A complete answer to the strong density problem for Sobolev spaces with values into manifolds

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Sobolev spaces with values into manifolds

Let $\mathcal{N}$ be a smooth Riemannian manifold, compact, connected, without boundary, isometrically embedded in $\mathbb{R}^\nu$.
Let $\Omega \subset \mathbb{R}^m$ be a smooth bounded open set, $1 \leq p < +\infty$, $0 < s < +\infty$.

Definition

$$W^{s,p}(\Omega; \mathcal{N}) = \{u \in W^{s,p}(\Omega; \mathbb{R}^\nu) : u(x) \in \mathcal{N} \quad \text{for almost every } x \in \Omega\}$$

Reminder : classical Sobolev spaces

Let $s = k + \sigma$ with $k \in \mathbb{N}$ and $\sigma \in [0, 1)$.

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^j u \in L^p(\Omega) \quad \text{for every } j \in \{1, \ldots, k\}\}$$

If $\sigma \in (0, 1)$,

$$W^{\sigma,p}(\Omega) = \left\{u \in L^p(\Omega) : \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{m+\sigma p}} \,\mathrm{d}x\mathrm{d}y < +\infty\right\}.$$

If $k \geq 1$,

$$W^{s,p}(\Omega) = \{u \in W^{k,p}(\Omega) : D^k u \in W^{\sigma,p}(\Omega)\}.$$

The strong density problem

Theorem

$C^\infty(\overline{\Omega})$ is dense in $W^{s,p}(\Omega)$

Question

Is $C^\infty(\overline{\Omega}; \mathcal{N})$ dense in $W^{s,p}(\Omega; \mathcal{N})$?

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A topological obstruction

For $1 \leq p < 2$, the map $u \in W^{1,p}(\mathbb{B}^2; \mathbb{S}^1)$ defined by

$$u(x) = \frac{x}{|x|}$$

cannot be approached by maps in $C^\infty(\overline{\mathbb{B}}^2; \mathbb{S}^1)$.

Theorem (Schoen-Uhlenbeck (1983), Bethuel-Zheng (1988), Escobedo (1988))

Assume that $sp < m$. If $C^\infty(\overline{\Omega}; \mathcal{N})$ is dense in $W^{s,p}(\Omega; \mathcal{N})$, then $\pi_{[sp]}(\mathcal{N}) = \{0\}$.

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The easy case $sp \geq m$

Proposition (Schoen-Uhlenbeck (1983), Brezis-Nirenberg (1995))

If $sp \geq m$, then $C^\infty(\overline{\Omega}; \mathcal{N})$ is dense in $W^{s,p}(\Omega; \mathcal{N})$.

There exists $\iota > 0$ such that the *nearest point projection* $\Pi \colon \mathcal{N} + B^\nu_\iota \to \mathcal{N}$ is well-defined and smooth.

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The strong density problem : $sp < m$

Question

Is $\mathcal{C}^\infty(\overline{\Omega}; \mathcal{N})$ dense in $W^{s,p}(\Omega; \mathcal{N})$ when $\pi_{[sp]}(\mathcal{N}) = \{0\}$?

Yes (when $\Omega = Q^m$) if

- $s = 1$ (Bethuel (1991));
- $s \in \mathbb{N}_*$ (Bousquet-Ponce-Van Schaftingen (2015));
- $0 < s < 1$ (Brezis-Mironescu (2015)).

Remaining case : $s > 1$, $s \notin \mathbb{N}_*$.

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A unified proof of this result for the whole range $0 < s < +\infty$ *via* the method of good and bad cubes.

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Good and bad cubes

Good cubes are all cubes such that

$$\frac{1}{\eta^{\frac{m}{sp}-1}}\|Du\|_{L^{sp}(\sigma^m)} \leq c\iota.$$

Other cubes are called bad cubes.

Let $\ell = [sp]$. The union of ℓ-faces of bad cubes is denoted by U_η^ℓ.

Thank you for your attention !