., Appell and Zabrejko [2, Lemma 9.5]), we have

$$(\Phi_\varepsilon)^* \tilde{\gamma} \rightarrow \varphi^* \tilde{\gamma} \text{ in } \mathcal{L}^1 \text{ as } \varepsilon \rightarrow 0.$$

Since $\tilde{\gamma}$ is an extension of γ , we have $\varphi^* \tilde{\gamma} = \varphi^* \gamma$ a.e. Letting $a \rightarrow 0$ in (4.5), we have

$$\langle T\varphi, \xi \rangle = \int_{\mathcal{M}} \varphi^* \gamma \wedge d\xi = - \int_{\mathcal{M} \times (0,\infty)} \Phi^*(d\tilde{\gamma}) \wedge d\tilde{\xi}.$$

This completes Step 2.

Step 3. Justification of (4.1) when $\varphi \in W^{1,k-1}(\mathcal{M}; \mathcal{F})$.

By (4.3), (4.4), and the definition of Φ , we have for a.e. $x \in \mathcal{M}$,

$$\begin{aligned} |\nabla \Phi(x, \varepsilon)| &\lesssim \int_{\mathcal{M}} |\varphi(y) - \varphi(x)| d\mathcal{H}^k(y) \\ &\quad + \int_{\mathcal{M}} \int_{\mathcal{M}} |\varphi(y) - \varphi(z)| d\mathcal{H}^k(y) d\mathcal{H}^k(z) \text{ if } \varepsilon_1 \leq \varepsilon < 2\varepsilon_1, \end{aligned} \quad (4.8)$$

$$\nabla \Phi(x, \varepsilon) = 0, \text{ if } \varepsilon \geq 2\varepsilon_1. \quad (4.9)$$

Combining (4.4), (4.8), and (4.9) with Hölder's inequality, we obtain the following estimate.

Lemma 4.2. *Let $0 < r < 1$ and $1 \leq q < \infty$. Then*

$$\int_{\mathcal{M} \times (0, \infty)} \varepsilon^{q(1-r)-1} |\nabla \Phi(x, \varepsilon)|^q \lesssim \int_{\mathcal{M}} \int_{\mathcal{M}} \frac{|\varphi(x) - \varphi(y)|^q}{[\text{dist}(x, y)]^{k+rq}} d\mathcal{H}^k(y) d\mathcal{H}^k(x).$$

Applying Lemma 4.2 with $r := 1 - 1/k$ and $q := k$, we obtain the estimate

$$\left| \int_{\mathcal{M} \times (0, \infty)} \Phi^*(d\tilde{\gamma}) \wedge d\tilde{\xi} \right| \lesssim |\varphi|_{W^{1-1/k, k}}^k |\xi|_{\text{Lip}}.$$

Step 4. Continuity of the right-hand side of (4.2) in $W^{1-1/k, k}(\mathcal{M}; \mathcal{F})$.

Consider $\varphi_j, \varphi: \mathcal{M} \rightarrow \mathcal{F}$ such that $\varphi_j \rightarrow \varphi$ in $W^{1-1/k, k}$ as $j \rightarrow \infty$. The converse to the dominated convergence theorem implies that there exists a map $G: \mathcal{M} \times \mathcal{M} \rightarrow [0, \infty]$ such that

$$\int_{\mathcal{M}} \int_{\mathcal{M}} \frac{[G(x, y)]^k}{[\text{dist}(x, y)]^{2k-1}} d\mathcal{H}^k(x) d\mathcal{H}^k(y) < \infty, \quad (4.10)$$

and, up to a subsequence,

$$|\varphi_j(x) - \varphi_j(y)| \leq G(x, y). \quad (4.11)$$

Let Φ_j be the corresponding maps associated with φ_j . Then, by the properties of convolution, for a.e. $x \in \mathcal{M}$ and $\forall \varepsilon > 0$, we have

$$\nabla \Phi_j(x, \varepsilon) \rightarrow \nabla \Phi(x, \varepsilon). \quad (4.12)$$

Combining dominated convergence with (4.10), (4.11), (4.12), (4.4), (4.8), (4.9), and the proof of Lemma 4.2 with $r := 1 - 1/k$ and $q = k$, we find that

$$\int_{\mathcal{M} \times (0, \infty)} \Phi_j^*(d\tilde{\gamma}) \wedge d\tilde{\xi} \rightarrow \int_{\mathcal{M} \times (0, \infty)} \Phi^*(d\tilde{\gamma}) \wedge d\tilde{\xi}.$$

Step 5. Density of $W^{1, k-1}(\mathcal{M}; \mathcal{F})$ in $W^{1-1/k, k}(\mathcal{M}; \mathcal{F})$.

By Theorems 3.1 and 3.2, we have

$$W^{1-1/k, k}(\mathcal{M}; \mathcal{F}) = \overline{\mathcal{R}_0}^{W^{1-1/k, k}} \subset \overline{W^{1, k-1}(\mathcal{M}; \mathcal{F})}^{W^{1-1/k, k}} \subset W^{1-1/k, k}(\mathcal{M}; \mathcal{F}),$$

which complete Step 5. □

4.2 Existence and uniqueness of $W^{s,p}$ lifting

In this section, we assume that $\pi: \mathcal{E} \rightarrow \mathcal{N}$ is a Riemannian covering map, where $\mathcal{E}$ is still complete Riemannian manifold, embedded into ν via a tubed embedding. Meanwhile, $\mathcal{M}$ is still a simply-connected, compact k -dimensional Lipschitz manifold. We first recall the “non-intersection” between two liftings, which is a counterpart of Lemma 2.8 in the Sobolev setting.

Lemma 4.3. *Let $0 < s < 1$ and $sp \geq 1$. If $\varphi, \varphi' \in W^{s,p}(\mathcal{M}; \mathcal{E})$ are two liftings of $f \in W^{s,p}(\mathcal{M}; \mathcal{N})$, then either (i) $\varphi = \varphi'$ a.e. or (ii) $\varphi \neq \varphi'$ a.e.*

Remark 4.4. When $\mathcal{M}$ is smooth, Lemma 4.3 has been proved by Mironescu and Van Schaftingen [30, Proposition 4.1]. $\square$

Proof of Lemma 4.3. Let

$$g(t) = \min \left\{ \frac{t}{\text{Inj}(\mathcal{N})}, 1 \right\}.$$

Define $h(x) := g \circ \text{dist}_{\mathcal{E}}(\varphi(x), \varphi'(x))$. Since

$$|h(x) - h(y)| \leq |g|_{\text{Lip}}(\text{dist}_{\mathcal{E}}(\varphi(x), \varphi(y)) + \text{dist}_{\mathcal{E}}(\varphi'(x), \varphi'(y))),$$

if $\varphi, \varphi' \in W^{s,p}(\mathcal{M}; \mathcal{E})$, then $h \in W^{s,p}(\mathcal{M}; \mathbb{R})$. On the other hand, combining the definition of h and Lemma 2.7, we have $h(x) \in \{0, 1\}$ for $\forall x \in \mathcal{M}$. Set

$$A = \{x \in \mathcal{M}: h(x) = 1\}.$$

Using the fact that $h \in W^{s,p}(\mathcal{M}; \mathbb{R})$, one has

$$\int_{\mathcal{M} \setminus A} \int_A \frac{1}{|x - y|^{k+sp}} d\mathcal{H}^k(x) d\mathcal{H}^k(y) < \infty. \quad (4.13)$$

If A satisfies (4.13), then we have either $\mathcal{H}^k(A) = 0$, or $\mathcal{H}^k(\mathcal{M} \setminus A) = 0$. (See [8, Appendix B]. There, A is a subset of a connected domain, but the argument applies to a Lipschitz manifold.) This concludes the proof. $\square$

We can prove the existence and uniqueness up to deck transformation of the lifting, which is a consequence of Lemmas 4.3 and 2.7.

Theorem 4.5. *Assume that $k \geq 3$, that the covering map $\pi: \mathcal{E} \rightarrow \mathcal{N}$ is normal, and that $\mathcal{E}$ is compact. Let $f \in W^{s,p}(\mathcal{M}; \mathcal{N})$, with $0 < s < 1$ and $p \geq 1$. Then,*

(i) *if $sp \geq k - 1$, there exists a lifting $\varphi \in W^{s,p}(\mathcal{M}; \mathcal{E})$ of f such that $\pi \circ \varphi = f$;*

(ii) if $sp > 1$, the lifting φ , if it exists, is unique up to a deck transformation.

Remark 4.6. In Theorem 4.5 above, the existence part (i) is restricted to $sp \geq k - 1$ because our argument relies on the density of the class $\mathcal{R}$, which so far is limited to $sp \geq k - 1$ in the context of mappings defined on Lipschitz manifolds. However, analogously to the situation of mappings defined on smooth domains, we expect (i) to remain valid in the larger range $sp \geq 2$. $\square$

Proof. For the existence of the lifting, we start with maps in the dense class $\mathcal{R}_0$ (see Section 3). Let $f \in \mathcal{R}_0$ be such that $f \in C^0(\mathcal{M} \setminus \mathcal{S}(f); \mathcal{N})$, where $\mathcal{S}(f)$ is a finite set. Since $k \geq 3$, the set $\mathcal{M} \setminus \mathcal{S}(f)$ is simply-connected, whence $f|_{\mathcal{M} \setminus \mathcal{S}(f)}$ has a continuous lifting $\varphi \in C^0(\mathcal{M} \setminus \mathcal{S}(f); \mathcal{E})$.

Next we use the chart structure introduced in Section 2.1. Consider $B_r(x) \subset \mathcal{M}$ such that $B_{4\sqrt{k}K_2r}(x) \subset U_i$ for some $i \in I$. Then $f \circ \psi_i \in W^{s,p}(V_i; \mathcal{N})$ and is continuous in $V_i \setminus \mathcal{S}_i(f)$, where $\mathcal{S}_i(f)$ is a union of finitely many points in V_i . Since $k \geq 3$, the set $V_i \setminus \mathcal{S}(f)$ is simply-connected. Hence, by [30, Lemma 3.4], the continuous lifting $\varphi \circ \psi_i$ is well defined in $Q_{\sqrt{k}r}(\psi_i^{-1}(x))$ and satisfies

$$|\varphi \circ \psi_i|_{W^{s,p}(Q_{\sqrt{k}r}(\psi_i^{-1}(x)))}^p \lesssim \left(\frac{\text{diam}(\mathcal{E})}{\text{Inj}(\mathcal{N})} \right)^p |f \circ \psi_i|_{W^{s,p}(Q_{\sqrt{k}r}(\psi_i^{-1}(x)))}^p$$

which implies that

$$|\varphi|_{W^{s,p}(B_r(x))}^p \lesssim \left(\frac{\text{diam}(\mathcal{E})}{\text{Inj}(\mathcal{N})} \right)^p |f|_{W^{s,p}(B_{3K_2r}(x))}^p.$$

Consider, for sufficiently small r , a cover $(B_r(x_j))_{j \in J}$ of $\mathcal{M}$. By [30, Lemma 3.5] (the argument there applies to Lipschitz manifolds) we have

$$\begin{aligned} |\varphi|_{W^{s,p}(\mathcal{M})}^p &\lesssim \sum_j \left(\frac{\text{diam}(\mathcal{E})}{\text{Inj}(\mathcal{N})} \right)^p |f|_{W^{s,p}(B_{3K_2r}(x_j))}^p \\ &\lesssim \left(\frac{\text{diam}(\mathcal{E})}{\text{Inj}(\mathcal{N})} \right)^p |f|_{W^{s,p}(\mathcal{M})}^p. \end{aligned} \tag{4.14}$$

For a general $f \in W^{s,p}(\mathcal{M}; \mathcal{N})$, consider a sequence $(f_\ell) \subset \mathcal{R}_0$ such that $f_\ell \rightarrow f$ in $W^{s,p}(\mathcal{M}; \mathcal{N})$. By (4.14) and the fact that $\mathcal{E}$ is compact, we obtain a sequence of liftings φ_ℓ of f_ℓ such that $(\varphi_\ell)_\ell$ is bounded in $W^{s,p}(\mathcal{M}; \mathcal{E})$. Then, possibly up to a subsequence, we have $\varphi_\ell \rightarrow \varphi$ in $W^{s,p}(\mathcal{M}; \mathcal{E})$ for some φ such that $\pi \circ \varphi = f$, which concludes the proof of existence.

We now turn to (ii). We first notice that, by definition, if $\tau \in \text{Aut}(\pi)$ is a deck transformation, then $\tau \circ \varphi$ is another lifting of f .

Next, assume that φ' is another lifting of f . Since φ and φ' belong to $W^{s,p}(\mathcal{M};\mathcal{E})$, for a.e. $x \in \mathcal{M}$, we have

$$\int_{\mathcal{M}} \frac{\text{dist}_{\mathcal{E}}(\varphi(x), \varphi(y))^p + \text{dist}_{\mathcal{E}}(\varphi'(x), \varphi'(y))^p}{|x - y|^{k+sp}} d\mathcal{H}^k(y) < \infty. \quad (4.15)$$

Let $x_0 \in \mathcal{M}$ be such that (4.15) holds with $x = x_0$. Let $\tau \in \text{Aut}(\pi)$ be the deck transformation such that $\tau \circ \varphi(x_0) = \varphi'(x_0)$. Then by Lemma 4.3, it suffices to find a set $A \subset \mathcal{M}$ with positive measure such that $\tau \circ \varphi(y) = \varphi'(y)$ for all $y \in A$. For this purpose, define

$$A_r := \{y \in B_r(x_0) : \text{dist}_{\mathcal{E}}(\varphi(x_0), \varphi(y)) \leq \text{Inj}(\mathcal{N})/2\},$$

and

$$A'_r := \{y \in B_r(x_0) : \text{dist}_{\mathcal{E}}(\varphi'(x_0), \varphi'(y)) \leq \text{Inj}(\mathcal{N})/2\}.$$

We rely on the following lemma (which, as it does not make use of the vector space structure on $W^{s,p}$, is valid in the larger range $p > 0$).

Lemma 4.7. *Assume that $0 < s < 1$ and $p > 0$. Let $\varphi \in W^{s,p}(\mathcal{M};\mathcal{E})$, and let $0 < c < 1$ and $M > 0$. Then, for every x_0 such that*

$$\int_{\mathcal{M}} \frac{\text{dist}_{\mathcal{E}}(\varphi(x_0), \varphi(y))^p}{|x - y|^{k+sp}} d\mathcal{H}^k(y) < \infty, \quad (4.16)$$

there exists R such that the set

$$A_r := \{y \in B_r(x_0) : \text{dist}_{\mathcal{E}}(\varphi(x_0), \varphi(y)) \leq M\}$$

satisfies

$$\mathcal{H}^k(A_r) \geq c \mathcal{H}^k(B_r(x_0)), \forall 0 < r \leq R. \quad (4.17)$$

Proof of Lemma 4.7. We argue by contradiction. Let $x_0 \in \mathcal{M}$ satisfy (4.16). For such x_0 , if (4.17) fails, one has

$$\begin{aligned} \int_{\mathcal{M}} \frac{\text{dist}_{\mathcal{E}}(\varphi(x_0), \varphi(y))^p}{|x_0 - y|^{k+sp}} d\mathcal{H}^k(y) &\geq \int_{B_r(x_0) \setminus A_r} \frac{\text{dist}_{\mathcal{E}}(\varphi(x_0), \varphi(y))^p}{|x_0 - y|^{k+sp}} d\mathcal{H}^k(y) \\ &\gtrsim (1 - c) \frac{M^p}{r^{sp}}, \forall r > 0, \end{aligned}$$

which contradicts (4.16). □

We continue the proof of Theorem 4.5. Applying Lemma 4.7 to both φ and φ' with $c = 2/3$ and $M = \text{Inj}(\mathcal{N})/2$, we have $\mathcal{H}^k(A_r \cap A'_r)/\mathcal{H}^k(B_r(x_0)) > 0$ for some $r > 0$. Then, for $\forall y \in A_r \cap A'_r$,

$$\begin{aligned} \text{dist}_{\mathcal{E}}(\tau \circ \varphi(y), \varphi'(y)) &\leq \text{dist}_{\mathcal{E}}(\tau \circ \varphi(y), \tau \circ \varphi(x_0)) + \text{dist}_{\mathcal{E}}(\varphi'(x_0), \varphi'(y)) \\ &= \text{dist}_{\mathcal{E}}(\varphi(y), \varphi(x_0)) + \text{dist}_{\mathcal{E}}(\varphi'(x_0), \varphi'(y)) \leq \text{Inj}(\mathcal{N}). \end{aligned}$$

By Lemma 2.7, we have

$$\begin{aligned} \text{dist}_{\mathcal{E}}(\tau \circ \varphi(y), \varphi'(y)) &= \text{dist}_{\mathcal{N}}(\pi \circ \tau \circ \varphi(y), \pi \circ \varphi'(y)) \\ &= \text{dist}_{\mathcal{N}}(\pi \circ \varphi(y), \pi \circ \varphi'(y)) \\ &= \text{dist}_{\mathcal{N}}(f(y), f(y)) = 0. \end{aligned}$$

This implies that both liftings $\tau \circ \varphi$ and φ' coincide in a subset of $\mathcal{M}$ with positive measure and thus, by Lemma 4.3, coincide in $\mathcal{M}$ a.e. $\square$

4.3 A selection result

The objective of this section is to prove that, under a normality assumption, whenever we are given a converging sequence of Sobolev maps into the base space, we can pick the liftings in such a way that they converge towards a lifting of the limiting map. This is a Sobolev counterpart of Theorem 2.12 for VMO liftings. Our main selection result reads as follows.

Theorem 4.8. *Assume that $k \geq 3$, that the covering map $\pi: \mathcal{E} \rightarrow \mathcal{N}$ is normal, and that $\mathcal{E}$ is compact. Let $0 < s < 1$ and $sp \geq k-1$. Then, for every sequence of maps $f_\ell \in W^{s,p}(\mathcal{M}; \mathcal{N})$ such that $f_\ell \rightarrow f$ in $W^{s,p}(\mathcal{M}; \mathcal{N})$, there exists a sequence of corresponding liftings $\varphi_\ell \in W^{s,p}(\mathcal{M}; \mathcal{E})$ such that $\varphi_\ell \rightarrow \varphi$ in $W^{s,p}(\mathcal{M}; \mathcal{E})$. Moreover, the conclusion still holds in the case where $0 < s < 1$ and $sp > 1$, provided that each f_ℓ admits a lifting in $W^{s,p}(\mathcal{M}; \mathcal{E})$.*

Before proving Theorem 4.8, we first look at the case where $\mathcal{M} = (0, 1)^k$.

Theorem 4.9. *Let $0 < s < 1$ and $sp \geq 2$. Assume that $f_\ell \rightarrow f$ in $W^{s,p}((0, 1)^k; \mathcal{N})$. Then, there exists a sequence of liftings (φ_ℓ) in $W^{s,p}((0, 1)^k; \mathcal{E})$ such that*

$$\varphi_\ell \rightarrow \varphi \text{ in } W^{s,p}((0, 1)^k; \mathcal{E}),$$

where φ is a lifting of f . Moreover, the conclusion still holds in the case where $0 < s < 1$ and $sp > 1$, provided that each f_ℓ admits a lifting in $W^{s,p}((0, 1)^k; \mathcal{E})$.

A crucial ingredient in our proof is the one from [30, Section 3.1], with at its core a reverse oscillation inequality. However, we need to slightly adapt it to the context of a

sequence of lifting, notably to be able to obtain uniform bounds along the statements and invoke dominated convergence arguments.

Proof. To start with, let us fix φ_ℓ a $W^{s,p}$ lifting of f_ℓ . When $1 < sp < 2$, this is provided by our assumption, while when $sp \geq 2$, the existence of such a lifting is always guaranteed by [30, Theorem 1].

For $j = 1, \dots, k$, $\hat{x}_j = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_k) \in (0, 1)^{k-1}$, and g a measurable map defined on $(0, 1)^k$, we define

$$g_{\hat{x}_j}(t) := g(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_k), \forall t \in (0, 1).$$

With this definition, for a.e. $\hat{x}_j \in (0, 1)^{k-1}$, we have $f_{\hat{x}_j} \in W^{s,p}((0, 1); \mathcal{N})$. Combining the seminorm equivalence (see, e.g., [1, Lemma 7.44])

$$|g|_{W^{s,p}((0,1)^k)}^p \sim \sum_{j=1}^k \int_{(0,1)^{k-1}} |g_{\hat{x}_j}|_{W^{s,p}((0,1))}^p d\hat{x}_j, \forall g: (0, 1)^k \rightarrow \mathbb{R}^v \quad (4.18)$$

with the convergence $f_\ell \rightarrow f$ in $W^{s,p}((0, 1)^k; \mathcal{N})$, we deduce that for every $1 \leq j \leq k$,

$$\lim_{\ell \rightarrow \infty} \int_{(0,1)^{k-1}} |(f_\ell - f)_{\hat{x}_j}|_{W^{s,p}((0,1))}^p d\hat{x}_j = 0.$$

Hence, the converse to the dominated convergence theorem implies that there exists a map

$$G_j: (0, 1) \times (0, 1) \times (0, 1)^{k-1} \rightarrow \mathbb{R}$$

such that, up to a subsequence,

$$|(f_\ell)_{\hat{x}_j}(a) - (f_\ell)_{\hat{x}_j}(b)|^p \leq G_j(a, b, \hat{x}_j), \forall 0 < a < b < 1, \quad (4.19)$$

and

$$\int_{(0,1)^{k-1}} \left(\iint_{(0,1)^2} \frac{G_j(a, b, \hat{x}_j)}{|a - b|^{1+sp}} da db \right) d\hat{x}_j < \infty. \quad (4.20)$$

We now provide a $W^{s,p}$ bound for φ_ℓ via (4.20).

For a.e. $\hat{x}_j \in (0, 1)^{k-1}$ fixed, since $sp > 1$, using Morrey's embedding $W^{s,p}((0, 1)) \hookrightarrow C^{0,s-1/p}((0, 1))$, there exists a map $h_\ell \in C^{0,s-1/p}((0, 1))$ (depending on $\hat{x}_j$) such that

$h_\ell = (\varphi_\ell)_{\hat{x}_j}$ a.e. and $\pi \circ h_\ell = (f_\ell)_{\hat{x}_j}$ a.e. By Lemma 2.7, for a.e. $0 < a < b < 1$, we have

$$\begin{aligned} \text{dist}_{\mathcal{E}}((\varphi_\ell)_{\hat{x}_j}(a), (\varphi_\ell)_{\hat{x}_j}(b)) &= \text{dist}_{\mathcal{E}}(h_\ell(a), h_\ell(b)) \\ &\leq \frac{\text{diam}(\mathcal{E})}{\text{Inj}(\mathcal{N})} \text{osc}_{[a,b]}(\pi \circ h_\ell). \end{aligned} \quad (4.21)$$

To control the right-hand side of (4.21), let $1/p < \sigma < s$. Since $\sigma p > 1$, we have

$$\text{osc}_{[a,b]}(\pi \circ h_\ell) \lesssim (b-a)^{\sigma-\frac{1}{p}} |\pi \circ h_\ell|_{W^{\sigma,p}((a,b))} = (b-a)^{\sigma-\frac{1}{p}} |(f_\ell)_{\hat{x}_j}|_{W^{\sigma,p}((a,b))}. \quad (4.22)$$

Combining (4.21), (4.22), and (4.19), we deduce that for a.e. $0 < a < b < 1$,

$$\begin{aligned} |(\varphi_\ell)_{\hat{x}_j}(a) - (\varphi_\ell)_{\hat{x}_j}(b)| &\lesssim (b-a)^{\sigma-\frac{1}{p}} \left(\int_a^b \int_a^b \frac{G_j(t, z, \hat{x}_j)}{|t-z|^{1+\sigma p}} dt dz \right)^{\frac{1}{p}} \\ &=: H_j(a, b, \hat{x}_j). \end{aligned} \quad (4.23)$$

Then,

$$\begin{aligned} &\int_{(0,1)^{k-1}} \left(\iint_{(0,1)^2} \frac{[H_j(a, b, \hat{x}_j)]^p}{|a-b|^{1+sp}} da db \right) d\hat{x}_j \\ &\lesssim \int_{(0,1)^{k-1}} \left(\iiint_{0 < a < t < z < b < 1} \frac{[G_j(t, z, \hat{x}_j)]}{|z-t|^{1+\sigma p}} \frac{1}{|a-b|^{2+(s-\sigma)p}} dt dz da db \right) d\hat{x}_j \\ &\lesssim \int_{(0,1)^{k-1}} \left(\iint_{(0,1)^2} \frac{G_j(t, z, \hat{x}_j)}{|z-t|^{1+\sigma p}} \iint_{-\infty < a < t < z < b < \infty} \frac{1}{|a-b|^{2+(s-\sigma)p}} da db dz dt \right) d\hat{x}_j \\ &\lesssim \int_{(0,1)^{k-1}} \left(\iint_{(0,1)^2} \frac{G_j(t, z, \hat{x}_j)}{|t-z|^{1+sp}} dt dz \right) d\hat{x}_j < \infty. \end{aligned} \quad (4.24)$$

Here, we have used (4.20) in the last inequality. Summing over j and combining with the fact that $\mathcal{E}$ is compact, we deduce that (φ_ℓ) is a bounded sequence in $W^{s,p}((0,1)^k; \mathcal{E})$. Consequently, possibly up to a further subsequence, there exists a lifting φ of f such that $\varphi_\ell \rightarrow \varphi$ (i) a.e.; (ii) strongly in $\mathcal{L}^p$; (iii) weakly in $W^{s,p}$; (iv) for $j = 1, \dots, k$,

$$\lim_{\ell \rightarrow \infty} \int_{(0,1)^{k-1}} |(\varphi_\ell - \varphi)_{\hat{x}_j}|_{W^{s,p}((0,1))}^p d\hat{x}_j = \int_{(0,1)^{k-1}} \lim_{\ell \rightarrow \infty} |(\varphi_\ell - \varphi)_{\hat{x}_j}|_{W^{s,p}((0,1))}^p d\hat{x}_j \quad (4.25)$$

(this relies on the dominated convergence theorem and (4.24)).

Next, we notice that for a.e. $\hat{x}_j$, we have (i) $(\varphi_\ell)_{\hat{x}_j} \rightarrow \varphi_{\hat{x}_j}$ a.e.; (ii) by repeating the

computation in (4.24) without the outer integral with respect to x_j ,

$$\iint_{(0,1)^2} \frac{[H_j(a, b, \hat{x}_j)]^p}{|a - b|^{1+sp}} da db \lesssim \iint_{(0,1)^2} \frac{G_j(a, b, \hat{x}_j)}{|a - b|^{1+sp}} da db < \infty.$$

Combining this with (4.23), (4.25), and the dominated convergence theorem, we deduce that

$$\lim_{\ell \rightarrow \infty} \int_{(0,1)^{k-1}} |(\varphi_\ell - \varphi)_{\hat{x}_j}|_{W^{s,p}((0,1))}^p d\hat{x}_j = 0. \quad (4.26)$$

Summing (4.26) over j and using the seminorm equivalence (4.18) again, we have $\varphi_\ell \rightarrow \varphi$ in $W^{s,p}((0,1)^k; \mathcal{E})$. Since the subsequence we chose satisfies $\varphi_\ell \rightarrow \varphi$ strongly in $\mathcal{L}^p$, we have $\varphi_\ell \rightarrow \varphi$ in $W^{s,p}((0,1)^k; \mathcal{E})$.

To conclude the proof, it suffices to make the whole sequence converge to φ in $W^{s,p}$, possibly after modifying the liftings *via* a deck transformation. For this purpose, we first note that

$$\sup_{\ell} \min\{\|\tau \circ \varphi_\ell - \varphi\|_{\mathcal{L}^p} : \tau \in \text{Aut}(\pi)\} < \infty.$$

(The minimum is achieved since $\text{Aut}(\pi)$ is finite provided that $\mathcal{E}$ is compact.) Let τ_ℓ be such that

$$\|\tau_\ell \circ \varphi_\ell - \varphi\|_{\mathcal{L}^p} = \min_{\tau \in \text{Aut}(\pi)} \{\|\tau \circ \varphi_\ell - \varphi\|_{\mathcal{L}^p}\}. \quad (4.27)$$

We claim that $\tau_\ell \circ \varphi_\ell \rightarrow \varphi$ in $W^{s,p}((0,1)^k; \mathcal{E})$. We argue by contradiction. Repeating the argument above, we can assume that there exists a subsequence $\tau_{\ell_\beta} \circ \varphi_{\ell_\beta} \rightarrow \varphi'$ with φ' another lifting of f . Let τ be the deck transformation provided by Theorem 4.5 (ii) satisfying $\tau \circ \varphi' = \varphi$. Then, by (4.27), we have

$$\begin{aligned} \|\tau_{\ell_\beta} \circ \varphi_{\ell_\beta} - \varphi\|_{\mathcal{L}^p} &\leq \|(\tau \circ \tau_{\ell_\beta}) \circ \varphi_{\ell_\beta} - \varphi\|_{\mathcal{L}^p} \\ &= \|\tau \circ (\tau_{\ell_\beta} \circ \varphi_{\ell_\beta}) - \tau \circ \varphi'\|_{\mathcal{L}^p} \\ &\lesssim \|\tau_{\ell_\beta} \circ \varphi_{\ell_\beta} - \varphi'\|_{\mathcal{L}^p} \rightarrow 0, \end{aligned}$$

which means that $\varphi = \varphi'$ a.e. We thus obtain a contradiction. This shows that $(\tau_\ell \circ \varphi_\ell)$ is the required sequence of liftings and concludes the proof. $\square$

We are now in the position to prove Theorem 4.8.

Proof of Theorem 4.8. Consider the chart structure $\{(U_i, V_i, \psi_i)\}_{i \in I}$, and assume that V_i is a k -dimensional cube $(0,1)^k$.

Let $f_\ell \rightarrow f$ in $W^{s,p}(\mathcal{M}; \mathcal{N})$. On each V_i , since $f_\ell \circ \psi_i \rightarrow f \circ \psi_i$ in $W^{s,p}((0,1)^k; \mathcal{N})$, applying Theorem 4.9, we get a corresponding sequence of liftings $\bar{\varphi}_\ell^i \rightarrow \bar{\varphi}^i$ in $W^{s,p}((0,1)^k; \mathcal{E})$. Define $\varphi^i := \bar{\varphi}^i \circ \psi_i^{-1}$ in U_i , and define φ_ℓ^i similarly. Then, we have

$$\varphi_\ell^i \rightarrow \varphi^i \text{ in } W^{s,p}(U_i; \mathcal{E}), \quad (4.28)$$

and φ_ℓ^i , respectively φ^i , is a lifting of $f_\ell^i := f_\ell|_{U_i}$, respectively $f^i := f|_{U_i}$.

Let $I_0 := \text{card}(I)$. Since $\mathcal{M}$ is connected, one can relabel $\{U_i\}_{i \in I}$ as $\{U_i\}_{i=1}^{I_0}$ in such a way that $(\bigcup_{i=1}^{k_0} U_i) \cap U_{k_0+1} \neq \emptyset$ for $\forall 1 \leq k_0 \leq I_0 - 1$. Let $\varphi_\ell \in W^{s,p}(\mathcal{M}; \mathcal{E})$ be the lifting of f_ℓ such that $\varphi_\ell = \varphi_\ell^1$ in U_1 .

First, using Theorem 4.5 (since the proof of uniqueness (ii) remains true when considering $\mathcal{M} = (0,1)^k$), on each U_i , we have $\varphi_\ell|_{U_i} = \tau_\ell^i \circ \varphi_\ell^i$ for some $\tau_\ell^i \in \text{Aut}(\pi)$. In particular, from the convergence of the φ_ℓ^i and the fact that deck transformations are isometries, we deduce that (φ_ℓ) is bounded in $W^{s,p}$ — the $\mathcal{L}^p$ bound follows, as usual, from the compactness of $\mathcal{E}$. Therefore, up to extraction of a subsequence, $\varphi_\ell \rightarrow \varphi$, where $\varphi \in W^{s,p}(\mathcal{M}; \mathcal{E})$ is a lifting of f .

Now that we know that the limiting map f admits a lifting as well, we come back to our main selection proof. Up to a deck transformation, we may assume that φ is the lifting such that $\varphi = \varphi^1$ in U_1 . Moreover, we can pick a deck transformation τ^i such that $\varphi|_{U_i} = \tau^i \circ \varphi^i$. The key argument to conclude the proof is the following lemma, which ensures that, when ℓ is sufficiently large, the deck transformation τ^i that brings φ^i onto $\varphi|_{U_i}$ also brings φ_ℓ^i onto $\varphi_\ell|_{U_i}$. This allows us to conclude by applying (4.28) on every U_i , up to the action of τ^i .

Lemma 4.10. *For ℓ sufficiently large, we have*

$$\varphi_\ell|_{U_i} = \tau_\ell^i \circ \varphi_\ell^i = \tau^i \circ \varphi_\ell^i, \forall i, \quad (4.29)$$

which implies that

$$\tau^i \circ \varphi_\ell^i|_{U_i \cap U_j} = \tau^j \circ \varphi_\ell^j|_{U_i \cap U_j} = \varphi_\ell|_{U_i \cap U_j}, \text{ if } U_i \cap U_j \neq \emptyset.$$

The proof of Lemma 4.10 is given below. Granted (4.29), we have, using again [30,

Lemma 3.5],

$$\begin{aligned}
\lim_{\ell \rightarrow \infty} \|\varphi_\ell - \varphi\|_{W^{s,p}(\mathcal{M})} &\lesssim \lim_{\ell \rightarrow \infty} \sum_{i \in I} \|\varphi_\ell - \varphi\|_{W^{s,p}(U_i)} \\
&\lesssim \lim_{\ell \rightarrow \infty} \sum_{i \in I} \|\tau_\ell^i \circ \varphi_\ell^i - \tau^i \circ \varphi^i\|_{W^{s,p}(U_i)} \\
&\lesssim \lim_{\ell \rightarrow \infty} \sum_{i \in I} \|\tau^i \circ \varphi_\ell^i - \tau^i \circ \varphi^i\|_{W^{s,p}(U_i)} = 0,
\end{aligned}$$

where in the last equality, we have used the fact that τ^i is smooth and an isometry. This concludes the proof of Theorem 4.8. $\square$

Proof of Lemma 4.10. First, we have $\varphi_\ell|_{U_1} = \text{id} \circ \varphi_\ell^1$. Assume now that (4.29) holds for $\forall i \leq i_0$ for some $i_0 \leq I_0$.

Since $(\bigcup_{i=1}^{i_0} U_i) \cap U_{i_0+1} \neq \emptyset$, there exists some U_i such that $U_i \cap U_{i_0+1}$ has positive measure. Let $B \subset U_i \cap U_{i_0+1}$ be an open simply-connected set. By the definition of τ_ℓ^i and τ^i , we have

$$\tau_\ell^{i_0+1} \circ \varphi_\ell^{i_0+1} = \tau_\ell^i \circ \varphi_\ell^i \text{ in } B, \text{ and } \tau^{i_0+1} \circ \varphi^{i_0+1} = \tau^i \circ \varphi^i \text{ in } B. \quad (4.30)$$

Combining (4.30) and the induction assumption, we can deduce that

$$\begin{aligned}
&\int_B [\text{dist}_{\mathcal{G}}(\tau_\ell^{i_0+1} \circ \varphi_\ell^{i_0+1}, \tau^{i_0+1} \circ \varphi^{i_0+1})]^p dx \\
&\leq \int_B [\text{dist}_{\mathcal{G}}(\tau_\ell^{i_0+1} \circ \varphi_\ell^{i_0+1}, \tau^{i_0+1} \circ \varphi^{i_0+1}) + \text{dist}_{\mathcal{G}}(\tau^{i_0+1} \circ \varphi^{i_0+1}, \tau^{i_0+1} \circ \varphi_\ell^{i_0+1})]^p dx \\
&\lesssim \int_B [\text{dist}_{\mathcal{G}}(\tau_\ell^i \circ \varphi_\ell^i, \tau^i \circ \varphi^i)]^p dx + \int_B [\text{dist}_{\mathcal{G}}(\varphi^{i_0+1}, \varphi_\ell^{i_0+1})]^p dx \\
&= \int_B [\text{dist}_{\mathcal{G}}(\tau^i \circ \varphi_\ell^i, \tau^i \circ \varphi^i)]^p dx + \int_B [\text{dist}_{\mathcal{G}}(\varphi^{i_0+1}, \varphi_\ell^{i_0+1})]^p dx \\
&= \int_B [\text{dist}_{\mathcal{G}}(\varphi_\ell^i, \varphi^i)]^p dx + \int_B [\text{dist}_{\mathcal{G}}(\varphi^{i_0+1}, \varphi_\ell^{i_0+1})]^p dx \rightarrow 0
\end{aligned}$$

as $\ell \rightarrow \infty$. Thus, for ℓ sufficiently large (depending on B), the set

$$\{x \in B: \text{dist}_{\mathcal{G}}(\tau_\ell^{i_0+1} \circ \varphi_\ell^{i_0+1}, \tau^{i_0+1} \circ \varphi_\ell^{i_0+1}) < \text{Inj}(\mathcal{N})\}$$

has positive measure. By Lemma 2.7 and Lemma 4.3, this implies that

$$\text{dist}_{\mathcal{G}}(\tau_\ell^{i_0+1} \circ \varphi_\ell^{i_0+1}, \tau^{i_0+1} \circ \varphi_\ell^{i_0+1}) = 0$$

a.e. in U_{i_0+1} . We then obtain (4.29) by induction. $\square$

4.4 Lifting and pullback of forms

As a direct consequence of Theorem 4.1 and Theorem 4.8, given a smooth closed k -form ω on $\mathcal{N}$ such that $\pi^*\omega = d\gamma$ for some smooth $(k-1)$ -form γ on $\mathcal{E}$, and given $\xi \in \text{Lip}(\mathcal{M})$, we can give a *robust* meaning of $\int_{\mathcal{M}} \xi f^*\omega$ *via* lifting. This is the content of our next result.

Theorem 4.11. *Let $k \geq 3$. Assume that the covering map $\pi: \mathcal{E} \rightarrow \mathcal{N}$ is normal and that $\mathcal{E}$ is compact. Let ω be a smooth closed k -form on $\mathcal{E}$ such that $\pi^*\omega = d\gamma$ for some smooth $(k-1)$ -form γ on $\mathcal{E}$, and let $\xi \in \text{Lip}(\mathcal{M})$. Then, the map*

$$W^{1,k}(\mathcal{M}; \mathcal{N}) \ni f \mapsto \int_{\mathcal{M}} \xi f^*\omega$$

has a “natural” extension to $W^{1-1/k,k}(\mathcal{M}; \mathcal{N})$, given by

$$\int_{\mathcal{M}} \xi f^*\omega := \frac{(-1)^k}{N} \sum_{\tau \in \text{Aut}(\pi)} \langle T(\tau \circ \varphi), \xi \rangle,$$

where $\varphi \in W^{1-1/k,k}(\mathcal{M}; \mathcal{E})$ is any lifting of f and $N := \text{card}(\text{Aut}(\pi))$. Moreover, this extension is continuous with respect to $W^{1-1/k,k}(\mathcal{M}; \mathcal{N})$.

Proof of Theorem 4.11. We first check that the definition is well-posed, i.e., that it does not depend on the choice of the lifting $\varphi \in W^{1-1/k,k}(\mathcal{M}; \mathcal{E})$.

Let $\varphi_1 \in W^{1-1/k,k}(\mathcal{M}; \mathcal{E})$ be another lifting of f . By Theorem 4.5, there exists $\tau_1 \in \text{Aut}(\pi)$ such that $\varphi_1 = \tau_1 \circ \varphi$. For any $\tau \in \text{Aut}(\pi)$, we have $\tau \circ \tau_1 \in \text{Aut}(\pi)$. Moreover, the map $\tau \mapsto \tau \circ \tau_1$ is a bijection on $\text{Aut}(\pi)$. Thus,

$$\sum_{\tau \in \text{Aut}(\pi)} \langle T(\tau \circ \varphi_1), \xi \rangle = \sum_{\tau \in \text{Aut}(\pi)} \langle T(\tau \circ \tau_1 \circ \varphi), \xi \rangle = \sum_{\tau \in \text{Aut}(\pi)} \langle T(\tau \circ \varphi), \xi \rangle.$$

Next, we prove that the formula coincides with $\int_{\mathcal{M}} \xi f^*\omega$ when $f \in W^{1,k}(\mathcal{M}; \mathcal{N})$.

Indeed, under the assumption that $d\gamma = \pi^*\omega$, we have

$$\begin{aligned}
(-1)^k \sum_{\tau \in \text{Aut}(\pi)} \langle T(\tau \circ \varphi), \xi \rangle &= (-1)^k \sum_{\tau \in \text{Aut}(\pi)} \int_{\mathcal{M}} (\tau \circ \varphi)^* \gamma \wedge d\xi \\
&= (-1)^k \sum_{\tau \in \text{Aut}(\pi)} \int_{\mathcal{M}} \varphi^*(\tau^* \gamma) \wedge d\xi \\
&= \sum_{\tau \in \text{Aut}(\pi)} \int_{\mathcal{M}} \xi \varphi^*(\tau^*(d\gamma)) \\
&= \sum_{\tau \in \text{Aut}(\pi)} \int_{\mathcal{M}} \xi (\pi \circ \varphi)^* \omega = N \int_{\mathcal{M}} \xi f^* \omega.
\end{aligned}$$

It remains to prove the continuity of the extension. Let $f_\ell \rightarrow f$ in $W^{1-1/k,k}(\mathcal{M}; \mathcal{N})$. By Theorem 4.8, there exists a sequence of corresponding liftings such that $\varphi_\ell \rightarrow \varphi$ in $W^{1-1/k,k}(\mathcal{M}; \mathcal{E})$. Then, for any $\tau \in \text{Aut}(\pi)$, using (j) the converse to the dominated convergence theorem, (jj) the fact that τ is a global isometry of $\mathcal{E}$, and (jjj) the dominated convergence theorem, we have $\tau \circ \varphi_\ell \rightarrow \tau \circ \varphi$ in $W^{1-1/k,k}(\mathcal{M}; \mathcal{E})$. Combining this with Theorem 4.1, we have

$$\lim_{\ell \rightarrow \infty} \langle T(\tau \circ \varphi_\ell), \xi \rangle = \langle T(\tau \circ \varphi), \xi \rangle,$$

which implies that the extension is continuous with respect to the $W^{1-1/k,k}(\mathcal{M}; \mathcal{N})$ convergence. $\square$

Proposition 4.12. *Under the same assumptions as in Theorem 4.11, if we further assume that the $(k-1)$ -th de Rham cohomology class $H_{\text{dR}}^{k-1}(\mathcal{E})$ of $\mathcal{E}$ is trivial, then $\forall f \in W^{1-1/k,k}(\mathcal{M}; \mathcal{N})$, $\int_{\mathcal{M}} \xi f^* \omega$ is independent of γ .*

Proof. Assume that γ_1 and γ_2 are two choices of a primitive of $\pi^*\omega$. Then, it follows from $d\gamma_1 - d\gamma_2 = \pi^*\omega - \pi^*\omega = 0$ that $\gamma_1 - \gamma_2$ is closed. From the assumption that $H_{\text{dR}}^{k-1}(\mathcal{E}) \simeq \{0\}$, we deduce that $\gamma_1 - \gamma_2 = d\eta$ for some $(k-2)$ -form η . The conclusion then follows from the fact that $T_{d\eta}(\tau \circ \varphi) = 0$ for every $\tau \in \text{Aut}(\pi)$ and $\varphi \in W^{1-1/k,k}(\mathcal{M}; \mathcal{E})$. $\square$

A Auxiliary results

This appendix is devoted to the proof of some technical lemmas that were used in the proofs of our main density results in Section 3.

Proof of Lemma 3.4. We prove the lemma when $0 < s < 1$; the case where $s = 1$ was settled by Bethuel (see [4, Lemma 6]).

Let

$$F: S(0, 1) \times (0, 1/2) \rightarrow \mathcal{F}, F(\omega, t) := f * \rho_t(\omega),$$

and $\tilde{F} := \Pi \circ F$, with the convolution defined as in (3.65). The following are straightforward: (i) $\tilde{F}$ is well-defined and $\mathcal{F}$ -valued (since f takes its values into $B_\delta(\mathfrak{f})$); (ii) $F, \tilde{F} \in (W^{s,p} \cap C^\infty)(S(0, 1) \times (0, 1/2))$ (since $f \in W^{s-1/p,p}$).

On the other hand, consider the map $\Psi: B_1(0) \setminus B_{1/2}(0) \rightarrow S(0, 1) \times (0, 1/2)$ defined by

$$\Psi(u) := (u/|u|, 1 - |u|).$$

Clearly, Ψ is bi-Lipschitz.

Let $0 \leq \eta \leq 1$ be a smooth cut-off function in $B_1(0)$, with $\eta(u) = 1$ when $|u| \geq 3/4$ and $\eta(u) = 0$ when $|u| \leq 2/3$. Define

$$g(u) := \eta(u)(\tilde{F} \circ \Psi(u) - \mathfrak{f}) + \mathfrak{f}.$$

Clearly: (i) $g \in W^{s,p} \cap C^\infty(B_1(0); B_\delta(\mathfrak{f}))$; (ii) f and g have the same trace on $S(0, 1)$.

Therefore, the map $G := f - g$ belongs to $W_0^{s,p}(B_1(0); B_{2\delta}(0))$.

For $r \leq 1/8$, let

$$G_r(u) := \begin{cases} G(u), & \text{if } |u| \leq 1 - 2r \\ G(2u + (2r - 1)u/|u|), & \text{if } 1 - 2r \leq |u| \leq 1 - r \\ 0, & \text{if } 1 - r \leq |u| \leq 2 \end{cases}$$

We have $G_r \in W_0^{s,p}(B_1(0); B_{2\delta}(0))$ and $G_r \rightarrow G$ in $W^{s,p}$ as $r \rightarrow 0$.

By convolution with a standard mollifying sequence, we obtain maps $G_{r,\varepsilon} := G_r * \rho_\varepsilon$ such that $G_{r,\varepsilon} \in (W^{s,p} \cap C_c^\infty)(B_1(0); B_{2\delta}(0))$ and $G_{r,\varepsilon} \rightarrow G_r$ as $\varepsilon \rightarrow 0$.

Hence, the map $h_{r,\varepsilon} := \Pi \circ [G_{r,\varepsilon} + g]$ satisfies: (i) $h_{r,\varepsilon}$ is in $W^{s,p}(B_1(0); B_{4\delta}(\mathfrak{f}) \cap \mathcal{F})$; (ii) $h_{r,\varepsilon}$ has the same trace as f on $S(0, 1)$; and (iii) $h_{r,\varepsilon} \rightarrow f$ as $\varepsilon \rightarrow 0$ and $r \rightarrow 0$. This concludes the proof of the lemma. $\square$

Proof of Lemma 3.7. Consider the homogeneous extension $H(f)$ of $f|_{S(0,1)}$ to $B_2(0)$. Since $sp = 1 < k$, we have $H(f) \in W^{s,p}(B_2(0); B_\delta(\mathfrak{f}))$ (see, [12, Lemma 3]).

Let $0 \leq \eta \leq 1$ be a smooth cut-off function in $B_1(0)$, with $\eta(u) = 1$ when $|u| \geq 3/4$ and $\eta(u) = 0$ when $|u| \leq 2/3$. Define

$$g(u) := \eta(u)(H(f)(u) - \mathfrak{f}) + \mathfrak{f}, \quad \forall u \in \overline{B_1(0)}.$$

Clearly, we have $g \in W^{s,p}(B_1(0); B_\delta(\mathfrak{f})) \cap C(\overline{B_1(0)})$.

Let $G := f - g$. Denote $W_{00}^{s,p}(B_1(0))$ the set of maps $h \in W^{s,p}(B_1(0))$ such that the map

$$\tilde{h}: \mathbb{R}^k \rightarrow \mathbb{R}, \tilde{h}(u) := \begin{cases} h(u), & \text{if } u \in B_1(0) \\ 0, & \text{otherwise} \end{cases}$$

is in $W^{s,p}(\mathbb{R}^k)$.

Step 1. The map G belongs to $W_{00}^{s,p}(B_1(0))$.

For this purpose, we first prove that the map

$$\tilde{G}: B_2(0) \rightarrow \mathcal{F}, G(u) := \begin{cases} f(u), & \text{if } u \in \overline{B_1(0)} \\ H(f)(u), & \text{otherwise} \end{cases}$$

belongs to $W^{s,p}(B_2(0))$. It suffices to check that

$$\int_{B_2(0) \setminus B_1(0)} \int_{B_1(0)} \frac{|f(u) - f(v/|v|)|^p}{|u - v|^{k+1}} du dv < \infty.$$

By (3.124) and the fact that

$$|u - v/|v|| \leq |u - v| + \text{dist}(v, S(0, 1)) \leq 2|u - v|$$

when $|u| \leq 1 \leq |v|$, we have

$$\begin{aligned} & \int_{B_2(0) \setminus B_1(0)} \int_{B_1(0)} \frac{|f(u) - f(v/|v|)|^p}{|u - v|^{k+1}} du dv \\ & \lesssim \int_{B_2(0) \setminus B_1(0)} \int_{B_1(0)} \frac{|f(u) - f(v/|v|)|^p}{|u - v/|v||^{k+1}} du dv \\ & \lesssim \int_{S(0,1)} \int_{B_1(0)} \frac{|f(u) - f(\omega)|^p}{|u - \omega|^{k+1}} du d\mathcal{H}^{k-1}(\omega) < \infty. \end{aligned}$$

Therefore, the map $\tilde{G} - H(f)$ belongs to $W^{s,p}(B_2(0))$. By the definition of G , we find that $G \in W_{00}^{s,p}(B_1(0))$.

Step 2. Approximating G with $C_c^\infty(B_1(0); B_{2\delta}(0))$ maps.

By Step 1, we are in position to repeat the proof of Lemma 3.4. We omit the details.

Step 3. Proof of (3.125).

We use the same notation as in the proof of Lemma 3.4. We note that

$$\begin{aligned}
& \int_{S(0,1)} \int_{B_1(0)} \frac{|h_{r,\varepsilon}(u) - f(u)|^p}{|u - \omega|^{k+sp}} du d\mathcal{H}^{k-1}(\omega) \\
& \lesssim \int_{S(0,1)} \int_{B_1(0)} \frac{|G_{r,\varepsilon}(u) - G(u)|^p}{|u - \omega|^{k+sp}} du d\mathcal{H}^{k-1}(\omega) \\
& \lesssim \int_{S(0,1)} \int_{B_1(0)} \frac{|G_{r,\varepsilon}(u) - G_r(u)|^p}{|u - \omega|^{k+sp}} du d\mathcal{H}^{k-1}(\omega) \\
& \quad + \int_{S(0,1)} \int_{B_1(0)} \frac{|G_r(u) - G(u)|^p}{|u - \omega|^{k+sp}} du d\mathcal{H}^{k-1}(\omega) =: \text{I} + \text{II}.
\end{aligned}$$

We first estimate I. For $\varepsilon \leq r/4$, we have $G_{r,\varepsilon}(u) = G_r(u) = 0$ when $1 - r/2 \leq |u| \leq 1$, and thus, for fixed r ,

$$\text{I} \lesssim \frac{1}{r^{k+sp}} \int_{B_{1-r/2}(0)} |G_{r,\varepsilon}(u) - G_r(u)|^p du \rightarrow 0, \text{ as } \varepsilon \rightarrow 0.$$

We now turn to II. By the definition of G_r , we have

$$\begin{aligned}
\text{II} &= \int_{S(0,1)} \int_{B_1(0) \setminus B_{1-2r}(0)} \frac{|G_r(u) - G(u)|^p}{|u - \omega|^{k+sp}} du d\mathcal{H}^{k-1}(\omega) \\
&\lesssim \int_{S(0,1)} \int_{B_1(0) \setminus B_{1-2r}(0)} \frac{|G(u)|^p}{|u - \omega|^{k+sp}} du d\mathcal{H}^{k-1}(\omega) \rightarrow 0, \text{ as } r \rightarrow 0,
\end{aligned}$$

where we use the fact that $G \in W_{00}^{s,p}(B_1(0))$ and Lebesgue's lemma in the last line. $\square$

Proof of Lemmas 3.5 and 3.6. For sufficiently small ε , define

$$G_\varepsilon : \overline{B_1(0)} \rightarrow \mathcal{N}_\delta, G_\varepsilon(u) := \begin{cases} g * \rho_{\varepsilon(1-|u|)}(u/|u|), & \text{if } u \in B_1(0) \\ g(u), & \text{if } u \in S(0,1) \end{cases},$$

with the convolution defined as in (3.65).

Step 1. It holds $G_\varepsilon \rightarrow H(g)$ in $W^{s,p}(B_1(0))$.

We first note that, by change of variables,

$$\begin{aligned}
& |G_\varepsilon - H(g)|_{W^{s,p}(B_1(0))}^p \\
& \lesssim \int_{S(0,1)} \int_0^1 \int_{S(0,1)} \int_0^t (t\tilde{t})^{k-1} \frac{|K(\omega, \tilde{\omega}, t, \tilde{t})|^p}{|t\omega - \tilde{t}\tilde{\omega}|^{k+sp}} d\tilde{t} d\mathcal{H}^{k-1}(\tilde{\omega}) dt d\mathcal{H}^{k-1}(\omega),
\end{aligned}$$

with

$$K(\omega, \tilde{\omega}, t, \tilde{t}) := (g * \rho_{\varepsilon(1-t)}(\omega) - g(\omega)) - (g * \rho_{\varepsilon(1-\tilde{t})}(\tilde{\omega}) - g(\tilde{\omega})).$$

Writing $\tilde{t} = at$ with $a \in (0, 1)$, we obtain

$$\begin{aligned} & |g_\varepsilon - H(f)|_{W^{s,p}(B_1(0))}^p \\ & \lesssim \int_{S(0,1)} \int_0^1 \int_{S(0,1)} \int_0^1 a^{k-1} t^{k-1-sp} \frac{|\tilde{K}(\omega, \tilde{\omega}, t, a)|^p}{|\omega - a\tilde{\omega}|^{k+sp}} da d\mathcal{H}^{k-1}(\tilde{\omega}) dt d\mathcal{H}^{k-1}(\omega), \end{aligned} \quad (\text{A.1})$$

with

$$\tilde{K}(\omega, \tilde{\omega}, t, a) := g * \rho_{\varepsilon(1-t)}(\omega) - g(\omega) - (g * \rho_{\varepsilon(1-at)}(\tilde{\omega}) - g(\tilde{\omega})).$$

To estimate the right-hand side of (A.1), we decompose $\tilde{K} = \tilde{K}_1 + \tilde{K}_2$, where

$$\begin{aligned} \tilde{K}_1(\omega, \tilde{\omega}, t) &:= g * \rho_{\varepsilon(1-t)}(\omega) - g(\omega) - (g * \rho_{\varepsilon(1-t)}(\tilde{\omega}) - g(\tilde{\omega})), \\ \tilde{K}_2(\tilde{\omega}, t, a) &:= g * \rho_{\varepsilon(1-t)}(\tilde{\omega}) - g * \rho_{\varepsilon(1-at)}(\tilde{\omega}). \end{aligned}$$

By [12, (12)–(14)], we have

$$\begin{aligned} & \int_{S(0,1)} \int_{S(0,1)} \int_0^1 a^{k-1} \frac{|\tilde{K}_1(\omega, \tilde{\omega}, t)|^p}{|\omega - a\tilde{\omega}|^{k+sp}} da d\mathcal{H}^{k-1}(\tilde{\omega}) dt d\mathcal{H}^{k-1}(\omega) \\ & \lesssim \|g * \rho_{\varepsilon(1-t)} - g\|_{W^{s,p}(S(0,1))}^p. \end{aligned} \quad (\text{A.2})$$

Using (A.2), dominated convergence theorem, and the fact that (see Lemma B.1)

$$\|g * \rho_{\varepsilon(1-t)}\|_{W^{s,p}(S(0,1))} \lesssim \|g\|_{W^{s,p}(S(0,1))}^p$$

and

$$g * \rho_{\varepsilon(1-t)} \rightarrow g \text{ in } W^{s,p}(S(0,1)), \text{ as } \varepsilon \rightarrow 0,$$

we have

$$\begin{aligned} & \int_{S(0,1)} \int_0^1 \int_{S(0,1)} \int_0^1 a^{k-1} t^{k-1-sp} \frac{|\tilde{K}_1(\omega, \tilde{\omega}, t)|^p}{|\omega - a\tilde{\omega}|^{k+sp}} da d\mathcal{H}^{k-1}(\tilde{\omega}) dt d\mathcal{H}^{k-1}(\omega) \\ & \lesssim \int_0^1 t^{k-1-sp} \|g * \rho_{\varepsilon(1-t)} - g\|_{W^{s,p}(S(0,1))}^p dt \rightarrow 0, \text{ as } \varepsilon \rightarrow 0. \end{aligned} \quad (\text{A.3})$$

We next estimate the contribution of $\tilde{K}_2$. For this purpose, we first note that

$$\int_{S(0,1)} \frac{1}{|\omega - a\tilde{\omega}|^{k+sp}} d\mathcal{H}^{k-1}(\omega) \lesssim \frac{1}{|\tilde{\omega} - a\tilde{\omega}|^{sp+1}} = \frac{1}{(1-a)^{sp+1}}. \quad (\text{A.4})$$

Therefore, (A.4) implies that

$$\begin{aligned} & \int_{S(0,1)} \int_0^1 a^{k-1} \frac{|\tilde{K}_2(\tilde{\omega}, t, a)|^p}{|\omega - a\tilde{\omega}|^{k+sp}} da d\mathcal{H}^{k-1}(\omega) \\ & \lesssim \int_0^1 a^{k-1} \frac{|g * \rho_{\varepsilon(1-t)}(\tilde{\omega}) - g * \rho_{\varepsilon(1-at)}(\tilde{\omega})|^p}{(1-a)^{1+sp}} da. \end{aligned} \quad (\text{A.5})$$

Denote $\tilde{B}_r(u)$ the geodesic ball with radius r centered at u on $S(0,1)$. Using (A.5) and a similar calculation as in the proof of Lemma B.2 (with the notation as in Appendix B), we have

$$\begin{aligned} & \int_{S(0,1)} \int_{S(0,1)} \int_0^1 a^{k-1} \frac{|\tilde{K}_2(\tilde{\omega}, t, a)|^p}{|\omega - a\tilde{\omega}|^{k+sp}} da d\mathcal{H}^{k-1}(\omega) d\mathcal{H}^{k-1}(\tilde{\omega}) \\ & \lesssim \int_{S(0,1)} \int_0^1 \int_{B_1(0)} a^{k-1} \frac{|g \circ R^{\omega'} \circ \psi(\varepsilon t(1-a)u) - g(\omega')|^p}{(1-a)^{1+sp}} du da d\mathcal{H}^{k-1}(\omega') \\ & \lesssim \int_{S(0,1)} \int_0^1 \int_{\tilde{B}_{\varepsilon t(1-a)}(\omega')} \left(\frac{a}{\varepsilon t}\right)^{k-1} \frac{|g(\omega) - g(\omega')|^p}{(1-a)^{k+sp}} d\mathcal{H}^{k-1}(\omega) da d\mathcal{H}^{k-1}(\omega') \\ & \lesssim \int_{S(0,1)} \int_{S(0,1)} \int_{|\omega' - \omega|/(\varepsilon t)}^1 \left(\frac{1-b}{\varepsilon t}\right)^{k-1} \frac{|g(\omega) - g(\omega')|^p}{b^{k+sp}} db d\mathcal{H}^{k-1}(\omega) d\mathcal{H}^{k-1}(\omega') \\ & \lesssim \int_{S(0,1)} \int_{S(0,1)} (\varepsilon t)^{sp} \frac{|g(\omega) - g(\omega')|^p}{|\omega' - v|^{k-1+sp}} d\mathcal{H}^{k-1}(\omega) d\mathcal{H}^{k-1}(\omega'). \end{aligned}$$

Therefore, we have

$$\begin{aligned} & \int_{S(0,1)} \int_0^1 \int_{S(0,1)} \int_0^1 a^{k-1} t^{k-1-sp} \frac{|\tilde{K}_2(\tilde{\omega}, t, a)|^p}{|\omega - a\tilde{\omega}|^{k+sp}} da d\mathcal{H}^{k-1}(\tilde{\omega}) dt d\mathcal{H}^{k-1}(\omega) \\ & \lesssim \int_0^1 t^{k-1} \varepsilon^{sp} |g|_{W^{s,p}(S(0,1))}^p dt \rightarrow 0, \text{ as } \varepsilon \rightarrow 0. \end{aligned} \quad (\text{A.6})$$

Consequently, we obtain

$$|G_\varepsilon - H(g)|_{W^{s,p}(B_1(0))}^p \rightarrow 0$$

as $\varepsilon \rightarrow 0$ via (A.1), (A.3), and (A.6).

Step 2. Proof of (3.123).

As in the previous step, we find that

$$\begin{aligned}
& \int_{S(0,1)} \int_{B_1(0)} \frac{|H(g)(u) - G_\varepsilon(u)|^p}{|u - \omega|^{k+sp}} du d\mathcal{H}^{k-1}(\omega) \\
& \lesssim \int_{B_1(0)} \frac{|H(g)(u) - G_\varepsilon(u)|^p}{(1 - |u|)^{1+sp}} du \\
& \lesssim \int_0^1 \int_{S(0,1)} \frac{t^{k-1} |g(\omega) - g * \rho_{\varepsilon(1-t)}(\omega)|^p}{(1-t)^{1+sp}} d\mathcal{H}^{k-1}(\omega) dt \\
& \lesssim \int_0^1 \int_{S(0,1)} \int_{\tilde{B}_{\varepsilon(1-t)}(\omega)} \left(\frac{t}{\varepsilon}\right)^{k-1} \frac{|g(\omega) - g(\tilde{\omega})|^p}{(1-t)^{k+sp}} d\mathcal{H}^{k-1}(\tilde{\omega}) d\mathcal{H}^{k-1}(\omega) dt \\
& \lesssim \int_{S(0,1)} \int_{S(0,1)} \int_{|\omega - \tilde{\omega}|/\varepsilon}^1 \left(\frac{1-a}{\varepsilon}\right)^{k-1} \frac{|g(\omega) - g(\tilde{\omega})|^p}{a^{k+sp}} d\mathcal{H}^{k-1}(\tilde{\omega}) d\mathcal{H}^{k-1}(\omega) da \\
& \lesssim \varepsilon^{sp} |g|_{W^{s,p}(S(0,1))}^p \rightarrow 0, \text{ as } \varepsilon \rightarrow 0,
\end{aligned}$$

whence (3.123). This concludes the proof. $\square$

Proof of Corollary 3.3. Assume that $\pi_{k-1}(\mathcal{F}) \simeq \{0\}$. It suffices to repeat the proof of Theorems 3.1 and 3.2, and add the following extra lemma after Step 6.

Lemma A.1. *Let $0 < s \leq 1$ and $1 \leq sp < k$. Assume that $\pi_{k-1}(\mathcal{F}) \simeq \{0\}$. Consider a map $g \in \text{Lip}(S(0,1); \mathcal{F})$, and let $H(g)$ be the homogeneous extension of g . Then, for every $\alpha > 0$, there exists $\bar{g} \in \text{Lip}(\bar{B}_1(0); \mathcal{F})$ such that: (i) $\bar{g} = g$ on $S(0,1)$; and (ii) we have*

$$\|H(g) - \bar{g}\|_{W^{s,p}(B_1(0))} \leq \alpha.$$

The proof of Lemma A.1 is given below.

Conversely, assuming that $\text{Lip}(\mathcal{M}; \mathcal{F})$ is dense in $W^{s,p}(\mathcal{M}; \mathcal{F})$, it suffices to prove that, for any map $g \in C^\infty(S(0,1); \mathcal{F})$, there exists a map $\bar{g} \in C^\infty(\bar{B}_1(0); \mathcal{F})$ such that $\bar{g} = g$ on $S(0,1)$. We divide the proof into two steps.

Step 1. Any $\bar{f} \in W^{s,p}(B_1(0); \mathcal{F})$ satisfying $\bar{f} = c_{\bar{f}}$ in $B_1(0) \setminus K$ for some compact $K \subset B_1(0)$ and $c_{\bar{f}} \in \mathcal{F}$ can be approximated with maps in $C^\infty(B_1(0); \mathcal{F})$.

Indeed, let $2r < r_0$ with r_0 as in (2.2), define (using the same notation as in Section 3)

$$f: \mathcal{M} \rightarrow \mathcal{F}, f(x) := \begin{cases} \bar{f}(\psi_i^{-1}(x)), & \text{if } x \in \psi_i(B_r(0)) \\ c_f, & \text{otherwise} \end{cases}.$$

Clearly, $f \in W^{s,p}(\mathcal{M}; \mathcal{F})$ and by assumption, it can be approximated with maps in $\text{Lip}(\mathcal{M}; \mathcal{F})$. We conclude this step by restricting the approximating maps in $\psi_i(B_{2r}(0))$ and using a standard mollifying procedure.

Step 2. Construction of $\bar{f}$ via g .

Consider the following two subsets of $B_2^{k-1}(0) \times (-3, 3)$:

$$D_1 := B_1^k(2e_k) \bigcup \{(u, t) \in B_2^{k-1}(0) \times (-3, 3) : t \geq |u| + 1\}$$

and

$$D_2 := B_1^k(-2e_k) \bigcup \{(u, t) \in B_2^{k-1}(0) \times (-3, 3) : t \leq -|u| - 1\},$$

where e_k is the k -vector of the canonical basis of $\mathbb{R}^k$. Define

$$\bar{g}_1 : D_1 \rightarrow \mathcal{F}, \bar{g}(u, t) := \begin{cases} g((u, t - 2)/|(u, t - 2)|), & \text{if } (u, t) \in B_1(2e_k) \\ g(\sqrt{1 - (t - 2)^2}(u/|u|), t - 2), & \text{otherwise} \end{cases}$$

and

$$\bar{g}_2 : D_2 \rightarrow \mathcal{F}, \bar{g}(u, t) := \begin{cases} g((u, 2 - t)/|(u, 2 - t)|), & \text{if } (u, t) \in B_1(-2e_k) \\ g(\sqrt{1 - (t - 2)^2}(u/|u|), 2 - t), & \text{otherwise} \end{cases}.$$

For every $u \in B_2(0)$, define $t_1 = t_1(u) := \min\{t : (u, t) \in D_1\}$ and $t_2 = t_2(u) := \max\{t : (u, t) \in D_2\}$. By definition of $\bar{g}_1$ and $\bar{g}_2$, we have $\bar{g}_1(u, t) = \bar{g}_2(u, t_2)$. Therefore, we obtain a map $\bar{g}$ by letting

$$\bar{g}(u, t) := \begin{cases} \bar{g}_1(u, t), & \text{if } (u, t) \in D_1 \\ \bar{g}_2(u, t), & \text{if } (u, t) \in D_2 \\ \bar{g}_1(u, t_1) = \bar{g}_2(u, t_2), & \text{otherwise} \end{cases}$$

Clearly, the map $\bar{g}$ we construct belongs to $W^{s,p}(B_2(0) \times (-3, 3); \mathcal{F})$ and $\text{Lip}_{\text{loc}}(B_2(0) \times (-3, 3) \setminus \{2e_k, -2e_k\}; \mathcal{F})$ and is constant on the boundary.

This construction is reminiscent of the celebrated *dipole construction*. More specifically, here, we construct a dipole by gluing together two homogeneous extensions face to face, a filling in the rest of a rectangle containing them in a compatible way, in such a way that the map is constant on the boundary of the rectangle. An illustration can be found on Figure 1 below. All black lines are level lines of the function we construct. We start with the map g on both gray circles (the copy of g on the circle below is a reflected version of the copy on the circle above). It is then extended radially in both balls enclosed in the circles, with the cost of having a discontinuity at both centers. This way, on any smaller circle centered around one of those two points, the map we construct is a scaled copy of g (or of its reflection). On the rest of the gray triangles, it is then extended propagating

horizontally the values from the circles. Finally, the rest of the rectangle is filled in by a vertical extension from the boundary of the region that was yet to be filled in. On the boundary of the rectangle, the map we have constructed takes the value that is taken by g at the north pole.

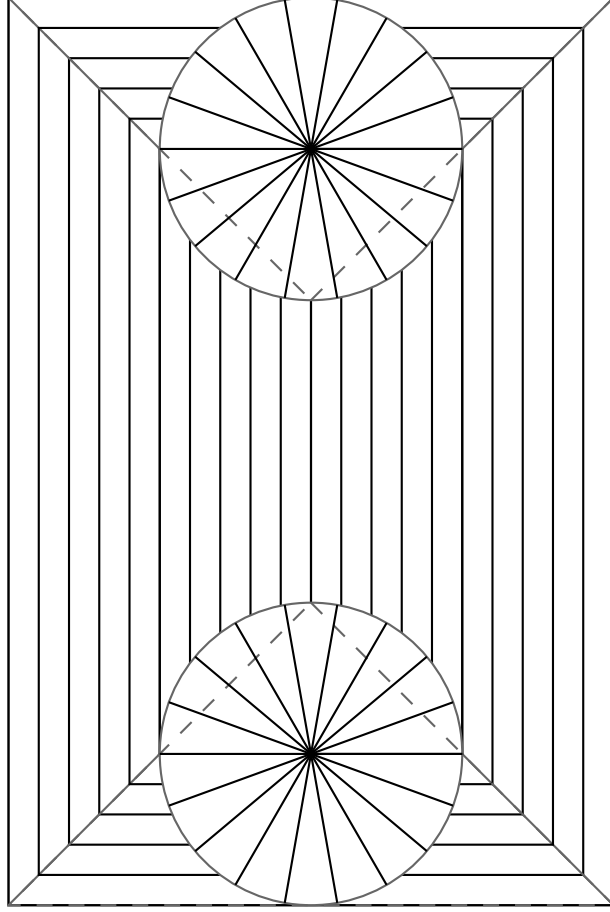


Figure 1: Dipole construction for a topological obstruction

By a scaling and constant extension argument, we obtain a map $\bar{f} \in W^{s,p}$ that satisfies the assumptions of the previous step. By Step 1, $\bar{f}$ can be approximated with smooth maps, which implies, by the construction of g_1 in $B_1(2e_k)$ and a standard slicing argument, that g is nullhomotopic. $\square$

Proof of Lemma A.1. We prove the case where $0 < s < 1$, the case where $s = 1$ can be treated similarly.

Since $\pi_{k-1}(\mathcal{F}) \simeq \{0\}$, there exists a map $\bar{h} \in \text{Lip}(\overline{B_1(0)}; \mathcal{F})$ such that $\bar{h} = g$ on $S(0, 1)$.

Let

$$\bar{g}_r: B_1(0) \rightarrow \mathcal{F}, \bar{g}_r(u) := \begin{cases} H(g)(u), & \text{if } r \leq |u| \leq 1 \\ \bar{h}(u/r), & \text{if } |u| \leq r \end{cases}.$$

We now estimate $|H(g) - \bar{g}_r|_{W^{s,p}}$. We first note that

$$\begin{aligned} |H(g) - \bar{g}_r|_{W^{s,p}(B_1(0))}^p &\lesssim |H(g)|_{W^{s,p}(B_r(0))}^p + |\bar{g}_r|_{W^{s,p}(B_r(0))}^p \\ &\quad + \int_{B_r(0)} \frac{|H(g)(u) - \bar{h}(u/r)|^p}{[\text{dist}(u, S(0, r))]^{sp}} du. \end{aligned} \quad (\text{A.7})$$

By changing variables, the first two of the last three terms in (A.7) converge to 0 as $r \rightarrow 0$. For the last term, we have

$$\begin{aligned} \int_{B_r(0)} \frac{|H(g)(u) - \bar{h}(u/r)|^p}{[\text{dist}(u, S(0, r))]^{sp}} du &= \int_{B_r(0)} \frac{|g(u/|u|) - \bar{h}(u/r)|^p}{[r - |u|]^{sp}} du \\ &\lesssim r^{k-sp} \int_{B_1(0)} \frac{|g(v/|v|) - \bar{h}(v)|^p}{[1 - |v|]^{sp}} dv \\ &\lesssim r^{k-sp} \int_{B_1(0)} [1 - |v|]^{(1-s)p} dv \rightarrow 0 \end{aligned}$$

as $r \rightarrow 0$, where the hidden constant in the last inequality depends on g and $\bar{h}$. This implies that $|H(g) - \bar{g}_r|_{W^{s,p}} \rightarrow 0$.

Using a similar argument, we obtain $H(g) \rightarrow \bar{g}_r$ in $\mathcal{L}^p$, and thus in $W^{s,p}$. The conclusion follows by taking r sufficiently small, depending on g itself. $\square$

B Mollifiers and maps on the sphere

In this section, we let $k \geq 2$ and $0 < s \leq 1$. Occasionally, we assume that $sp \geq k - 1$. Consider a map $h \in W^{s,p}(S(0, 1); \mathcal{F})$, with $S(0, 1)$ being the $(k - 1)$ -dimensional sphere embedded into $\mathbb{R}^k$. Our goal is to define a suitable convolution procedure, adapted to the setting of mappings defined *on* the sphere. This raises some challenges compared to the flat setting of mappings on a Euclidean space, but we are nevertheless able to take profit of the symmetry of the geometry of the sphere.

To start with, let $\mathcal{O}$ be the north pole of the sphere. Denote $\tilde{B}_r(\omega)$ the geodesic ball centered at ω and of radius r on $S(0, 1)$. Let ρ be a radial mollifier in $\mathbb{R}^{k-1}$ with support in $B_1(0)$.

Let $\psi: B_1(0) \rightarrow \tilde{B}_1(\mathcal{O})$ be a parametrization of $\tilde{B}_1(\mathcal{O})$ such that: (i) ψ is a diffeomorphism; (ii) $\text{dist}(\psi(u), \mathcal{O}) = |u|$ for any $u \in B_1(0)$; (iii) for any rotation R of $\mathbb{R}^k$ around

the axis $0x_k$, we have $R \circ \psi(u) = \psi(R(u, 0))$; and (iv) ψ maps any segment $[0, x)$, for $x \in \partial B_1(0)$, onto a great circle (which will therefore be a longitude).

For $\omega \in S(0, 1)$, let R^ω be any rotation of $\mathbb{R}^k$ around the origin that sends $\mathcal{O}$ to ω . Define, for $0 < t < 1$,

$$h_t(\omega) = h * \rho_t(\omega) := \int_{B_1(0)} h \circ R^\omega \circ \psi(u) \rho_t(u) \, du, \forall 0 < t \leq 1.$$

Clearly, the map h_t does not depend on the choice of the rotation R^ω , thanks to the properties of ψ and the fact that ρ is radial.

In what follows, the constants in “ $\lesssim$ ” do not depend on t .

Lemma B.1. *For $k \geq 2$ and $0 < s \leq 1$, there exists $0 < t_0 < 1$ sufficiently small, depending only on k , such that we have*

$$|h_t|_{W^{s,p}(S(0,1))} \lesssim |h|_{W^{s,p}(S(0,1))}, \forall 0 < t \leq t_0$$

and

$$h_t \rightarrow h \text{ in } W^{s,p}(S(0,1)) \text{ as } t \rightarrow 0.$$

Proof. We consider only the case where $0 < s < 1$; the case where $s = 1$ is similar. To estimate the $W^{s,p}$ seminorm, we start with the case where $\text{dist}(\omega, \omega') < 1/4$. Let $\tilde{R}_\omega^{\omega'}$ be the (unique) rotation of $\mathbb{R}^k$ around the $(k-2)$ -plane orthogonal to the plane $0\omega\omega'$ that sends ω to ω' via the (minimizing) geodesic between the two points. For any $\omega_1 \in S(0, 1)$, it holds

$$\text{dist}(\omega, \omega') = \text{dist}(\omega, \tilde{R}_\omega^{\omega'}(\omega)) \geq \text{dist}(\omega_1, \tilde{R}_\omega^{\omega'}(\omega_1)).$$

(This is readily seen on a picture. We omit the analytic proof.) We also, in the course of this proof, make a canonical choice for the rotation R^ω . More specifically, if $\mathcal{O}'$ denotes the south pole of the sphere and $\omega \in S(0, 1) \setminus \tilde{B}_1(\mathcal{O}')$, we let $\tilde{R}^\omega = \tilde{R}_\omega^\omega$. Given ω , we set $R^{\omega'} := \tilde{R}_\omega^{\omega'} \circ \tilde{R}^\omega$. Let us observe importantly that

$$\begin{aligned} \Theta: (S(0, 1) \setminus \tilde{B}_1(\mathcal{O}')) \times (S(0, 1) \setminus \tilde{B}_1(\mathcal{O}')) \times S(0, 1) &\rightarrow S(0, 1) \times S(0, 1), \\ (\omega, \omega', \omega_0) &\mapsto (\tilde{R}^\omega(\omega_0), \tilde{R}_\omega^{\omega'} \tilde{R}^\omega(\omega_0)) \end{aligned}$$

is a smooth map, with $\Theta(\cdot, \cdot, \mathcal{O}) = \text{id}_{(S(0,1) \setminus \tilde{B}_1(\mathcal{O}')) \times (S(0,1) \setminus \tilde{B}_1(\mathcal{O}'))}$. Therefore, there exists $\eta_1 > 0$, depending only on k , such that, for every $\omega_0 \in \tilde{B}_{\eta_1}(\mathcal{O})$, $\Theta(\cdot, \cdot, \omega_0)$ is a diffeomorphism onto its image, with Jacobian bounded from below by a constant independent of $\omega_0 \in B_{\eta_1}(\mathcal{O})$. (This relies on the fact that embeddings are an open set in the C^1 topology,

see [25, Theorem 1.4, Section 2.1], and on the fact that the Jacobian is continuous with respect to the C^1 topology.)

We write

$$h_t(\omega') = \int_{B_1(0)} h \circ R^{\omega'} \circ \psi(u) \rho_t(u) \, du = \int_{B_1(0)} h \circ \tilde{R}_{\omega'}^{\omega'} \circ \tilde{R}^{\omega} \circ \psi(u) \rho_t(u) \, du.$$

This implies that, choosing $t_0 \leq \eta_1$,

$$\begin{aligned} & \int_{S(0,1) \setminus B_1(\mathcal{O}')} \int_{\tilde{B}_{1/4}(\omega)} \frac{|h_t(\omega) - h_t(\omega')|^p}{|\omega - \omega'|^{k+sp}} \, d\mathcal{H}^{k-1}(\omega') \, d\mathcal{H}^{k-1}(\omega) \\ & \lesssim \int_{S(0,1) \setminus B_1(\mathcal{O}')} \int_{\tilde{B}_{1/4}(\omega)} \int_{B_t(0)} \frac{|h \circ \tilde{R}^{\omega} \circ \psi(u) - h \circ R^{\omega'} \circ \psi(u)|^p}{|\omega - \omega'|^{k+sp}} \, du \, d\mathcal{H}^{k-1}(\omega') \, d\mathcal{H}^{k-1}(\omega) \\ & \lesssim \int_{B_t(0)} \int_{S(0,1) \setminus B_1(\mathcal{O}')} \int_{\tilde{B}_{1/4}(\omega)} \frac{|h \circ \tilde{R}^{\omega} \circ \psi(u) - h \circ R^{\omega'} \circ \psi(u)|^p}{|\tilde{R}^{\omega} \circ \psi(u) - R^{\omega'} \circ \psi(u)|^{k+sp}} \, d\mathcal{H}^{k-1}(\omega') \, d\mathcal{H}^{k-1}(\omega) \, du \\ & \lesssim \int_{B_t(0)} \int_{S(0,1)} \int_{\tilde{B}_1(\tilde{\omega})} \frac{|h(\tilde{\omega}) - h(\tilde{\omega}')|^p}{|\tilde{\omega} - \tilde{\omega}'|^{k+sp}} \, d\mathcal{H}^{k-1}(\tilde{\omega}') \, d\mathcal{H}^{k-1}(\tilde{\omega}) \, du \lesssim |h|_{W^{s,p}(S(0,1))}^p. \end{aligned}$$

In the last line, we have changed variables as

$$(\tilde{\omega}, \tilde{\omega}') = \Theta(\omega, \omega', \psi(u)) = (\tilde{R}^{\omega}(\psi(u)), \tilde{R}_{\omega'}^{\omega'} \circ \tilde{R}^{\omega}(\psi(u))),$$

taking profit of the fact that Θ is a diffeomorphism with lower bounded Jacobian with respect to its two first variables for the choice of $\psi(u)$ as its last variable, since $\psi(u) \in B_{\eta_1}(\mathcal{O})$ for $u \in B_t(0)$ with $t \leq t_0 \leq \eta_1$.

Meanwhile, if R is any fixed rotation sending $\mathcal{O}$ to $\mathcal{O}'$, and if $\tilde{\tilde{R}}^{\omega}$ is a choice of rotation sending $\mathcal{O}'$ to ω , then we also have

$$h_t(\omega) = \int_{B_1(0)} h \circ \tilde{\tilde{R}}^{\omega} \circ R \circ \psi(u) \rho_t(u) \, du.$$

Repeating the above argument, swapping the roles of the north and south poles, as $R \circ \psi(u) \in B_{\eta_1}(\mathcal{O}')$ for $u \in B_t(0)$, we arrive at

$$\int_{S(0,1) \setminus B_1(\mathcal{O})} \int_{\tilde{B}_{1/4}(\omega)} \frac{|h_t(\omega) - h_t(\omega')|^p}{|\omega - \omega'|^{k+sp}} \, d\mathcal{H}^{k-1}(\omega') \, d\mathcal{H}^{k-1}(\omega) \lesssim |h|_{W^{s,p}(S(0,1))}^p.$$

On the other hand, when $\text{dist}(\omega, \omega') \geq 1/4$, using Poincaré's inequality, we have (with

$$c := \int_{S(0,1)} h$$

$$\begin{aligned}
& \int_{S(0,1)} \int_{S(0,1) \setminus \tilde{B}_{1/4}(\omega)} \frac{|h_t(\omega') - h_t(\omega)|^p}{|\omega' - \omega|^{k+sp}} d\mathcal{H}^{k-1}(\omega') d\mathcal{H}^{k-1}(\omega) \\
& \lesssim \int_{S(0,1)} |h_t(\omega) - c|^p d\mathcal{H}^{k-1}(\omega) \\
& \lesssim \int_{S(0,1)} \int_{B_t(0)} |h \circ \tilde{R}^\omega \circ \psi(u) - c|^p du d\mathcal{H}^{k-1}(\omega) \\
& \lesssim \int_{S(0,1)} \int_{\tilde{B}_t(\omega)} |h(\tilde{\omega}) - c|^p d\mathcal{H}^{k-1}(\tilde{\omega}) d\mathcal{H}^{k-1}(\omega) \\
& \lesssim \int_{S(0,1)} |h(\tilde{\omega}) - c|^p d\mathcal{H}^{k-1}(\tilde{\omega}) \lesssim |h|_{W^{s,p}(S(0,1))}^p.
\end{aligned}$$

Therefore, we have

$$|h_t|_{W^{s,p}(S(0,1))} \lesssim |h_t|_{W^{s,p}(S(0,1))}.$$

A similar calculation implies that

$$\|h_t\|_{\mathcal{L}^p(S(0,1))} \lesssim \|h_t\|_{\mathcal{L}^p(S(0,1))}.$$

We now turn to the approximation property of our mollification procedure. Due to the nonlinear setting in which we work, it does not seem straightforward to adapt the usual (simple) argument for mollifiers on a Euclidean space. Instead, we rely on an indirect argument.

We start with the following two classical observations: (i) $\text{Lip}(S(0,1))$ is dense in $W^{s,p}(S(0,1))$ (this can be shown by reducing to a linear setting *via* a partition of unity); and (ii) given the linear estimate we just proved for our mollification construction, it suffices to prove the approximation property on a dense class. Gathering both these observations, we can assume from now that $h \in \text{Lip}(\mathcal{M})$.

Let us first notice that, if $\text{dist}(\omega, \omega') \leq 1/4$, then $\text{dist}(\tilde{R}_\omega^{\omega'}(\omega_0), \omega_0) \lesssim \text{dist}(\omega, \omega')$ for every $\omega_0 \in S(0,1)$. Therefore, choosing any rotation R^ω for ω and choosing $\tilde{R}_\omega^{\omega'} \circ R^\omega$ for ω' , we obtain

$$|h_t(\omega') - h_t(\omega)| \leq \int_{B_t(0)} |h \circ \tilde{R}_\omega^{\omega'} \circ R^\omega \circ \psi(u) - h \circ R^\omega \circ \psi(u)| du \lesssim \text{dist}(\omega, \omega'),$$

where the hidden constant depends on the Lipschitz constant of h . The same obviously remains true for points that are further apart, invoking a finite covering argument. This shows that our mollifying procedure does not increase the Lipschitz seminorm, up to a

constant factor.

Meanwhile, we clearly have $h_t \rightarrow h$ uniformly on $S(0, 1)$ as $t \rightarrow 0$.

We now fix $\varepsilon > 0$, and define the set

$$A_{\varepsilon,t} := \{(\omega, \omega') \in S(0, 1) \times S(0, 1) : |h_t(\omega') - h(\omega')|^p + |h_t(\omega) - h(\omega)|^p \geq \varepsilon |\omega - \omega'|^{k+sp}\}.$$

We let

$$I(\omega, \omega') := \frac{|h_t(\omega') - h_t(\omega) - h(\omega') + h(\omega)|^p}{|\omega' - \omega|^{k+sp}}.$$

On $A_{\varepsilon,t}$, we use the estimate

$$I(\omega, \omega') \lesssim \frac{|h_t(\omega') - h_t(\omega)|^p}{|\omega' - \omega|^{k+sp}} + \frac{|h(\omega') - h(\omega)|^p}{|\omega' - \omega|^{k+sp}} \lesssim |\omega' - \omega|^{(1-s)p-k},$$

and on $(S(0, 1) \times S(0, 1)) \setminus A_{\varepsilon,t}$, we use the estimate

$$I(\omega, \omega') \lesssim \frac{|h_t(\omega') - h(\omega')|^p}{|\omega' - \omega|^{k+sp}} + \frac{|h_t(\omega) - h(\omega)|^p}{|\omega' - \omega|^{k+sp}} \leq \varepsilon.$$

Therefore, we obtain

$$|h_t - h|_{W^{s,p}(S(0,1))}^p \lesssim \varepsilon + \iint_{A_{\varepsilon,t}} |\omega' - \omega|^{(1-s)p-k}.$$

As $h_t \rightarrow h$ uniformly when $t \rightarrow 0$, we have $|A_{\varepsilon,t}| \rightarrow 0$ for every fixed $\varepsilon > 0$. Hence, by virtue of Lebesgue's lemma,

$$\limsup_{t \rightarrow 0} |h_t - h|_{W^{s,p}(S(0,1))}^p \lesssim \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $h_t \rightarrow h$ in $W^{s,p}(S(0, 1))$ as $t \rightarrow 0$ — the $\mathcal{L}^p$ convergence is directly implied by the uniform convergence. This concludes the proof. $\square$

Next, provided that the projection below makes sense, we define

$$F : S(0, 1) \times (0, t_0) \rightarrow \mathcal{F}, F(\omega, t) := \Pi \circ h_t(\omega).$$

We now study the oscillation of $F(\cdot, t)$, which plays a crucial role in Sections 3.3–3.5.

Using Poincaré's inequality, we have (with $c := \int_{S(0,1)} h$)

$$\begin{aligned}
& |h_t(\omega) - h_t(\omega')| \\
& \lesssim \int_{B_t(0)} |h \circ R^\omega \circ \psi(u) - h \circ R^{\omega'} \circ \psi(u)| \rho_t(u) \, du \\
& \lesssim \left(\int_{B_t(0)} |h \circ R^\omega \circ \psi(u) - c|^p \, du \right)^{1/p} + \left(\int_{B_t(0)} |h \circ R^{\omega'} \circ \psi(u) - c|^p \, du \right)^{1/p} \quad (\text{B.1}) \\
& \lesssim \left(\frac{1}{t^{k-1}} \int_{S(0,1)} |h(\tilde{\omega}) - c|^p \, d\mathcal{H}^{k-1}(\tilde{\omega}) \right)^{1/p} \lesssim t^{-(k-1)/p} |h|_{W^{s,p}(S(0,1))}.
\end{aligned}$$

The third property is the following estimate of the distance between h_t and $\mathcal{F}$. For $\omega \in S(0,1)$, using the fact that $\int_{\mathbb{R}^{k-1}} \rho_t = 1$ and a calculation similar to (B.1) (with $c := \int_{B_t(0)} h \circ R^\omega \circ \psi$), we have

$$\begin{aligned}
& \text{dist}(h_t(\omega), \mathcal{F}) \\
& \lesssim \int_{\tilde{B}_t(\omega)} |h_t(\omega) - h(\tilde{\omega})| \, d\mathcal{H}^{k-1}(\tilde{\omega}) \\
& \lesssim \int_{B_t(0)} \int_{B_t(0)} |h \circ R^\omega \circ \psi(u) - h \circ R^\omega \circ \psi(v)| \, du \, dv \quad (\text{B.2}) \\
& \lesssim \left(\int_{B_t(0)} |h \circ R^\omega \circ \psi(u) - c|^p \, du \right)^{1/p} \lesssim t^{s-(k-1)/p} |h|_{W^{s,p}(S(0,1))}.
\end{aligned}$$

As a consequence, if we further assume that $sp \geq k-1$, then there exists a sufficiently small t_0 (depending on h), such that F is well-defined.

The last property is the estimate of $|F|_{W^{s,p}(S(0,1) \times (0,t_0); \mathcal{F})}$.

Lemma B.2. *Assume that $sp \geq k-1$. We have*

$$|F|_{W^{s,p}(S(0,1) \times (0,t_0); \mathcal{F})} \lesssim t_0^{1/p} |h|_{W^{s,p}(S(0,1); \mathcal{F})}. \quad (\text{B.3})$$

Proof. We consider only the case where $0 < s < 1$. We first note the following slicing

formula, which relies on Tonelli's theorem:

$$\begin{aligned}
& |F|_{W^{s,p}(S(0,1) \times (0,t_0))}^p \\
& \lesssim \int_{S(0,1)} \int_0^{t_0} \int_{S(0,1)} \int_0^{t_0} \frac{|F(\omega, t) - F(\omega', t')|^p}{[|\omega - \omega'| + |t - t'|]^{k+sp}} dt' d\mathcal{H}^{k-1}(\omega') dt d\mathcal{H}^{k-1}(\omega) \\
& \lesssim \int_0^{t_0} \left(\int_{S(0,1)} \int_{S(0,1)} \int_0^{t_0} \frac{|F(\omega, t') - F(\omega', t')|^p}{[|\omega - \omega'| + |t - t'|]^{k+sp}} dt d\mathcal{H}^{k-1}(\omega') d\mathcal{H}^{k-1}(\omega) \right) dt' \\
& \quad + \int_{S(0,1)} \left(\int_0^{t_0} \int_0^{t_0} \int_{S(0,1)} \frac{|F(\omega, t) - F(\omega, t')|^p}{[|\omega - \omega'| + |t - t'|]^{k+sp}} d\mathcal{H}^{k-1}(\omega') dt' dt \right) d\mathcal{H}^{k-1}(\omega) \\
& \lesssim \int_0^{t_0} \left(\int_{S(0,1)} \int_{S(0,1)} \frac{|F(\omega, t') - F(\omega', t')|^p}{|\omega - \omega'|^{k-1+sp}} d\mathcal{H}^{k-1}(\omega') d\mathcal{H}^{k-1}(\omega) \right) dt' \\
& \quad + \int_{S(0,1)} \left(\int_0^{t_0} \int_0^{t_0} \frac{|F(\omega, t) - F(\omega, t')|^p}{|t - t'|^{1+sp}} dt' dt \right) d\mathcal{H}^{k-1}(\omega) \\
& = \int_0^{t_0} |F(\cdot, t')|_{W^{s,p}(S(0,1))}^p dt' + \int_{S(0,1)} |F(\omega, \cdot)|_{W^{s,p}((0,t_0))}^p d\mathcal{H}^{k-1}(\omega).
\end{aligned}$$

In view of Lemma B.1, in order to conclude the proof, it suffices to control the last term in the above formula by $t_0 |h|_{W^{s,p}(S(0,1))}^p$. For this purpose, we first note that, when $\omega \in S(0, 1)$, we have

$$\begin{aligned}
& |h_{t'}(\omega) - h_t(\omega)| \\
& = \left| \int_{B_{t'}(0)} h \circ \tilde{R}^\omega \circ \psi(u) \rho_{t'}(u) du - \int_{B_t(0)} h \circ \tilde{R}^\omega \circ \psi(u) \rho_t(u) du \right| \\
& = \left| \int_{B_1(0)} h \circ \tilde{R}^\omega \circ \psi(t'v) \rho(v) dv - \int_{B_1(0)} h \circ \tilde{R}^\omega \circ \psi(tv) \rho(v) dv \right| \\
& \lesssim \int_{B_1(0)} |h \circ \tilde{R}^\omega \circ \psi(t'v) - h \circ \tilde{R}^\omega \circ \psi(tv)| dv.
\end{aligned}$$

We now observe that

$$\tilde{R}^\omega \circ \psi(t'v) = \tilde{R}_\omega^{R^\omega \circ \psi(tv)} \circ \tilde{R}^\omega \circ \psi((t' - t)v) \tag{B.4}$$

for every $v \in B_1(0)$ and $0 \leq t < t' \leq 1$. Indeed, all the points $\psi(sv)$ for $0 \leq s \leq 1$ are aligned along a common great circle by assumption on ψ . Therefore, since rotations preserve great circles, they are mapped into another great circle by $\tilde{R}^\omega$, that passes through ω since $\mathcal{O} = \psi(0)$ is mapped to ω by $\tilde{R}^\omega$. Now, the point $\tilde{R}^\omega \circ \psi((t' - t)v)$ is a point which lies along this great circle at distance $(t' - t)|v|$ from ω , as rotations preserve geodesic distances and $\psi((t' - t)v)$ lies at distance $(t' - t)|v|$ from 0. Again by

construction, the rotation $\tilde{R}_\omega^{R^\omega \circ \psi(tv)}$ preserves this great circle, and moves points on it by a distance $\text{dist}(\omega, R^\omega \circ \psi(tv)) = t|v|$ away from ω . Therefore, $\tilde{R}_\omega^{R^\omega \circ \psi(tv)} \circ \tilde{R}^\omega \circ \psi((t' - t)v)$ is the point on this great circle at distance $(t' - t)|v| + t|v| = t'|v|$ away from ω . But this point is nothing else than $\tilde{R}^\omega \circ \psi(t'v)$, by virtue of the exact same reasoning as for $\tilde{R}^\omega \circ \psi((t' - t)v)$, which concludes the proof of (B.4).

Using: (j) observation (B.4); (jj) the change of variables $\tilde{\omega} = \tilde{R}^\omega \circ \psi(tv)$, which has lower bounded Jacobian (arguing as in the proof of Lemma B.1), and letting $\tilde{R}^{\tilde{\omega}} = \tilde{R}_\omega^{\tilde{\omega}} \circ \tilde{R}^\omega$; (jjj) the fact that the quantity in the third line below is independent of the special choice of $\tilde{R}^{\tilde{\omega}}$; and (jjjj) Tonelli's theorem, we obtain

$$\begin{aligned}
& \int_{S(0,1) \setminus \tilde{B}_1(\mathcal{O})} \int_0^{t_0} \int_0^{t_0} \frac{|h_{t'}(\omega) - h_{t''}(\omega)|^p}{|t - t'|^{1+sp}} dt' dt d\mathcal{H}^{k-1}(\omega) \\
& \lesssim \int_{B_1(0)} \int_0^{t_0} \int_0^{t_0} \int_{S(0,1) \setminus \tilde{B}_1(\mathcal{O})} \frac{|h \circ \tilde{R}^\omega \circ \psi(t'v) - h \circ \tilde{R}^\omega \circ \psi(tv)|^p}{|t - t'|^{1+sp}} \\
& \quad \times d\mathcal{H}^{k-1}(\omega) dt' dt dv \\
& \lesssim \int_{B_1(0)} \int_0^{t_0} \int_0^{t_0} \int_{S(0,1)} \frac{|h \circ \tilde{R}^{\tilde{\omega}} \circ \psi((t' - t)v) - h(\tilde{\omega})|^p}{|t - t'|^{1+sp}} d\mathcal{H}^{k-1}(\tilde{\omega}) dt' dt dv \\
& \lesssim \int_{S(0,1)} \int_0^{t_0} \int_0^{t_0} \int_{B_{(t-t')(0)}} \frac{|h \circ R^{\tilde{\omega}} \circ \psi(u) - h(\tilde{\omega})|^p}{|t - t'|^{k+sp}} du dt' dt d\mathcal{H}^{k-1}(\tilde{\omega}) \\
& \lesssim t_0 \int_{S(0,1)} \int_{B_{t_0}(0)} \frac{|h \circ R^{\tilde{\omega}} \circ \psi(u) - h(\tilde{\omega})|^p}{|u|^{k-1+sp}} du d\mathcal{H}^{k-1}(\tilde{\omega}) \\
& \lesssim t_0 \int_{S(0,1)} \int_{\tilde{B}_{t_0}(\tilde{\omega})} \frac{|h(\omega) - h(\tilde{\omega})|^p}{|\omega - \tilde{\omega}|^{k-1+sp}} d\mathcal{H}^{k-1}(\omega) d\mathcal{H}^{k-1}(\tilde{\omega}) \leq t_0 |h|_{W^{s,p}(S(0,1))}^p.
\end{aligned}$$

(The change from $\tilde{R}^{\tilde{\omega}}$ to $R^{\tilde{\omega}}$ is required as otherwise there would still be a dependency in ω via $\tilde{R}_\omega^{\tilde{\omega}}$.) The part $S(0,1) \setminus \tilde{B}_1(\mathcal{O}')$ can be estimated similarly, as in the proof of Lemma B.1. Thus, we obtain the desired estimate (B.3). $\square$

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