Pullback of closed forms by low regularity maps to manifolds, and applications

Antoine Detaille, Petru Mironescu, and Kai Xiao

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Abstract

For maps f in the Sobolev space $W^{1,k}(\mathbb{B}^N;\mathscr{N})$, with $\mathscr{N}$ a closed manifold, Bethuel, Coron, Demengel, and Hélein highlighted the importance, in approximation problems, of the pullbacks $f^*\omega$ of smooth closed k-forms ω on $\mathscr{N}$. When $\mathscr{N}$ is a sphere-like manifold and $k \le p < k+1$, they proved that a $W^{1,p}$ map to $\mathscr{N}$ can be strongly approximated with smooth maps to $\mathscr{N}$ if and only if all its corresponding pullbacks are closed currents. We extend this result to $W^{s,p}$ maps, with $0 < s < 1$. In the process, we adapt the Brezis–Nirenberg theory of homotopical invariants to VMO maps on metric measure spaces, establish the existence and some main properties of integral invariants for VMO maps on Lipschitz manifolds, prove the existence of distributional pullbacks by fractional Sobolev maps and obtain some of their properties, including various slicing formulas, and characterize the closure of smooth maps in terms of restrictions on generic skeletons.

1 Introduction

The topics we investigate here are related to the matter of the strong density of smooth maps in Sobolev spaces to manifolds. To fix the ideas, consider the space

$$W^{s,p}(\mathbb{B}^N;\mathscr{N}) := \{f \in W^{s,p}(\mathbb{B}^N)\colon f(x) \in \mathscr{N}\},$$

AD, PM & KX. Universite Claude Bernard Lyon 1, CNRS, Centrale Lyon, INSA Lyon, Université Jean Monnet, ICJ UMR5208, 69622 Villeurbanne, France
AD antoine.detaille@univ-lyon1.fr
PM mironescu@math.univ-lyon1.fr
KX xiao@math.univ-lyon1.fr
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where $\mathscr{N}$ is an embedded closed manifold. In general, $C^\infty(\overline{\mathbb{B}}^N;\mathscr{N})$ *is not* dense in $W^{s,p}(\mathbb{B}^N;\mathscr{N})$. This observation goes back to Schoen and Uhlenbeck [62], who noticed that, e.g., the map $x \mapsto x/|x|$ belongs to the space $W^{1,2}(\mathbb{B}^3;\mathbb{S}^2)$ but cannot be approximated, in this space, with smooth $\mathbb{S}^2$-valued maps. This raises two natural questions: (Q1) characterize $(s, p, N, \mathscr{N})$ such that strong density holds; (Q2) if strong density does not hold, characterize the $W^{s,p}$ maps which can be strongly approximated with smooth maps.

The first remarkable contribution in connection with (Q1) is due to Bethuel [5], who proved that, in $W^{1,p}(\mathbb{B}^N;\mathscr{N})$, there is strong density of $C^\infty(\overline{\mathbb{B}}^N;\mathscr{N})$ if and only if: (i) either $p \geq N$; (ii) or $p < N$ and the homotopy group $\pi_{\lfloor p \rfloor}(\mathscr{N})$ is trivial. Several subsequent contributions (Bethuel and Zheng [9], Escobedo [31], Hajłasz [40], Bethuel [6], Rivière [60], Bousquet [13], Mucci [52], Bousquet, Ponce, and Van Schaftingen [15, 16, 17], Brezis and Mironescu [21], and Detaille [27]) led to the following final answer to (Q1): $C^\infty(\overline{\mathbb{B}}^N;\mathscr{N})$ is strongly dense in $W^{s,p}(\mathbb{B}^N;\mathscr{N})$ if and only if: (i) either $sp \geq N$; (ii) or $sp < N$ and the homotopy group $\pi_{\lfloor sp \rfloor}(\mathscr{N})$ is trivial. The answer is also known when $\mathbb{B}^N$ is replaced with a general smooth bounded domain in $\mathbb{R}^N$ (Hang and Lin [42], Detaille [27]).

Concerning (Q2), the picture is not yet complete. Again, the first significant contribution is due to Bethuel [4], who considered maps $f \in W^{1,2}(\mathbb{B}^3;\mathbb{S}^2)$. For such maps, one can define the distributional Jacobian (in the sense of Ball [3]), $\operatorname{Jac} f$, which can be interpreted as the exterior differential $\mathrm{d}[f^*\omega]$ of the pullback $f^*\omega$ of a volume form ω on $\mathbb{S}^2$. The main theorem in [4] asserts that f can be strongly approximated with smooth $\mathbb{S}^2$-valued maps if and only if $\operatorname{Jac} f = 0$. A far-reaching generalization of this result was announced by Bethuel, Coron, Demengel, and Hélein [8]. A fruitful contribution of [8] is to highlight the important role played by the pullback of forms by Sobolev maps. The relevant object here is $f^*\omega$, where ω is a smooth k-form on $\mathscr{N}$ and $f \in W^{1,k}(\mathbb{B}^N;\mathscr{N})$; clearly, this is a k-form with $\mathscr{L}^1$ coefficients. A second significant contribution was to coin the importance of the following topological assumption on $\mathscr{N}$:

$$\left[\int_{\mathbb{S}^k} g^*\omega = 0, \forall \text{ smooth closed } k\text{-form } \omega \text{ on } \mathscr{N}\right] \implies g \text{ is nullhomotopic.} \tag{A}$$

The main result in [8] asserts that, under the assumptions: (i) $k \leq p < k+1 \leq N$; (ii) the closed manifold $\mathscr{N}$ satisfies (A), a map $f \in W^{1,p}(\mathbb{B}^N;\mathscr{N})$ can be strongly approximated with smooth $\mathscr{N}$-valued maps if and only if, for each smooth *closed* k-form on $\mathscr{N}$, we have $\mathrm{d}[f^*\omega] = 0$ in the sense of distributions. (More precisely, in the sense of currents.)

In [8], the authors present the main lines of proof of the above result. One of its main ingredients is the characterization of strongly approximable maps *via* the homotopy type of their restrictions to "generic" k-dimensional skeletons. This type of characterization has been subsequently formalized by Hang and Lin [42]. However, a rigorous proof of the validity of such characterizations (in the $W^{1,p}$ setting) has only been achieved very recently by Bousquet, Ponce, and Van Schaftingen [18]. This leads to a full proof of the results announced in [8].

In our work, we obtain, in fractional Sobolev spaces $W^{s,p}$ with $0 < s < 1$, full counterparts of the above described results. In addition to the aforementioned difficulties, we have to cope with the fact that the pullback $f^*\omega$ has no obvious meaning when $s < 1$.

We next describe our contributions and how they fit together to prove our main result.

VMO *and homotopy.* Brezis and Nirenberg [24] carried out a systematic study of the homotopy classes naturally associated with $\mathrm{VMO}(\mathscr{M};\mathscr{N})$, where $\mathscr{M}$, respectively $\mathscr{N}$, is a smooth compact manifold, respectively closed manifold. In our setting, a relevant $\mathscr{M}$ is the boundary of a cube. In Section 2, we establish the counterparts of the results in [24] in the rather general case where $\mathscr{M}$ is a compact metric measure space with a doubling measure. This seems a natural generalization and we hope that it is of independent interest. In particular, the mollifiers that we construct may prove useful in other contexts.

Integral invariants. In Section 3, we consider non-smooth versions of the integral invariants of the form $\mathscr{I}(f) := \int_{\mathscr{M}} f^*\omega$, where $\mathscr{M}$ and $\mathscr{N}$ are smooth closed manifolds, $f\colon \mathscr{M} \to \mathscr{N}$ is smooth, $\mathscr{M}$ is k-dimensional, and ω is a smooth closed k-form on $\mathscr{N}$. In the smooth case, it is well-known that this is a homotopical invariant acting on de Rham cohomology classes. We extend this result to Lipschitz closed manifolds $\mathscr{M}$. Here, we opted for a completely elementary approach, avoiding geometric measure theory language and tools. We hope that making this part of the text low tech and essentially self-contained was worth a few extra pages.

Estimates for $\mathscr{I}(f)$. A first major difficulty in the proof of the main theorem arises in the estimate of $\mathscr{I}(f)$. When $\mathscr{M} = \mathscr{N} = \mathbb{S}^k$ and $f \in W^{s,p}$, with $sp = k$, this has been obtained in [11]. In Section 4, we extend the result in [11] to general $\mathscr{M}$ and $\mathscr{N}$.

The distribution $f^*\omega$. In Section 5, we investigate whether one can naturally associate a distribution with $f^*\omega$. This topic has been originally addressed by Brezis and Nguyen [23] when $\mathscr{M} = \mathscr{N} = \mathbb{S}^k$ and ω is the standard volume form. We obtain counterparts of their results in the general case. We hope that this provides a muggle's approach to some "magical" identities in [23]. This route will be further pursued in [29].

A higher dimensional version of $\mathscr{I}(f)$. A second major difficulty in the proof of the main theorem is related to the definition of the exterior differential $\mathrm{d}[f^*\omega]$ when $\dim \mathscr{M} > k$.

(In our case, $\mathscr{M}$ is typically a ball of dimension $> k$.) Unlike the analysis in Sections 2 and 3, which naturally involves VMO maps, in this setting the right regularity of maps is Sobolev. Our first main result in Section 6 provides, roughly speaking, a *robust* definition for $d[f^*\omega]$ when $f \in W^{s,p}(\mathscr{M};\mathscr{N})$, $\dim \mathscr{M} > k$, and $sp = k$. This generalizes a result in [11], which corresponds to $\mathscr{M} = \mathbb{S}^{k+1}$, $\mathscr{N} = \mathbb{S}^k$, and ω the standard volume form on $\mathbb{S}^k$. (See also [41, 14].) A similar direction of research was also investigated, using the language of geometric measure theory, by Giaquinta, Modica, and Souček [37] for $W^{1/2,2}$ maps with values into $\mathbb{S}^1$, by Giaquinta and Mucci [38] for $W^{1/2,2}$ maps into more general targets, and by Mucci [53] for $W^{1/p,p}$ maps with $p > 1$. These latter contributions are in line with the theory of *Cartesian currents*, developed by Giaquinta, Modica, and Souček in $W^{1,p}$, and culminating with the monograph [35, 36]. Our approach is purely analytical and avoids geometric measure theory.

We next show that, at least when f is sufficiently nice, $d[f^*\omega]$ encodes the singular set of f and the topology carried by the singularities. The first result of this type is due to Brezis, Coron, and Lieb [19]. For other similar results, see Jerrard and Soner [45, Theorem 1.2], Alberti, Baldo, and Orlandi [1, Theorem 3.8], and Bousquet [13, Proposition 1]. The result we prove was initially obtained by Giaquinta, Modica, and Souček [36, Section 4.2, Theorem 1]. However, the reader may find instructive our different approach, relying only on an iterated use of the Stokes formula.

Slicing. A third major difficulty arises from the disintegration of $d[f^*\omega]$. When $f \in W^{1,k}$, a simple application of the Fubini theorem allows to recover $d[f^*\omega]$ from its $(k+1)$-dimensional slices. In the fractional Sobolev setting, a similar disintegration formula was obtained Mironescu, Russ, and Sire [51, Lemma 3.12] when $\mathscr{N} = \mathbb{S}^1$ and ω is the standard volume form. In Section 6.4, we prove such a formula in the general case.

A first answer to (Q2). In this context, it is more convenient to work with maps defined on $\mathbb{R}^N$ (with $0 < s < 1$ and $1 \le k \le sp < k+1 \le N$). A main result in Section 6.5, Theorem 6.16, asserts that a map $f \in W^{s,p}(\mathbb{R}^N;\mathscr{N})$ is approximable with smooth $\mathscr{N}$-valued maps if and only if, on "sufficiently many" grids, its restriction to the boundaries of $(k+1)$-dimensional cubes is nullhomotopic. This relies on approximation techniques devised in [21]. A specific feature of the case $0 < s < 1$ (as opposed to the case where s is an integer, investigated in [18]), is the conceptually simpler approach for strong density proposed in [21], which substantially simplifies our task, especially when we have to quantify the notion of "genericity".

Providing a rigorous proof of Theorem 6.16 is one of the main contributions of our work.

A second answer to (Q2). In Section 6.6, we prove the fractional counterpart of the main result in [8]. More specifically, we prove that, if (i) $1 \le k \le sp < k+1 \le N$; (ii) the closed manifold $\mathscr{N}$ satisfies (A), a map $f \in W^{s,p}(\mathbb{B}^N; \mathscr{N})$ can be strongly approximated with smooth $\mathscr{N}$-valued maps if and only if, for each smooth *closed* k-form on $\mathscr{N}$, we have $\mathrm{d}[f^*\omega] = 0$ in the sense of distributions.

The proof follows the strategy in [8] and relies on all the above analytical tools and results. Its three main steps are: *Step 1.* Starting from *higher-dimensional integral invariants* and using a dimensional reduction relying on *slicing*, we determine the *integral invariants* on the boundaries of $(k+1)$-dimensional cubes. *Step 2.* Using assumption (A) and the value of the integral invariants computed in the first step, we obtain a *homotopical information* on the restrictions of f to the boundaries of $(k+1)$-dimensional cubes. *Step 3.* We conclude using the homotopical information obtained in Step 2 and the *first answer to* (Q2).

When $\mathscr{N} = \mathbb{S}^k$, the above result takes the following simpler form: a map $f \in W^{s,p}(\mathscr{M}; \mathbb{S}^k)$ is approximable with smooth $\mathbb{S}^k$-valued maps if and only if $\operatorname{Jac} f = 0$, where $\operatorname{Jac} f$ is the distributional Jacobian introduced in [11] and [14]. This result was announced in Mucci [54]. As in our approach, the proof in [54] follows the main lines in Bethuel, Coron, Demengel, and Hélein [8], with a sketch of the slicing argument.

About assumption (A). Assumptions in the spirit of (A) are crucial in various contributions subsequent to [8], including, but not only, Giaquinta, Modica, and Souček [35, 36], Pakzad and Rivière [57], Giaquinta and Mucci [39], Canevari and Orlandi [26], and Bousquet, Ponce, and Van Schaftingen [18]. In Appendix A, we clarify how assumption (A) compares with the ones in the aforementioned references.

2 Homotopy classes of VMO maps on doubling metric measure spaces

In this section, with no claim of originality: (a) $\mathscr{M}$ is a compact doubling metric measure space (see below); (b) $\mathscr{N}$ is a closed manifold. We carefully adapt to this setting the results of Brezis and Nirenberg [24] concerning the existence and some basic properties of the homotopy classes of the space $\mathrm{VMO}(\mathscr{M}; \mathscr{N})$. (In [24, 25], $\mathscr{M}$ is a compact manifold.)

More specifically, we assume that $\mathscr{M}$ is a compact metric space endowed with a non-trivial (finite) Borel measure μ satisfying the doubling property

$$\exists\, C_{\mathscr{M}} < \infty \text{ such that } 0 < \frac{\mu(B_{2r}(x))}{\mu(B_r(x))} \le C_{\mathscr{M}},\ \forall\, x \in \mathscr{M}, \forall\, r > 0. \tag{2.1}$$

(The balls we consider are open, but we could have also considered closed balls.)

The prototypical example of $\mathscr{M}$ we have in mind is $\mathscr{M} = \partial C^m$, where C^m is a cube in $\mathbb{R}^m$, with dist the geodesic or Euclidean distance and μ the $(m-1)$-dimensional Hausdorff measure.

Throughout this section, we assume that the doubling condition (2.1) *holds.* This is a crucial condition. In contrast, $\mathscr{M}$ is assumed to be compact mainly in order to stay on the "safe side" for all the statements in this section; in many of them, we could have assumed that $\mathscr{M}$ is merely bounded or totally bounded.

We note that the boundedness of $\mathscr{M}$ and the doubling condition (2.1) imply that there exists some $C_r > 0$ such that

$$\mu(B_r(x)) \geq C_r, \forall\, x \in \mathscr{M}. \tag{2.2}$$

We also note also that, since $\mathscr{M}$ is bounded, we have the following straightforward property:

$$\text{if (2.1) holds for any } 0 < r \leq r_0 \text{ and any } x, \text{ then (2.1) holds for any } r \text{ and } x. \tag{2.3}$$

2.1 BMO and VMO on doubling metric measure spaces

We first define BMO. For $f \in \mathscr{L}^1(\mathscr{M}) = \mathscr{L}^1(\mathscr{M};\mathbb{R})$, we define the seminorm

$$|f|_{\mathrm{BMO}} = \sup_{x\in\mathscr{M},\, 0<\varepsilon\leq\varepsilon_0} \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |f(y) - f(z)|\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(z), \tag{2.4}$$

and let

$$\mathrm{BMO} = \mathrm{BMO}(\mathscr{M}) = \mathrm{BMO}(\mathscr{M};\mathbb{R}) := \{f \in \mathscr{L}^1(\mathscr{M})\colon |f|_{\mathrm{BMO}} < \infty\}.$$

Similarly for maps in $\mathscr{L}^1(\mathscr{M};\mathbb{R}^n)$.

In the above definition, $\varepsilon_0 > 0$ is a fixed constant. *"By default", we let* $\varepsilon_0 := \operatorname{diam}(\mathscr{M})$ (if $\mathscr{M}$ contains at least two points), but, under the mild assumption that $\mathscr{M}$ is connected, ε_0 could be any positive number (see below).

We first establish a variant of [24, Lemma A.1].

Lemma 2.1. (1) *Assume that* $\varepsilon_0 \geq \operatorname{diam}(\mathscr{M})$. *Then there exists a finite constant* $C = C(\mathscr{M},\mu)$ *such that*

$$\|f\|_1 \leq C|f|_{\mathrm{BMO}} + \left|\int_{\mathscr{M}} f\right|, \forall\, f \in \mathrm{BMO}. \tag{2.5}$$

(2) *Assume that* $\mathscr{M}$ *is connected. Then* (2.5) *holds for some finite constant* $C = C(\mathscr{M},\mu,\varepsilon_0)$.

Proof of item (2). We will use the following straightforward property: (P) if $\mathscr{M}$ is connected, then any measurable function locally constant a.e. is actually constant a.e.

With no loss of generality, we may consider only functions with zero integral. We argue by contradiction. Assume that there exists a sequence $(f_j) \subset \mathrm{BMO}$ such that

$$\int_{\mathscr{M}} f_j = 0, \, |f_j|_{\mathrm{BMO}} \to 0, \text{ and } \|f_j\|_1 = 1.$$

Since $\mathscr{M}$ is compact, we can cover $\mathscr{M}$ with a finite number of balls $B_{\varepsilon_0}(x_i)$, $1 \le i \le N$. For fixed i, we have

$$⨍_{B_{\varepsilon_0}(x_i)} \left| f_j(y) - ⨍_{B_{\varepsilon_0}(x_i)} f_j \right| \mathrm{d}\mu(y) \le |f_j|_{\mathrm{BMO}} \to 0. \tag{2.6}$$

Using (2.6) and $\|f_j\|_1 = 1$, we find that $\left(⨍_{B_{\varepsilon_0}(x_i)} f_j\right)_j$ is bounded (for every fixed i). From the above, we deduce that, up to a subsequence: (j) $\left(⨍_{B_{\varepsilon_0}(x_i)} f_j\right)_j$ converges to some constant a_i; (jj) on each $B_{\varepsilon_0}(x_i)$, f_j converges to a_i a.e. and in $\mathscr{L}^1$. Since $(B_{\varepsilon_0}(x_i))_{1 \le i \le N}$ is an open cover of $\mathscr{M}$, all the constants a_i are equal (by the property (P)), so that $f_j \to a_1$ in $\mathscr{L}^1(\mathscr{M})$. Since $\int_{\mathscr{M}} f_j = 0$, we find that $a_1 = 0$, and thus $f_j \to 0$ in $\mathscr{L}^1(\mathscr{M})$. This contradicts the assumption $\|f_j\|_1 = 1$. □

Proof of item (1). The proof is essentially the same as above. Property (P) is not needed in this setting since, for any $x_i \in \mathscr{M}$, we have $\mathscr{M} = B_{\varepsilon_0}(x_i)$. □

Corollary 2.2. *Assume that $\mathscr{M}$ is connected. Then two different values of ε_0 yield equivalent seminorms on* BMO.

Proof. In view of (2.5), it suffices to prove that, if $r_0 < \varepsilon_0$, then we have, for some finite $C = C(r_0, \varepsilon_0)$,

$$⨍_{B_\rho(x)} ⨍_{B_\rho(x)} |f(y) - f(z)| \, \mathrm{d}\mu(y) \, \mathrm{d}\mu(z) \le C\|f\|_1,$$
$$\forall f \in \mathscr{L}^1 \text{ s.t. } \int_{\mathscr{M}} f = 0, \forall x \in \mathscr{M}, \forall r_0 < \rho \le \varepsilon_0.$$

With ρ as above and C_r as in (2.2), this follows from

$$\begin{aligned}
&\fint_{B_\rho(x)} \fint_{B_\rho(x)} |f(y) - f(z)| \, \mathrm{d}\mu(y) \, \mathrm{d}\mu(z) \\
&\qquad = \frac{1}{[\mu(B_\rho(x))]^2} \int_{B_\rho(x)} \int_{B_\rho(x)} |f(y) - f(z)| \, \mathrm{d}\mu(y) \, \mathrm{d}\mu(z) \\
&\qquad \leq \frac{1}{[\mu(B_\rho(x))]^2} \int_{B_\rho(x)} \int_{B_\rho(x)} [|f(y)| + |f(z)|] \, \mathrm{d}\mu(y) \, \mathrm{d}\mu(z) \\
&\qquad = \frac{2}{\mu(B_\rho(x))} \|f\|_1 \leq \frac{2}{C_{r_0}} \|f\|_1. \qquad \square
\end{aligned}$$

We now turn to VMO and its basic characterizations and properties, in the spirit of Sarason [61]. For $f \in \mathrm{BMO}$ and $r > 0$, define

$$M_r(f) := \sup_{x \in \mathscr{M}, 0 < s \leq r} \fint_{B_s(x)} \fint_{B_s(x)} |f(y) - f(z)| \, \mathrm{d}\mu(y) \, \mathrm{d}\mu(z) \leq |f|_{\mathrm{BMO}},$$

and

$$M_0(f) := \lim_{r \searrow 0} M_r(f).$$

We denote $\mathrm{VMO} = \mathrm{VMO}(\mathscr{M}) = \mathrm{VMO}(\mathscr{M}; \mathbb{R})$ the closure of continuous functions with respect to the BMO seminorm, i.e.,

$$\mathrm{VMO} := \overline{C(\mathscr{M})/\mathbb{R}}^{|\cdot|_{\mathrm{BMO}}}. \tag{2.7}$$

Similarly for $\mathrm{VMO}(\mathscr{M}; \mathbb{R}^n)$. We also denote

$$\mathrm{dist}(f, \mathrm{VMO}) := \inf_{g \in \mathrm{VMO}} |f - g|_{\mathrm{BMO}}.$$

We next introduce an approximation procedure adapted to the study of VMO under the doubling condition (2.1). For $x, y \in \mathscr{M}$ and $\varepsilon > 0$, let

$$\rho(x, \varepsilon, y) := [\varepsilon - \mathrm{dist}(x, y)]_+, \; K(x, \varepsilon) := \left(\int_{\mathscr{M}} \rho(x, \varepsilon, y) \, \mathrm{d}\mu(y) \right)^{-1}, \tag{2.8}$$

and set

$$f_\varepsilon(x) := K(x, \varepsilon) \int_{\mathscr{M}} \rho(x, \varepsilon, y) f(y) \, \mathrm{d}\mu(y) = \fint_{\mathscr{M}} f \, \mathrm{d}[\rho(x, \varepsilon, \cdot)\mu]. \tag{2.9}$$

For further use, let us note the following straightforward inequalities.

Lemma 2.3. *We have, for every $x \in \mathscr{M}$ and $\varepsilon > 0$,*

$$\frac{\varepsilon}{2}\chi_{B_{\varepsilon/2}(x)} \le \rho(x,\varepsilon,\cdot) \le \varepsilon\chi_{B_\varepsilon(x)}, \tag{2.10}$$

$$\frac{\varepsilon}{2}\mu(B_{\varepsilon/2}(x)) \le \int_{\mathscr{M}} \rho(x,\varepsilon,y)\,\mathrm{d}\mu(y) \le \varepsilon\mu(B_\varepsilon(x)), \tag{2.11}$$

$$\frac{1}{\varepsilon\mu(B_\varepsilon(x))} \le K(x,\varepsilon) \le \frac{2C_{\mathscr{M}}}{\varepsilon\mu(B_\varepsilon(x))}, \tag{2.12}$$

$$\frac{1}{2\mu(B_\varepsilon(x))}\chi_{B_{\varepsilon/2}(x)} \le K(x,\varepsilon)\rho(x,\varepsilon,\cdot) \le \frac{2C_{\mathscr{M}}}{\mu(B_\varepsilon(x))}\chi_{B_\varepsilon(x)}. \tag{2.13}$$

Proof. The inequality (2.10) is clear. Integrating (2.10) yields (2.11). Then, (2.12) follows from (2.11) and the doubling assumption (2.1). Finally, (2.13) is a consequence of (2.10) and (2.12). □

The next result is crucial for the existence of well-behaved homotopy classes.

Lemma 2.4. *The map $\mathscr{M} \times (0,\infty) \ni (x,\varepsilon) \mapsto f_\varepsilon(x)$ is continuous.*

Proof. Since f is integrable and $\mathscr{M}$ is compact, it suffices to prove that $K(x,\varepsilon)\rho(x,\varepsilon,y)$ is continuous with respect to (x,ε,y). Clearly, $\rho(x,\varepsilon,y)$ is continuous with respect to (x,ε,y). On the other hand, we have

$$\infty > \varepsilon\mu(\mathscr{M}) \ge \int_{\mathscr{M}} \rho(x,\varepsilon,y)\,\mathrm{d}\mu(y) \ge \frac{\varepsilon}{2}\mu(B_{\varepsilon/2}(x)) > 0,$$

so that $K(x,\varepsilon)$ is well-defined and continuous with respect to (x,ε). □

We have the following versions of [24, Lemma A.5, Corollary 1].

Lemma 2.5. *There exists a finite constant $A = A(\mathscr{M},\mu)$ such that*

$$M_0(f) \le \operatorname{dist}(f,\mathrm{VMO}) \le AM_0(f),\ \forall\, f \in \mathrm{BMO}, \tag{2.14}$$

and

$$|f - f_\varepsilon|_{\mathrm{BMO}} \le AM_{2\varepsilon}(f),\ \forall\, f \in \mathrm{BMO}, \forall\, 0 < \varepsilon \le \varepsilon_0/2. \tag{2.15}$$

In particular, we have

$$\mathrm{VMO} = \{f \in \mathrm{BMO}\colon M_0(f) = 0\}. \tag{2.16}$$

Corollary 2.6. *For $f \in \mathrm{VMO}$, we have $f_\varepsilon \in \mathrm{VMO}$ and $f_\varepsilon \to f$ in BMO as $\varepsilon \to 0$.*

We will often use Corollary 2.6 in conjunction with the following observation.

Lemma 2.7. *For $f \in \mathscr{L}^1(\mathscr{M})$, we have $f_\varepsilon \to f$ in $\mathscr{L}^1$ as $\varepsilon \to 0$.*

The proof of Lemma 2.5 relies on the following straightforward variant of [24, Lemma A.6].

Lemma 2.8. *For any given numbers $0 < r \le \rho$, any ball $B_\rho(x) \subset \mathscr{M}$ can be covered with a finite number K of balls $B_r(x_i)$ with $x_i \in B_\rho(x)$, $i = 1, \dots, K$, such that $\operatorname{dist}(x_i, x_j) \ge r$ for $i \ne j$ and*

$$\sum_{i=1}^{K} \mu(B_r(x_i)) \le (C_{\mathscr{M}})^2 \mu(B_\rho(x)).$$

(The number K may depend on r, ρ, and x.)

Proof of Lemma 2.8. Since $\mathscr{M}$ is compact, there exists a (finite) maximal collection of disjoint balls $B_{r/2}(x_i)$, $1 \le i \le K$, with centers x_i in $B_\rho(x)$. For any point $x' \in B_\rho(x) \setminus \bigcup_{i=1}^{K} B_{r/2}(x_i)$, there exists some i_0 such that $\operatorname{dist}(x', x_{i_0}) < r$ (for otherwise we can add $B_{r/2}(x')$ to the collection, which contradicts its maximality). Therefore, we have

$$B_\rho(x) \subset \bigcup_{i=1}^{K} B_r(x_i).$$

Since

$$B_{r/2}(x_i) \subset B_{\rho+r/2}(x) \subset B_{2\rho}(x)$$

and thus

$$\sum_{i=1}^{K} \mu(B_{r/2}(x_i)) \le \mu(B_{2\rho}(x)),$$

the doubling assumption (2.1) yields

$$\sum_{i=1}^{K} \mu(B_r(x_i)) \le C_{\mathscr{M}} \sum_{i=1}^{K} \mu(B_{r/2}(x_i)) \le C_{\mathscr{M}} \mu(B_{2\rho}(x)) \le (C_{\mathscr{M}})^2 \mu(B_\rho(x)). \qquad \square$$

Proof of Lemma 2.5. We first prove that

$$M_0(f) \le \operatorname{dist}(f, \mathrm{VMO}), \forall f \in \mathrm{BMO}. \tag{2.17}$$

Clearly, if $f, g \in \mathrm{BMO}$, then, for any $r \geq 0$,

$$M_r(f) \leq M_r(f-g) + M_r(g). \tag{2.18}$$

On the other hand, if $g \in C(\mathcal{M})$, then g is uniformly continuous and therefore $M_0(g) = 0$. Letting $r \to 0$ in (2.18), we find that

$$M_0(f) \leq M_0(f-g) \leq |f-g|_{\mathrm{BMO}}, \forall\, f \in \mathrm{BMO}, \forall\, g \in C(\mathcal{M}). \tag{2.19}$$

Inequality (2.17) follows from (2.19) and the definition of VMO.

We next assume that (2.15) holds. Then, combined with Lemma 2.4, it implies that

$$\operatorname{dist}(f, \mathrm{VMO}) \leq A M_{2\varepsilon}(f), \forall\, f \in \mathrm{BMO},\ \forall\, \varepsilon > 0. \tag{2.20}$$

Letting $\varepsilon \to 0$ in (2.20) yields the second inequality in (2.14).

Therefore, it suffices to establish (2.15), which amounts to the existence of some finite A, independent of f and of ε and r as below, such that

$$\begin{gathered}\fint_{B_r(x)} \fint_{B_r(x)} |(f - f_\varepsilon)(y) - (f - f_\varepsilon)(z)| \, \mathrm{d}\mu(y) \, \mathrm{d}\mu(z) \leq A M_{2\varepsilon}(f), \\ \forall\, 0 < \varepsilon \leq \varepsilon_0/2, \forall\, 0 < r \leq \varepsilon_0.\end{gathered} \tag{2.21}$$

Proof of (2.21) *when* $r \leq \varepsilon$. We first note that

$$\begin{aligned}&\fint_{B_r(x)} \fint_{B_r(x)} |f(y) - f_\varepsilon(y) - f(z) + f_\varepsilon(z)| \, \mathrm{d}\mu(y) \, \mathrm{d}\mu(z) \\ &\leq \fint_{B_r(x)} \fint_{B_r(x)} |f(y) - f(z)| \, \mathrm{d}\mu(y) \, \mathrm{d}\mu(z) + \sup_{y,z \in B_r(x)} |f_\varepsilon(y) - f_\varepsilon(z)| \\ &\leq M_r(f) + \sup_{y,z \in B_r(x)} |f_\varepsilon(y) - f_\varepsilon(z)|.\end{aligned} \tag{2.22}$$

In order to estimate the latter quantity in (2.22), we start from the identity

$$\begin{aligned}&f_\varepsilon(y) - f_\varepsilon(z) \\ &= f_\varepsilon(y) \int_{\mathcal{M}} K(z, \varepsilon) \rho(z, \varepsilon, \eta) \, \mathrm{d}\mu(\eta) - f_\varepsilon(z) \int_{\mathcal{M}} K(y, \varepsilon) \rho(y, \varepsilon, \xi) \, \mathrm{d}\mu(\xi) \\ &= \int_{\mathcal{M}} \int_{\mathcal{M}} K(y, \varepsilon) K(z, \varepsilon) \rho(y, \varepsilon, \xi) \rho(z, \varepsilon, \eta) [f(\xi) - f(\eta)] \, \mathrm{d}\mu(\xi) \, \mathrm{d}\mu(\eta).\end{aligned} \tag{2.23}$$

Combining (2.23) and (2.13) we obtain, for $y, z \in B_r(x)$,

$$
\begin{aligned}
&|f_\varepsilon(y) - f_\varepsilon(z)| \\
&\leq \frac{4(C_{\mathscr{M}})^2}{\mu(B_\varepsilon(y))\mu(B_\varepsilon(z))} \int_{B_\varepsilon(z)} \int_{B_\varepsilon(y)} |f(\xi) - f(\eta)| \, \mathrm{d}\mu(\xi) \, \mathrm{d}\mu(\eta) \\
&\leq 4(C_{\mathscr{M}})^6 \fint_{B_{2\varepsilon}(x)} \fint_{B_{2\varepsilon}(x)} |f(\xi) - f(\eta)| \, \mathrm{d}\mu(\xi) \, \mathrm{d}\mu(\eta) \leq 4(C_{\mathscr{M}})^6 M_{2\varepsilon}(f),
\end{aligned}
\tag{2.24}
$$

where, in the last line, we use the fact that $B_{2\varepsilon}(x) \subset B_{4\varepsilon}(y)$ and thus, thanks to (2.1),

$$
\frac{\mu(B_{2\varepsilon}(x))}{\mu(B_\varepsilon(y))} = \frac{\mu(B_{2\varepsilon}(x))}{\mu(B_{4\varepsilon}(y))} \frac{\mu(B_{4\varepsilon}(y))}{\mu(B_\varepsilon(y))} \leq (C_{\mathscr{M}})^2.
\tag{2.25}
$$

Combining (2.22) and (2.24), we obtain, for $0 < r \leq \varepsilon$,

$$
\fint_{B_r(x)} \fint_{B_r(x)} |f(y) - f_\varepsilon(y) - f(z) - f_\varepsilon(z)| \, \mathrm{d}\mu(y) \, \mathrm{d}\mu(z) \leq (4(C_{\mathscr{M}})^6 + 1) M_{2\varepsilon}(f).
\tag{2.26}
$$

Proof of (2.21) *when* $r \geq \varepsilon$. By Lemma 2.8 and the doubling assumption (2.1), $B_r(x)$ can be covered by a finite number of $B_\varepsilon(x_i)$ such that

$$
\sum_i \mu(B_{2\varepsilon}(x_i)) \leq (C_{\mathscr{M}})^3 \mu(B_r(x)).
\tag{2.27}
$$

Using successively (2.13), (2.25), and (2.27) , we have

$$
\begin{aligned}
&\fint_{B_r(x)} \fint_{B_r(x)} |f(y) - f_\varepsilon(y) - f(z) + f_\varepsilon(z)| \, \mathrm{d}\mu(y) \, \mathrm{d}\mu(z) \\
&\leq 2 \fint_{B_r(x)} |f(y) - f_\varepsilon(y)| \, \mathrm{d}\mu(y) \leq \frac{2}{\mu(B_r(x))} \sum_i \int_{B_\varepsilon(x_i)} |f(y) - f_\varepsilon(y)| \, \mathrm{d}\mu(y) \\
&= \frac{2}{\mu(B_r(x))} \sum_i \int_{B_\varepsilon(x_i)} \left| \int_{B_\varepsilon(y)} K(y, \varepsilon) \rho(y, \varepsilon, z) [f(y) - f(z)] \, \mathrm{d}\mu(z) \right| \mathrm{d}\mu(y) \\
&\leq \frac{2}{\mu(B_r(x))} \sum_i \int_{B_\varepsilon(x_i)} \int_{B_\varepsilon(y)} K(y, \varepsilon) \rho(y, \varepsilon, z) |f(y) - f(z)| \, \mathrm{d}\mu(z) \, \mathrm{d}\mu(y) \\
&\leq \frac{4C_{\mathscr{M}}}{\mu(B_r(x))} \sum_i \int_{B_\varepsilon(x_i)} \int_{B_\varepsilon(y)} \frac{1}{\mu(B_\varepsilon(y))} |f(y) - f(z)| \, \mathrm{d}\mu(z) \, \mathrm{d}\mu(y) \\
&\leq \frac{4(C_{\mathscr{M}})^3}{\mu(B_r(x))} \sum_i \int_{B_{2\varepsilon}(x_i)} \int_{B_{2\varepsilon}(x_i)} \frac{1}{\mu(B_{2\varepsilon}(x_i))} |f(y) - f(z)| \, \mathrm{d}\mu(z) \, \mathrm{d}\mu(y) \\
&\leq \frac{4(C_{\mathscr{M}})^3}{\mu(B_r(x))} \sum_i \mu(B_{2\varepsilon}(x_i)) M_{2\varepsilon}(f) \leq 4(C_{\mathscr{M}})^6 M_{2\varepsilon}(f).
\end{aligned}
\tag{2.28}
$$

Combining (2.26) and (2.28), we obtain

$$|f - f_\varepsilon|_{\text{BMO}} \le (4(C_{\mathscr{M}})^6 + 1)M_{2\varepsilon}(f), \forall f \in \text{BMO}, \forall 0 < \varepsilon \le \varepsilon_0/2,$$

so that (2.21) and (2.15) hold with $A := 4(C_{\mathscr{M}})^6 + 1$. □

Proof of Corollary 2.6. Combine (2.15) and Lemma 2.4. □

Proof of Lemma 2.7. Set

$$T_\varepsilon(f) := f_\varepsilon, \forall f \in \mathscr{L}^1(\mathscr{M}), \forall \varepsilon > 0.$$

Clearly: (j) T_ε is linear; (jj) if $f \in C(\mathscr{M})$, then, as $\varepsilon \to 0$, $T_\varepsilon(f) \to f$ uniformly, and thus in $\mathscr{L}^1(\mathscr{M})$. In order to conclude (*via* (j), (jj), and density), it suffices to find some finite constant C such that

$$\|T_\varepsilon\|_{\mathscr{L}(\mathscr{L}^1(\mathscr{M});\mathscr{L}^1(\mathscr{M}))} \le C, \forall \varepsilon > 0. \tag{2.29}$$

Estimate (2.29) follows from (2.13), which yields

$$\begin{aligned}
\|T_\varepsilon(f)\|_1 &\le \int_{\mathscr{M}} \int_{\mathscr{M}} K(x,\varepsilon)\rho(x,\varepsilon,y)|f(y)|\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x) \\
&\le 2C_{\mathscr{M}} \int_{\mathscr{M}} \frac{1}{\mu(B_\varepsilon(x))} \int_{B_\varepsilon(x)} |f(y)|\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x) \\
&= 2C_{\mathscr{M}} \int_{\mathscr{M}} |f(y)| \int_{B_\varepsilon(y)} \frac{1}{\mu(B_\varepsilon(x))}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \\
&\le 2C_{\mathscr{M}} \int_{\mathscr{M}} |f(y)| \int_{B_\varepsilon(y)} \frac{1}{\mu(B_\varepsilon(x))} \frac{\mu(B_{2\varepsilon}(x))}{\mu(B_\varepsilon(y))}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \\
&\le 2(C_{\mathscr{M}})^2 \int_{\mathscr{M}} |f(y)| \int_{B_\varepsilon(y)} \frac{1}{\mu(B_\varepsilon(y))}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) = 2(C_{\mathscr{M}})^2\|f\|_1,
\end{aligned}$$

where we have used the obvious inclusion $B_\varepsilon(y) \subset B_{2\varepsilon}(x)$ and the assumption (2.1). □

2.2 Homotopy classes of VMO($\mathscr{M};\mathscr{N}$)

If $\mathscr{N} \subset \mathbb{R}^n$, we naturally define

$$\text{VMO}(\mathscr{M};\mathscr{N}) := \{f \in \text{VMO}(\mathscr{M};\mathbb{R}^n) : f(x) \in \mathscr{N}, \forall x \in \mathscr{M}\},$$

and similarly for BMO($\mathscr{M};\mathscr{N}$).

Of interest to us is (only) the case where $\mathscr{N}$ is a closed submanifold of $\mathbb{R}^n$. One could consider the more general situation of an abstract compact Riemannian manifold, and naturally define $\mathrm{VMO}(\mathscr{M};\mathscr{N})$ or $\mathrm{BMO}(\mathscr{M};\mathscr{N})$ by isometrically embedding $\mathscr{N}$ into some $\mathbb{R}^n$. It turns out that the definition does not depend on the choice of the embedding (as we next explain), and thus we can fix once for all the embedding and consider $\mathscr{N}$ as a subset of $\mathbb{R}^n$. To justify this independence, we note that, since $\mathscr{N}$ is compact, if $\Phi_j\colon \mathscr{N} \to \mathscr{N}_j \subset \mathbb{R}^{n_j}$, $j = 1, 2$, are isometric embeddings, then the geodesic and Euclidean distance on each $\mathscr{N}_j$ are equivalent, and thus the transition map $\Phi := \Phi_2 \circ \Phi_1^{-1}\colon \mathscr{N}_1 \to \mathscr{N}_2$ is bi-Lipschitz. The independence of $\mathrm{VMO}(\mathscr{M};\mathscr{N})$ or $\mathrm{BMO}(\mathscr{M};\mathscr{N})$ on the choice of the embedding is then clear, from (2.4).

In view of the above, *from now on, we assume that $\mathscr{N}$ is a smooth closed manifold embedded in $\mathbb{R}^n$. We also recall that we assume that $\mathscr{M}$ is compact and satisfies the doubling condition* (2.1).

We first note the following simple result.

Lemma 2.9. *For every integrable map $f\colon \mathscr{M} \to \mathscr{N}$, we have*

$$\mathrm{dist}(f_\varepsilon(x), \mathscr{N}) \le 2C_{\mathscr{M}} M_\varepsilon(f), \forall\, x \in \mathscr{M}, \forall\, \varepsilon > 0. \tag{2.30}$$

Proof. For every $y \in \mathscr{M}$, we have $\mathrm{dist}(f_\varepsilon(x), \mathscr{N}) \le |f_\varepsilon(x) - f(y)|$, so that (using (2.13))

$$\begin{aligned}\mathrm{dist}(f_\varepsilon(x), \mathscr{N}) &\le \fint_{B_\varepsilon(x)} |f(y) - f_\varepsilon(x)|\, \mathrm{d}\mu(y)\\ &\le \fint_{B_\varepsilon(x)} \int_{B_\varepsilon(x)} K(x,\varepsilon)\rho(x,\varepsilon,z)|f(y) - f(z)|\, \mathrm{d}\mu(z)\, \mathrm{d}\mu(y)\\ &\le 2C_{\mathscr{M}} \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |f(y) - f(z)|\, \mathrm{d}\mu(z)\, \mathrm{d}\mu(y) \le 2C_{\mathscr{M}} M_\varepsilon(f).\end{aligned}$$

□

We next introduce a convenient projection on $\mathscr{N}$.

Definition 2.10. For sufficiently small $\delta = \delta(\mathscr{N})$, we let Π be the nearest point projection from $\mathscr{N}_\delta := \{z \in \mathbb{R}^n \colon \mathrm{dist}(z, \mathscr{N}) \le \delta\}$ to $\mathscr{N}$.

Here, δ is chosen such that Π is well-defined, smooth, and has bounded derivatives.

In what follows, δ is implicitly assumed to be sufficiently small such that Π has all the above properties.

By (2.30) and (2.16), for each $f \in \mathrm{VMO}(\mathscr{M};\mathscr{N})$, there exists some $\varepsilon_1 = \varepsilon_1(f)$ such that

$$f_\varepsilon(x) \in \mathscr{N}_\delta, \forall\, x \in \mathscr{M}, \forall\, 0 < \varepsilon \le \varepsilon_1. \tag{2.31}$$

Therefore, if we set

$$f^\varepsilon := \Pi \circ f_\varepsilon \colon \mathscr{M} \to \mathscr{N}, \tag{2.32}$$

then f^ε is well-defined, $\forall\, 0 < \varepsilon \le \varepsilon_1$. By Lemma 2.4, if (2.1) holds, then the mapping

$$(0, \varepsilon_1] \ni \varepsilon \mapsto f^\varepsilon \in C(\mathscr{M}; \mathscr{N}) \tag{2.33}$$

is continuous, and therefore the following definition makes sense.

Definition 2.11. For $f \in \mathrm{VMO}(\mathscr{M}; \mathscr{N})$, we define the *homotopy class* $[f]$ *of* f by $[f] := [f^\varepsilon]$ for small ε, i.e.,

$$[f] = \{h \in C(\mathscr{M}; \mathscr{N}) \colon h \sim f^\varepsilon \text{ for some } \ \varepsilon \le \varepsilon_1\}. \tag{2.34}$$

Two maps $f, g \in \mathrm{VMO}(\mathscr{M}; \mathscr{N})$ are *homotopic* (and this is denoted $f \sim g$) if $[f] = [g]$.

We first note that (by (2.33)), in (2.34), it is equivalent to ask that $h \sim f^\varepsilon$ for *some* ε or *each* ε. We next note that, when f is continuous, $[f]$ is the classical homotopy class of f. Indeed, in this case $f_\varepsilon \to f$, and therefore $f^\varepsilon \to f$ uniformly as $\varepsilon \to 0$, so that the claim follows from the stability of the homotopy classes.

We next prove the fundamental fact that the homotopy class is stable under $\mathrm{BMO} \cap \mathscr{L}^1$ convergence (analogue of [24, Theorem 1]).

Proposition 2.12. *Let* $f \in \mathrm{VMO}(\mathscr{M}; \mathscr{N})$. *Let* $\varepsilon_2 = \varepsilon_2(f) \le \varepsilon_0$ *be such that* $M_{\varepsilon_2}(f) \le \delta/(8C_{\mathscr{M}})$. *Then, with* C_r *as in* (2.2), *we have*

$$\begin{aligned}[][g \in \mathrm{VMO}(\mathscr{M}; \mathscr{N}), |g - f|_{\mathrm{BMO}} \le \delta/(8C_{\mathscr{M}}), \|f - g\|_1 \le \delta C_{\varepsilon_2}/(4C_{\mathscr{M}})] \implies \\ [g^\varepsilon \sim f^\varepsilon, 0 < \varepsilon \le \varepsilon_2].\end{aligned} \tag{2.35}$$

In particular, under the assumptions of (2.35), *we have* $g \sim f$.

Corollary 2.13. *If* $(f_j) \subset \mathrm{VMO}(\mathscr{M}; \mathscr{N})$, $f \in \mathrm{VMO}(\mathscr{M}; \mathscr{N})$, *and* $f_j \to f$ *in* $\mathrm{BMO} \cap \mathscr{L}^1$, *then, for large* j, $f_j \sim f$.

Proof of Proposition 2.12. Let g satisfy the assumptions of (2.35). By (2.18), we have $M_{\varepsilon_2}(g) \le \delta/(4C_{\mathscr{M}})$, $\forall\, 0 < \varepsilon \le \varepsilon_2$, and thus (by (2.30))

$$f_\varepsilon(x),\ g_\varepsilon(x) \in \mathscr{N}_{\delta/2}, \forall\, x \in \mathscr{M}, \forall\, 0 < \varepsilon \le \varepsilon_2. \tag{2.36}$$

In order to complete the proof, we prove that $g^{\varepsilon_2} \sim f^{\varepsilon_2}$. For this purpose, it suffices to establish the estimate

$$\|g_{\varepsilon_2} - f_{\varepsilon_2}\|_\infty \le \delta/2. \tag{2.37}$$

Indeed, granted (2.37), we have, thanks to (2.36),

$$(1-t)f_{\varepsilon_2}(x) + t g_{\varepsilon_2}(x) = f_{\varepsilon_2}(x) + t(g_{\varepsilon_2}(x) - f_{\varepsilon_2}(x)) \in \mathcal{N}_\delta, \forall\, x \in \mathcal{M}, \forall\, 0 \le t \le 1,$$

and thus $[0,1] \ni t \mapsto \Pi((1-t)f_{\varepsilon_2} + t g_{\varepsilon_2})$ is a homotopy between f^{ε_2} and g^{ε_2}.

But, we note that (2.37) follows, under the assumptions of (2.35), from

$$|g_{\varepsilon_2}(x) - f_{\varepsilon_2}(x)| \le \frac{2C_{\mathcal{M}}}{\mu(B_{\varepsilon_2}(x))}\|g-f\|_1 \le \frac{2C_{\mathcal{M}}}{C_{\varepsilon_2}}\|g-f\|_1 \le \delta/2,$$

where we have used (2.13). □

Although we will not use the next result in what follows, we state it since it gives some insight concerning Definition 2.11.

Lemma 2.14. *For $f, g \in \mathrm{VMO}(\mathcal{M};\mathcal{N})$, we have*

$$f \sim g \Leftrightarrow [\exists\, F \in C([0,1];(\mathrm{VMO}\cap\mathcal{L}^1)(\mathcal{M};\mathcal{N}))\ \text{s.t.}\ F_0 = f \text{ and } F_1 = g].$$

Here, we use the standard notation $F_t \coloneqq F(t,\cdot)$.

Proof. "$\Rightarrow$" Since $f \sim g$, by definition, there exists some sufficiently small $\overline{\varepsilon}$ such that $f^\varepsilon \sim g^\varepsilon$ for every $\varepsilon \le \overline{\varepsilon}$, which implies that there exists a continuous map $H\colon [0,1] \to C(\mathcal{M};\mathcal{N})$ such that $H_0 = f^{\overline{\varepsilon}}$ and $H_1 = g^{\overline{\varepsilon}}$. Then define F as follows:

$$F_t \coloneqq \begin{cases} f, & \text{if } t = 0 \\ f^t, & \text{if } 0 < t \le \overline{\varepsilon} \\ H_{(t-\overline{\varepsilon})/(1-2\overline{\varepsilon})}, & \text{if } \overline{\varepsilon} \le t \le 1-\overline{\varepsilon} \\ g^{1-t}, & \text{if } 1-\overline{\varepsilon} \le t < 1 \\ g, & \text{if } t = 1 \end{cases}.$$

Since $C(\mathcal{M};\mathcal{N}) \hookrightarrow (\mathrm{VMO}\cap\mathcal{L}^1)(\mathcal{M};\mathcal{N})$, $t \mapsto F_t$ belongs to $F \in C((0,1);(\mathrm{VMO}\cap\mathcal{L}^1)(\mathcal{M};\mathcal{N}))$. In order to prove the continuity of F on $[0,1]$ and complete the proof of "$\Rightarrow$", it therefore suffices to check the continuity at $t = 0$ and $t = 1$. For this purpose, we rely on Corollary 2.6, Lemma 2.7, and the fact that the superposition with Lipschitz functions is continuous in VMO (see Brezis and Nirenberg [24, Lemma A.8]).

"$\Leftarrow$" By Proposition 2.12, the map $t \mapsto [F_t]$ is locally constant. By a standard argument, it is constant, whence the conclusion. □

A final result in this section concerns maps such that $|f|_{\mathrm{BMO}} \ll 1$.

Proposition 2.15. *There exists some positive constant $C = C(\mathscr{M}, \mathscr{N})$ such that*

$$[f \in \mathrm{VMO}(\mathscr{M}; \mathscr{N}), |f|_{\mathrm{BMO}} \leq C] \implies f \sim \xi \text{ for some point } \xi \in \mathscr{N}. \tag{2.38}$$

If, in addition, $\mathscr{N}$ is connected, then (2.38) holds for any $\xi \in \mathscr{N}$.

Proof. We may assume that $\varepsilon_0 = \operatorname{diam} \mathscr{M}$ (see Corollary 2.2). Let $f \in \mathrm{VMO}(\mathscr{M}; \mathscr{N})$. Since $\mathscr{M} = B_{\varepsilon_0}(x)$, $\forall\, x \in \mathscr{M}$, there exists some $z = z(f) \in \mathscr{M}$ such that

$$⨍_{\mathscr{M}} |f(y) - f(z)| \, \mathrm{d}\mu(y) \leq |f|_{\mathrm{BMO}}. \tag{2.39}$$

Set $\xi := f(z) \in \mathscr{N}$. From Proposition 2.12 (with the constant map ξ playing the role of f and $\varepsilon_2 := \varepsilon_0 = \operatorname{diam} \mathscr{M}$) and (2.39), we find that (2.38) holds, provided $C \leq \min\left(\dfrac{\delta}{8C_{\mathscr{M}}}, \dfrac{C_{\varepsilon_0} \delta \mu(\mathscr{M})}{4C_{\mathscr{M}}}\right)$. □

3 Integral invariants for VMO maps to manifolds

In this section, again with no claim of originality, we assume that: (a) $\mathscr{M}$ is a Lipschitz k-dimensional manifold embedded into some $\mathbb{R}^m$, endowed with a finite bi-Lipschitz chart structure, considered as a metric subspace of $\mathbb{R}^m$ and endowed with the natural measure, i.e., the k-dimensional Hausdorff measure $\mathscr{H}^k$; (b) $\mathscr{N}$ is a closed smooth manifold embedded into some $\mathbb{R}^n$; (c) ω is a smooth *closed* k-form on $\mathscr{N}$. (For $\mathscr{M}$, the prototypical example we have in mind is $\mathscr{M} = \partial C^{k+1}$, with C^{k+1} a cube in $\mathbb{R}^{k+1}$.) The main objective here is to give, when $\mathscr{M}$ is compact, a robust meaning to $\int_{\mathscr{M}} f^* \omega$ when $f \in \mathrm{VMO}(\mathscr{M}; \mathscr{N})$.

To be more specific, the instrumental definition of Lipschitz manifolds we adopt here is the following.

Definition 3.1. A *k-dimensional finite chart structure* on $\mathscr{M} \subset \mathbb{R}^m$ is a finite family $\{(U_i, V_i, \varphi_i)\}_{i \in I}$ such that:

(i) U_i is open in $\mathscr{M} \subset \mathbb{R}^m$, $\forall\, i \in I$, and $\bigcup_{i \in I} U_i$ is a cover of $\mathscr{M}$;

(ii) V_i is an open subset of $\mathbb{R}^k$, $\forall\, i \in I$;

(iii) $\varphi_i \colon V_i \to U_i$ is bi-Lipschitz, $\forall\, i \in I$.

A *(k-dimensional) Lipschitz manifold* is a set $\mathscr{M}$ embedded into some $\mathbb{R}^m$ and endowed with a k-dimensional finite chart structure (in the sense of Definition 3.1).

Considering a *finite* chart structure is a matter of convenience. As we will see, working with Lipschitz maps requires excluding exceptional null sets, and we wanted to avoid working with infinite unions of null sets. In practice, $\mathscr{M}$ will most of the time be compact, so that considering a finite chart structure is not a real limitation. Another not so common feature is the *bi-Lipschitz character of the* φ_i*'s* (this condition is clearly satisfied, at least locally, in the smooth case). This is also a matter of convenience, for avoiding using the decomposition of rectifiable sets as images of bi-Lipschitz maps (see, e.g., Federer [33, Lemma 3.2.18]).

Smooth closed manifolds are examples of such $\mathscr{M}$'s. More generally, if $\mathscr{M}$ is bi-Lipschitz homeomorphic with some smooth closed manifold $\mathscr{M}'$, then $\mathscr{M}'$ naturally induces a chart structure on $\mathscr{M}$. This includes, as special cases, ∂C^{k+1}, and more generally, ∂B, where B is a ball for some norm in $\mathbb{R}^{k+1}$, and even more generally, boundaries of convex bodies in $\mathbb{R}^{k+1}$. Indeed, such boundaries are bi-Lipschitz homeomorphic with the Euclidean unit sphere $\mathbb{S}^k$. (See, e.g., Section 3.4 below for more details.)

This section is organized as follows. First, we prove, in Section 3.1, that $\mathscr{M}$ as above, when compact and endowed with the natural distance and measure, fits into the framework developed in Section 2.1. Next, in Sections 3.2–3.6, we carefully adapt notions as the tangent space, the differential, and the calculus with forms (exterior calculus, pull-back, integration on oriented manifolds) to the context of Lipschitz manifolds. Since our final purpose is to establish integral estimates associated with such forms, we adopt an analytic point of view, working mainly in local coordinates. While consistent with the smooth case, this approach has the advantage of making obvious the main properties of the calculus with forms. Finally, in Section 3.7, which is at the heart of this part, we define $\int_{\mathscr{M}} f^*\omega$ when $\mathscr{M}$ is compact, $f \in \mathrm{VMO}(\mathscr{M};\mathscr{N})$, and ω is a closed smooth k-form on $\mathscr{N}$, and prove that this quantity is a homotopical invariant. In Section 3.8, we consider the special case of $W^{1,k}$ maps and prove that, as expected, in this case $\int_{\mathscr{M}} f^*\omega$ is a genuine integral.

Although most of the results we establish in Sections 3.2–3.6 can be derived from more general advanced assertions from geometric measure theory (we have in mind in particular the analysis on rectifiable sets and on finite perimeter sets, as in [33, Section 3.2], and the homological integration in [33, Chapter 4]), we have opted for a low tech and essentially self-contained exposition that does not require any knowledge of geometric measure theory.

3.1 Compact Lipschitz manifolds are doubling metric measure spaces

In this short section, $\mathscr{M}$ is compact and is endowed with a finite chart structure in the sense of Definition 3.1. We establish the following result.

Lemma 3.2. *We endow $\mathscr{M}$ with the Euclidean distance (or any distance induced by a norm on $\mathbb{R}^m$) and with the Hausdorff measure $\mathscr{H}^k$. Then $\mathscr{M}$ satisfies the doubling condition (2.1).*

If $\mathscr{M}$ is connected, then the same holds for the geodesic distance.

Proof. Since all the above distances are equivalent to the Euclidean distance on $\mathscr{M}$ (for the geodesic distance, this follows from Definition 3.1 (iii)), it suffices to consider the Euclidean distance $|\cdot|$. Let $0 < K_1 \le K_2 < \infty$ be such that

$$K_1|v-w| \le |\varphi_i(v)-\varphi_i(w)| \le K_2|v-w|, \forall\, i, \forall\, v, w \in V_i. \tag{3.1}$$

We claim that, if $B \subset U_i$ is a Borel set, then

$$(K_1)^k \mathscr{H}^k(\varphi_i^{-1}(B)) \le \mathscr{H}^k(B) \le (K_2)^k \mathscr{H}^k(\varphi_i^{-1}(B)). \tag{3.2}$$

Indeed, (3.2) clearly follows from: (i) the fact that a K-Lipschitz map $\varphi\colon A \subset \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ can be extended to a K-Lipschitz map to the whole $\mathbb{R}^{n_1}$ (Kirszbraun's theorem); (ii) the fact that, if $\varphi\colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ is K-Lipschitz, then

$$\mathscr{H}^s(\varphi(B)) \le K^s \mathscr{H}^s(B), \forall\, s > 0, \forall\, B \subset \mathbb{R}^{n_1} \text{ a Borel set.}$$

Let r_0 be such that for every $x \in \mathscr{M}$, there exists some i such that $B_{r_0}(x) \subset U_i$. Let $x \in \mathscr{M}$. If $r \le r_0$ and i are such that $B_r(x) \subset U_i$, we write $x = \varphi_i(v)$ for some $v \in V_i$. By (3.1), we have

$$B_{r/K_2}(v) \subset \varphi_i^{-1}(B_r(x)) \subset B_{r/K_1}(v). \tag{3.3}$$

Combining (3.2) and (3.3), we find that

$$\mathscr{H}^k(B_r(x)) \sim r^k, \forall\, 0 < r \le r_0, \forall\, x \in \mathscr{M}. \tag{3.4}$$

We conclude *via* (3.4) and (2.3). □

From now on, any compact Lipschitz k-manifold is implicitly assumed to be endowed with the Euclidean distance and the k-dimensional Hausdorff measure.

3.2 Tangent spaces on Lipschitz manifolds

We are here in the setting of Definition 3.1 and $\mathscr{M}$ need not be compact. Coordinates of points in $\mathbb{R}^k$ and $\mathbb{R}^m$ appear as superscripts, e.g., $v = (v^1, \ldots, v^k)$. The differential of a map φ at v is denoted $\mathrm{D}_v\varphi$. The canonical basis in $\mathbb{R}^k$ is denoted $\{e_1, \ldots, e_k\}$.

For every $i \in I$ and almost every $v \in V_i$, φ_i is differentiable at v (by Rademacher's theorem). If $x = \varphi_i(v)$ for such v, then we set

$$\left.\frac{\partial}{\partial v^\ell}\right|_x := \frac{\partial \varphi_i}{\partial v^\ell}(v) = \mathrm{D}_v\varphi_i(e_\ell) = \text{ (in short) } \frac{\partial}{\partial v^\ell} \text{ or } \frac{\partial}{\partial v_i^\ell},\ \ell = 1, \ldots, k, \tag{3.5}$$

$$T_x\mathscr{M} := \mathrm{D}_v\varphi_i(\mathbb{R}^k) = \operatorname{span}\left\{\left.\frac{\partial}{\partial v^\ell}\right|_x : \ell = 1, \ldots, k\right\}. \tag{3.6}$$

We first note that the above definitions are consistent with the ones for differentiable manifolds. We next check that $T_x\mathscr{M}$ enjoys two basic expected properties.

Lemma 3.3. *We have* $\dim T_x\mathscr{M} = k$.

Lemma 3.4. *The definition of $T_x\mathscr{M}$ does not depend on i.*

Proof of Lemma 3.3. We have to prove that $\mathrm{D}_v\varphi_i$ is one-to-one. Let $K_1 > 0$ be such that

$$|\varphi_i(v) - \varphi_i(w)| \geq K_1|v - w|, \forall\, w \in V_i. \tag{3.7}$$

By (3.7), we have

$$|\mathrm{D}_v\varphi_i(\xi)| = \lim_{t\to 0}\left|\frac{\varphi_i(v + t\xi) - \varphi_i(v)}{t}\right| \geq K_1|\xi|, \forall\, \xi \in \mathbb{R}^k,$$

whence the conclusion. □

Proof of Lemma 3.4. Assume that $x = \varphi_i(v_i) = \varphi_j(v_j)$, with φ_i, respectively φ_j, differentiable at v_i, respectively v_j. It suffices to prove that $\mathrm{D}_{v_i}\varphi_i(\mathbb{R}^k)$ and $\mathrm{D}_{v_j}\varphi_j(\mathbb{R}^k)$ have the same unit sphere. By the proof of Lemma 3.3, $\mathrm{D}_{v_i}\varphi_i$ and $\mathrm{D}_{v_j}\varphi_j$ are one-to-one. In view of Definition 3.1, the conclusion of the lemma follows from the following

Claim. Let $V \subset \mathbb{R}^k$ be an open set. Let $\varphi\colon V \to \varphi(V) \subset \mathbb{R}^m$ be such that: (i) $0 \in V$ and $\varphi(0) = 0$; (ii) φ is differentiable at the origin; (iii) $\mathrm{D}_0\varphi$ is one-to-one; (iv) φ is a homeomorphism. Then, for w a unit vector of $\mathbb{R}^m$, we have

$$w \in \mathrm{D}_0\varphi(\mathbb{R}^k) \iff \exists\,(x_j) \subset \varphi(V) \setminus \{0\} \text{ s.t. } x_j \to 0 \text{ and } \frac{x_j}{|x_j|} \to w. \tag{3.8}$$

To establish the claim, let first w be a unit vector in $\mathrm{D}_0\varphi(\mathbb{R}^k)$. Let $\xi \in \mathbb{R}^k$ be such that $\mathrm{D}_0\varphi(\xi) = w$. Then, for large j, $x_j := \varphi(j^{-1}\xi)$ belongs to $\varphi(V) \setminus \{0\}$ and satisfies $x_j \to 0$ and $\dfrac{x_j}{|x_j|} \to w$. (Here, we do not use the assumptions (iii) and (iv).) For the reverse inclusion, let (x_j) be as in (3.8). Write $x_j = \varphi(v_j)$, with $v_j \in V \setminus \{0\}$. By the assumption (iv), we have $v_j \to 0$. Write $v_j = t_j\xi_j$, with $t_j > 0$, $\xi_j \in \mathbb{R}^k$, $|\xi_j| = 1$, $t_j \to 0$. Up to a subsequence, we may assume that $\xi_j \to \xi$. By the assumption (iii), we have $\mathrm{D}_0\varphi(\xi) \neq 0$, and then one easily sees that

$$w = \lim_j \frac{\varphi(t_j\xi_j)}{|\varphi(t_j\xi_j)|} = \frac{\mathrm{D}_0\varphi(\xi)}{|\mathrm{D}_0\varphi(\xi)|} = \mathrm{D}_0\varphi(\xi/|\mathrm{D}_0\varphi(\xi)|). \qquad \square$$

In what follows, we implicitly consider only *regular* points $x \in \mathscr{M}$, i.e., points x such that, if $x = \varphi_i(v_i)$ for some i, then φ_i is differentiable at v_i. By the above, the complement of the regular points is an $\mathscr{H}^k$-null set, and the tangent space at any regular point $x \in U_i$ is expressed *via* (3.5)–(3.6).

Remark 3.5. A digression about measurability issues. Given a locally Lipschitz function $g\colon V \to \mathbb{R}$, where V is an open set in $\mathbb{R}^k$, the exceptional set A of points where g is not differentiable is a Borel set. Moreover, the gradient (and thus the differential) $V \setminus A \ni x \mapsto \nabla g(x) \in \mathbb{R}^k$ is a Borel function. Both these properties are well-known to experts, but we could not find a reference. They may be derived, for example, by following the proof of Rademacher's theorem (see, e.g., Evans and Gariepy [32, Section 3.1]), which implicitly contains explicit formulas for A and for ∇g allowing to check their Borel measurability.

In what follows, we do not discuss anymore measurability issues but, following this remark, it is easy to prove that all the forms and functions we construct below are Borel measurable and defined up to an $\mathscr{H}^k$-null Borel set. $\square$

3.3 Lipschitz maps on $\mathscr{M}$: differential and pullback of forms

Here, we are again in the setting of Definition 3.1 and $\mathscr{M}$ need not be compact. We consider (locally) Lipschitz maps defined on $\mathscr{M}$, since this setting is sufficient for most of the applications we have in mind (see, however, Section 3.8 for $W^{1,k}$ maps), but with more effort some of the results below can be extended to approximately differentiable maps.

Given a locally Lipschitz function $f\colon \mathscr{M} \to \mathbb{R}$, we define, for $\mathscr{H}^k$-a.e. $x \in \mathscr{M}$, $\mathrm{d}_x f$ as follows.

Definition 3.6. Let $x = \varphi_i(v) \in \mathscr{M}$ be a regular point such that $f \circ \varphi_i$ is differentiable at

$v \in V_i$. We define

$$\mathrm{d}_x f \colon T_x\mathscr{M} \to \mathbb{R},\ \mathrm{d}_x f(\mathrm{D}_v\varphi_i(\xi)) := \mathrm{D}_v(f \circ \varphi_i)(\xi), \forall\, \xi \in \mathbb{R}^k. \tag{3.9}$$

Similarly when $f \colon \mathscr{M} \to \mathbb{R}^n$.

We note that the above definition is consistent with the one for smooth manifolds and, by Rademacher's theorem, $\mathrm{d}_x f$ is defined except on an $\mathscr{H}^k$-null set. (This null set depends on f.)

We first check that the definition is correct, in the sense that it is independent of the chart. This is a straightforward consequence of the chain rule combined with the following result.

Lemma 3.7. *Let $x = \varphi_i(v_i) = \varphi_j(v_j) \in \mathscr{M}$ be a regular point. Let $W_j := \varphi_j^{-1}(U_i \cap U_j)$ and $W_i := \varphi_i^{-1}(U_i \cap U_j)$. Then*

$$\varphi := \varphi_i^{-1} \circ \varphi_j \colon W_j \to W_i$$

is differentiable at v_j.

Proof. With no loss of generality, we may assume that $U_i = U_j$, $v_i = v_j = 0$, and $\varphi_i(0) = \varphi_j(0) = 0$. By Lemmas 3.3 and 3.4, there exists a unique (linear, bijective) map $A \colon \mathbb{R}^k \to \mathbb{R}^k$ such that

$$\mathrm{D}_0\varphi_j(a) = \mathrm{D}_0\varphi_i(Aa), \forall\, a \in \mathbb{R}^k. \tag{3.10}$$

For $w \in V_j$, let $y := \varphi_i^{-1}(\varphi_j(w)) \in V_i$. The conclusion of the lemma follows from the equality

$$y = Aw + o(|w|) \text{ as } w \to 0, \tag{3.11}$$

that we next prove.

By the assumption (iii) in Definition 3.1, we have

$$|y| \sim |w| \text{ as } w \to 0. \tag{3.12}$$

Next, using: (i) (3.10); (ii) the equation $\varphi_j(w) = \varphi_i(y)$ under the self-explaining form $\mathrm{D}_0\varphi_j(w) + o(|w|) = \mathrm{D}_0\varphi_i(y) + o(|y|)$; (iii) the equivalence (3.12), we find that

$$\begin{aligned} \mathrm{D}_0\varphi_i(Aw) + o(|w|) = \mathrm{D}_0\varphi_j(w) + o(|w|) &= \mathrm{D}_0\varphi_i(y) + o(|y|) \\ &= \mathrm{D}_0\varphi_i(y) + o(|w|). \end{aligned} \tag{3.13}$$

We obtain (3.11) from (3.13), (3.12), and the fact that $D_0\varphi_i$ is one-to-one. ◻

Remark 3.8. Let $\mathscr{N}$ be a C^1-submanifold of $\mathbb{R}^n$ and V be an open subset of $\mathbb{R}^k$. Assume that $g\colon V \to \mathbb{R}^n$ is differentiable at some point $v \in V$ and that $g(V) \subset \mathscr{N}$. It is straightforward that g, seen as a map from V to $\mathscr{N}$, is differentiable at v, and that $D_v g(\mathbb{R}^k) \subset T_{g(v)}\mathscr{N}$.

This consideration leads to the following. Let $f\colon \mathscr{M} \to \mathscr{N}$ be locally Lipschitz. Then, at each regular point $x = \varphi_i(v) \in \mathscr{M}$ such that $f \circ \varphi_i$ is differentiable at v, we have $d_x f\colon T_x\mathscr{M} \to T_{f(x)}\mathscr{N}$. ◻

As in the smooth case, we associate with $x = \varphi_i(v) \in U_i$ its "coordinates"

$$x^\ell = x_i^\ell = x^\ell(x) := v^\ell,\ \ell = 1, \dots, k.$$

The maps $U_i \ni x \mapsto x^\ell \in \mathbb{R}$ are Lipschitz. Moreover, one sees (from (3.9)) that, at each regular point,

$$d_x x_i^\ell \left(\frac{\partial}{\partial v_i^{\ell'}}\bigg|_x \right) = \delta_{\ell\ell'},\ 1 \le \ell, \ell' \le k. \tag{3.14}$$

Therefore, when $x \in \mathscr{M}$ is a regular point and $1 \le p \le k$, an alternate form $\eta = \eta(x)$ of order p (in short, a p-form) on $T_x\mathscr{M}$ can be uniquely written as

$$\begin{aligned}\eta(x) &= \sum_{1 \le \ell_1 < \ell_2 < \dots < \ell_p \le k} \eta^i_{\ell_1,\dots,\ell_p}(x)\, d_x x_i^{\ell_1} \wedge \dots \wedge d_x x_i^{\ell_p} \\ &= \text{(in short)} \sum_{1 \le \ell_1 < \ell_2 < \dots < \ell_p \le k} \eta_{\ell_1,\dots,\ell_p}(x)\, d_x x^{\ell_1} \wedge \dots \wedge d_x x^{\ell_p}.\end{aligned} \tag{3.15}$$

More specifically, for every $f_1, \dots, f_p \in \mathrm{Lip}(\mathscr{M};\mathscr{N})$ and $\xi_1, \dots, \xi_p \in T_x\mathscr{M}$, we have

$$d_x f_1 \wedge \dots \wedge d_x f_p(\xi_1, \cdots, \xi_p) = \det(d_x f_i(\xi_j)). \tag{3.16}$$

Combining (3.16) with (3.14) and (3.15), this implies that

$$\eta_{\ell_1,\dots,\ell_p}(x) = \eta(x)\left(\frac{\partial}{\partial v^{\ell_1}}\bigg|_x, \dots, \frac{\partial}{\partial v^{\ell_p}}\bigg|_x \right). \tag{3.17}$$

Definition 3.9. If $A \subset \mathscr{M}$ is an $\mathscr{H}^k$-null Borel subset such that $\mathscr{M} \setminus A$ consists of regular points, and if, for each $x \in \mathscr{M} \setminus A$, we are given a p-form $\eta(x)$ as in (3.15), we say that η is Borel measurable, respectively bounded, if the (locally defined) coefficients $\eta_{\ell_1,\dots,\ell_p}$ are Borel measurable, respectively bounded.

As in the smooth case, one checks, using: (i) Lemma 3.7; (ii) the bi-Lipschitz character of the chart system; (iii) the chain rule, that the Borel measurable or bounded character of η does not depend on the chart.

In what follows, we consider only forms that are implicitly defined up to an $\mathscr{H}^k$-null Borel set $A \subset \mathscr{M}$ as in Definition 3.9.

We next define the pullback of forms in the two special cases we are interested in.

Definition 3.10. If η is a p-form on $\mathscr{M}$ defined at a regular point $x = \varphi_i(v) \in \mathscr{M}$, we set

$$(\varphi_i)^*\eta(v)(\xi_1, \ldots, \xi_p) := \eta(x)(\mathrm{D}_v\varphi_i(\xi_1), \ldots, \mathrm{D}_v\varphi_i(\xi_p)), \forall\, \xi_1, \ldots, \xi_p \in \mathbb{R}^k. \tag{3.18}$$

Definition 3.11. Let $\mathscr{N}$ be a C^1-submanifold of $\mathbb{R}^n$. Let ω be a p-form on $\mathscr{N}$ (defined everywhere) and $f\colon \mathscr{M} \to \mathscr{N}$ be locally Lipschitz. If $x = \varphi_i(v) \in \mathscr{M}$ is a regular point such that $f \circ \varphi_i$ is differentiable at v, we set

$$f^*\omega(x)(y_1, \ldots, y_p) := \omega(f(x))(\mathrm{d}_x f(y_1), \ldots, \mathrm{d}_x f(y_p)), \forall\, y_1, \ldots, y_p \in T_x\mathscr{M}. \tag{3.19}$$

We note that (3.19) does not depend on i.

Clearly, $(\varphi_i)^*\eta$ is a p-form on V_i, while $f^*\omega$ is a p-form on $\mathscr{M}$. Moreover, assuming f Lipschitz and $\mathscr{N}$ compact, if η (respectively ω) is Borel measurable or bounded, then so is $(\varphi_i)^*\eta$ (respectively $f^*\omega$).

On the other hand, with f and ω as above, one can classically define, at each regular point $x = \varphi_i(v) \in \mathscr{M}$ such that $f \circ \varphi_i$ is differentiable at v,

$$\begin{aligned}(f \circ \varphi_i)^*\omega(v)(\xi_1, \ldots, \xi_p) := \omega(f(x))(\mathrm{D}_v(f \circ \varphi_i)(\xi_1), \ldots, \mathrm{D}_v(f \circ \varphi_i)(\xi_p)),\\ \forall\, \xi_1, \ldots, \xi_p \in \mathbb{R}^k.\end{aligned} \tag{3.20}$$

Using successively (3.18), (3.9), and (3.20), we find that

$$(\varphi_i)^*(f^*\omega) = (f \circ \varphi_i)^*\omega \quad \mathscr{H}^k\text{-a.e. on } V_i. \tag{3.21}$$

Let us note the following obvious consequence of the discussions in this section.

Lemma 3.12. *Assume $\mathscr{N}$ compact and $f\colon \mathscr{M} \to \mathscr{N}$ Lipschitz. Let ω be a (everywhere defined) bounded Borel p-form on $\mathscr{N}$. Then $f^*\omega$ is a bounded Borel p-form on $\mathscr{M}$.*

3.4 Orientation

Definition 3.13. The finite chart structure in Definition 3.1 defines an orientation on $\mathscr{M}$ if, for each i and j, $\det \mathrm{D}_v(\varphi_j^{-1} \circ \varphi_i) > 0$ for a.e. $v \in \varphi_i^{-1}(U_i \cap U_j)$.

We say that $\mathscr{M}$ is *oriented* whenever we are given a chart structure as above.

As in the case of differentiable manifolds, an orientation allows to define, for $\mathscr{H}^k$-a.e. $x \in \mathscr{M}$, the notion of direct basis of $T_x\mathscr{M}$. On the other hand, if $\varphi_i^{-1}(U_i \cap U_j)$ is connected (which is equivalent to requiring that $U_i \cap U_j$ itself is connected, since φ_i is bi-Lipschitz), then the sign of $v \mapsto \det \mathrm{D}_v(\varphi_j^{-1} \circ \varphi_i)$ is constant almost everywhere on $\varphi_i^{-1}(U_i \cap U_j)$. For this (not so obvious) property, the reader may refer to Federer [33, Corollary 4.1.26].

A basic class of oriented Lipschitz manifolds is given by the bi-Lipschitz images of smooth oriented manifolds.

Example 3.14. Assume that $\mathscr{M}'$ is a smooth closed oriented manifold, and that $\mathscr{M} = g(\mathscr{M}')$ for some bi-Lipschitz map $g \colon \mathscr{M}' \to \mathscr{M}$. Then g naturally induces a structure of oriented manifold on $\mathscr{M}$. Indeed, let the orientation of $\mathscr{M}'$ be given by a finite atlas $\{(U_i', V_i', \varphi_i')\}_{i\in I}$. Then, clearly, $\{(g(U_i'), V_i', g \circ \varphi_i')\}_{i\in I}$ endows $\mathscr{M}$ with a finite chart structure. This structure defines an orientation. Indeed, for every i and j, we find, using the fact that the atlas on $\mathscr{M}'$ defines an orientation, that

$$\det \mathrm{D}_v((g \circ \varphi_j')^{-1} \circ (g \circ \varphi_i')) = \det \mathrm{D}_v((\varphi_j')^{-1} \circ \varphi_i') > 0$$

for each $v \in (g \circ \varphi_i')^{-1}(g(U_i') \cap g(U_j')) = (\varphi_i')^{-1}(U_i' \cap U_j')$. □

We now give more insight about the orientation induced in Example 3.14. Motivated by the applications we have in mind, we focus on the particular case where $\mathscr{M}'$ is a sphere and $\mathscr{M}$ is the boundary of a convex body (though the same study could be performed, at the cost of more technicality, for the boundary of a Lipschitz open set). This class of examples is sufficiently large to include as a particular instance the case where $\mathscr{M}$ is the boundary of a cube, which will be of crucial importance for us in the sequel.

Example 3.15. Recall that a convex body in $\mathbb{R}^m$ is a compact convex subset C of $\mathbb{R}^m$ with nonempty interior. Without loss of generality, we may assume that $0 \in \operatorname{int} C$. Consider the *Minkowsky gauge* λ_C associated with C,

$$\lambda_C \colon \mathbb{R}^m \to \mathbb{R}_+,\ \lambda_C(y) := \inf\left\{t > 0\colon \frac{1}{t}y \in C\right\},\ \forall\, y \in \mathbb{R}^m. \tag{3.22}$$

The following properties are well-known (and straightforward):

$$\lambda_C \text{ is positively 1-homogeneous,} \tag{3.23}$$

$$\text{when } y \neq 0, \text{ the inf in (3.22) is actually a min, and } \frac{1}{\lambda_C(y)}y \in \partial C, \tag{3.24}$$

$$\lambda_C \text{ is convex (and thus locally Lipschitz).} \tag{3.25}$$

Set

$$\Phi_C \colon \mathbb{R}^m \to \mathbb{R}^m, \Phi_C(y) := \begin{cases} \dfrac{|y|}{\lambda_C(y)} y = \dfrac{1}{\lambda_C(y/|y|)} y, & \text{if } y \neq 0 \\ 0, & \text{if } y = 0 \end{cases}, \tag{3.26}$$

$$\Psi_C \colon \mathbb{R}^m \to \mathbb{R}^m, \Psi_C(x) := \begin{cases} \lambda_C(x/|x|)x, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}. \tag{3.27}$$

It is straightforward (using (3.23)–(3.24)) that: (j) $\Phi_C(\overline{\mathbb{B}}^m) = C$; (jj) $\Phi_C(\mathbb{S}^{m-1}) = \partial C$; (jjj) Ψ_C is the reciprocal of Φ_C. Moreover, using: (i) the definitions (3.26)–(3.27); (ii) (3.25); (iii) standard properties of products and superpositions of locally Lipschitz maps, we find that Φ_C and Ψ_C are locally Lipschitz. Combining the above, we find that Φ_C is a bi-Lipschitz homeomorphism between $\overline{\mathbb{B}}^m$ and C, whose restriction g to $\mathbb{S}^{m-1}$ is a bi-Lipschitz homeomorphism between $\mathbb{S}^{m-1}$ and ∂C. Thus, ∂C fits Example 3.14, with $k := m-1$, $\mathscr{M}' := \mathbb{S}^{m-1}$, and $g := \Phi_{C|\mathbb{S}^{m-1}}$.

Assume that $\mathbb{S}^{m-1}$ is oriented consistently with Stokes' formula on $\mathbb{B}^m$, that is, for every $y = \varphi(v) \in \mathbb{S}^{m-1}$ in the codomain of a chart φ,

$$\text{the } m\text{-tuple } \left(y, \frac{\partial \varphi}{\partial v^1}(v), \dots, \frac{\partial \varphi}{\partial v^{m-1}}(v)\right) \text{ is a direct basis of } \mathbb{R}^m. \tag{3.28}$$

Consider on ∂C the induced parametrization $\psi := g \circ \varphi$. We claim that ψ is consistent with Stokes' formula on C: if $x = \psi(v)$ and ψ is differentiable at v, then

$$\text{the } m\text{-tuple } \left(x, \frac{\partial \psi}{\partial v^1}(v), \dots, \frac{\partial \psi}{\partial v^{m-1}}(v)\right) \text{ is a direct basis of } \mathbb{R}^m. \tag{3.29}$$

Indeed, if we set $t(v) := \dfrac{1}{\lambda_C(\varphi(v))} > 0$ then: (j) (by (3.26)) $\psi = t\varphi$; (jj) ψ is differentiable at v if and only if t is differentiable at v; (jjj) condition (3.29) is equivalent to

$$\det\left(t\varphi, \frac{\partial t}{\partial v^1}\varphi + t\frac{\partial \varphi}{\partial v^1}, \dots, \frac{\partial t}{\partial v^{m-1}}\varphi + t\frac{\partial \varphi}{\partial v^{m-1}}\right) > 0 \tag{3.30}$$

(where the above determinant is evaluated at v). We complete the proof of claim (3.29) by combining (3.30) with the fact that (3.28) is equivalent to

$$\det\left(\varphi, \frac{\partial \varphi}{\partial v^1}, \dots, \frac{\partial \varphi}{\partial v^{m-1}}\right) > 0.$$

□

Example 3.16. Let us now consider the special case where C is a cube aligned with the coordinate axes. For simplicity, we let $C = [-1,1]^m$, but the considerations below, in particular the description of the orientation of the faces, do not depend on this specific choice. Clearly, $\lambda_C(y) = |y|_\infty$ and g is smooth in a neighborhood of $y \in \mathbb{S}^{m-1}$ provided $|y^j| \neq |y^\ell|$ when $j \neq \ell$. Thus, the procedure described in Example 3.15 provides an orientation on ∂C, with parametrizations that are smooth in the interiors of the faces of ∂C. Consider, e.g., the open face

$$F := \{x = (x', 1) \colon x' \in \mathbb{R}^{m-1},\, |x'|_\infty < 1\}.$$

Then, clearly, $T_x \partial C = \mathbb{R}^{m-1} \times \{0\}$, $\forall\, x \in F$. Moreover, in view of claim (3.29), we have

$$\begin{aligned} &((e_1, 0), \ldots, (e_{m-1}, 0)) \in (\mathbb{R}^{m-1} \times \{0\})^{m-1} \text{ is a direct basis of } T_x \partial C \\ &\qquad\qquad\qquad \iff (-1)^{m-1} \det(e_1, \ldots, e_{m-1}) > 0. \end{aligned}$$

Similar considerations apply to the other faces (see also Example 3.18). □

3.5 Integral of forms

In this section, we briefly check that "everything goes as expected" for the integral of k-forms; this crucially relies on the area formula (instead of the standard change of variables formula). We assume that: (a) $\mathscr{M}$ is a compact k-dimensional Lipschitz manifold oriented by a finite chart structure $\{(U_i, V_i, \varphi_i)\}_{i \in I}$; (b) (ξ_i) is a Lipschitz partition of unity subordinated to the covering (U_i).

Let η be a Borel k-form defined in some Borel subset $\widetilde{U}_i$ of U_i. By (3.15), we may uniquely write, for $\mathscr{H}^k$-a.e. $x \in \widetilde{U}_i$,

$$\eta(x) = \alpha_i(x)\, \mathrm{d}_x x_i^1 \wedge \cdots \wedge \mathrm{d}_x x_i^k, \tag{3.31}$$

with α_i a Borel function.

As in the smooth case, we have the following result.

Lemma 3.17. *If η is defined in $U_i \cap U_j$ and α_i is integrable on $U_i \cap U_j$, then α_j is integrable on $U_j \cap U_i$ and (with W_i and W_j as in Lemma 3.7)*

$$\int_{W_i} \alpha_i \circ \varphi_i = \int_{W_j} \alpha_j \circ \varphi_j. \tag{3.32}$$

Proof. If $\varphi \colon W_j \to W_i$ is the transition map in Lemma 3.7, using: (i) Lemma 3.7; (ii) the chain rule; (iii) the exterior calculus rules; (iv) the fact that $\mathscr{M}$ is oriented; (v) the area formula (for the last line), we find (as in the smooth case)

$$\alpha_j(\varphi_j(v)) = \alpha_i(\varphi_j(v)) \det \mathrm{D}_v\varphi \text{ for } \mathscr{H}^k\text{-a.e. } v \in W_j,$$

and finally

$$\begin{aligned}\int_{W_j} \alpha_j(\varphi_j(v))\,\mathrm{d}v &= \int_{W_j} \alpha_i(\varphi_j(v)) \det \mathrm{D}_v\varphi\,\mathrm{d}v = \int_{W_j} \alpha_i(\varphi_j(v))\,|\det \mathrm{D}_v\varphi|\,\mathrm{d}v \\ &= \int_{W_j} \alpha_i(\varphi_i(\varphi(v)))\,|\det \mathrm{D}_v\varphi|\,\mathrm{d}v = \int_{W_i} \alpha_i(\varphi_i(w))\,\mathrm{d}w. \qquad \square\end{aligned}$$

Assume next that η is an $\mathscr{L}^1$ k-form on $\mathscr{M}$, in the sense that, for each i, $\alpha_i \circ \varphi_i$ is integrable on V_i. Using Lemma 3.17, we see that the definition

$$\int_{\mathscr{M}} \eta \coloneqq \sum_i \int_{V_i} (\xi_i\alpha_i) \circ \varphi_i = \sum_i \int_{V_i} (\xi_i \circ \varphi_i)(\alpha_i \circ \varphi_i) \tag{3.33}$$

is correct, in the sense that it does not depend on the choice of the chart structure, and yields a finite real number. Moreover, this definition is consistent with the one in the classical setting.

Example 3.18. Consider the special case where $\mathscr{M}$ is the boundary of a cube C as in Example 3.16, say $C = [-1,1]^{k+1}$. We will establish an explicit formula for the integral of a form on ∂C. Consider the open faces

$$F_{\ell,\pm} \coloneqq \{(x^1, \dots, x^{\ell-1}, \pm 1, x^{\ell+1}, \dots, x^{k+1}) \in \partial C : x^j \in (-1,1), \forall\, j \neq \ell\} \sim (-1,1)^k,$$

and $F \coloneqq \bigcup_{\ell,\pm} F_{\ell,\pm}$, so that $\partial C \setminus F$ is an $\mathscr{H}^k$-null set. Consider a sequence $(\zeta_j) \subset C_c^\infty(F;[0,1])$ such that $\zeta_j(x) \to 1$ for $\mathscr{H}^k$-a.e. $x \in \partial C$. If η is an $\mathscr{L}^1$ k-form on ∂C, then, by dominated convergence,

$$\int_{\partial C} \eta = \lim_j \int_{\partial C} \zeta_j \eta = \lim_j \sum_{\ell,\pm} \int_{F_{\ell,\pm}} \zeta_j \eta \chi_{F_{\ell,\pm}} = \sum_{\ell,\pm} \int_{F_{\ell,\pm}} \eta, \tag{3.34}$$

where $F_{\ell,\pm}$ is equipped with the orientation induced by ∂C.

Moreover, in $F_{\ell,\pm}$ (with $\pm$ fixed) we may write $\eta = \alpha_{\ell,\pm}\,\widehat{\mathrm{d}x^\ell}$, with the convention

$$\widehat{\mathrm{d}x^\ell} \coloneqq \mathrm{d}x^1 \wedge \cdots \mathrm{d}x^{\ell-1} \wedge \mathrm{d}x^{\ell+1} \wedge \cdots \wedge \mathrm{d}x^{k+1} \tag{3.35}$$

and $\alpha_{\ell,\pm} \in \mathscr{L}^1$.

As explained in Example 3.16, $(-e_1, \dots, -e_{\ell-1}, e_{\ell+1}, \dots, e_{k+1})$ is a direct basis of $T_x\partial C$,

$\forall\, x \in F_{\ell,+}$ (and a similar formula holds for $F_{\ell,-}$). Combining this with (3.34) and (3.35), we find that

$$\int_{F_{\ell,+}} \eta = \int_{F_{\ell,+}} \alpha_{\ell,+}\, \widehat{\mathrm{d}x^\ell} = (-1)^{\ell-1} \int_{F_{\ell,+}} \alpha_{\ell,+}.$$

Similarly for $\int_{F_{\ell,-}} \alpha$. Finally, we obtain

$$\int_{\partial C} \eta = \sum_\ell (-1)^{\ell-1} \Big(\int_{(-1,1)^k} \alpha_{\ell,+}\, \widehat{\mathrm{d}x^\ell} - \int_{(-1,1)^k} \alpha_{\ell,-}\, \widehat{\mathrm{d}x^\ell} \Big), \tag{3.36}$$

where we have identified $F_{\ell,\pm}$ with $(-1,1)^k$ with the standard orientation. □

Lemma 3.19. *Let $f\colon \mathscr{M} \to \mathscr{N}$ be a Lipschitz map. Let ω be a (everywhere defined) bounded Borel k-form on $\mathscr{N}$. Then*

$$\int_{\mathscr{M}} f^*\omega = \sum_i \int_{V_i} (\xi_i \circ \varphi_i)\, (f \circ \varphi_i)^*\omega.$$

Proof. Let $\eta := f^*\omega$ and $(e_1, \ldots, e_k)$ be the canonical basis of $\mathbb{R}^k$. In view of the definition (3.33), it suffices to check that, for every i, the function α_i associated with $f^*\omega$ as in (3.31) satisfies

$$\alpha_i(\varphi_i(v)) = (f \circ \varphi_i)^*\omega(v)(e_1, \ldots, e_k) \text{ for } \mathscr{H}^k\text{-a.e. } v \in V_i. \tag{3.37}$$

At a regular point $x = \varphi_i(v) \in U_i$ such that $f \circ \varphi_i$ is differentiable at v, we have, *via*: (i) (3.17); (ii) (3.19) and (3.5); (iii) (3.20),

$$\begin{aligned} \alpha_i(x) &= (f^*\omega)(x)\Big(\frac{\partial}{\partial v_i^1}\Big|_x, \ldots, \frac{\partial}{\partial v_i^k}\Big|_x \Big) \\ &= \omega(f(x))(\mathrm{D}_v(f \circ \varphi_i)(e_1), \ldots, \mathrm{D}_v(f \circ \varphi_i)(e_k)) \\ &= (f \circ \varphi_i)^*\omega(v)(e_1, \ldots e_k), \end{aligned}$$

whence (3.37). □

Remark 3.20. Let us note a variant of the above considerations and definitions if, instead of $\mathscr{M}$, we consider the product $\widetilde{\mathscr{M}} := \mathscr{M} \times J$ with $J = (a,b) \subset \mathbb{R}$ a non-empty open interval. Clearly, if $\mathscr{M}$ has a (oriented) finite chart structure, then $\widetilde{\mathscr{M}}$ has a natural (oriented) finite chart structure, by setting

$$\widetilde{V}_i := V_i \times J, \widetilde{U}_i := U_i \times J, \widetilde{\varphi}_i(v,t) := (\varphi_i(v), t), \forall\, v \in V_i, \forall\, t \in J. \tag{3.38}$$

Assume that $\mathscr{M}$ is compact and I is bounded. Given a bounded Borel $(k+1)$-form η on $\widetilde{\mathscr{M}}$, we write, in $\widetilde{U}_i$, $\eta(x) = \alpha_i(x,t)\,\mathrm{d}_x x_i^1 \wedge \cdots \wedge \mathrm{d}_x x_i^k \wedge \mathrm{d}t$, and naturally set

$$\int_{\widetilde{\mathscr{M}}} \eta := \sum_i \int_{V_i \times J} (\xi_i \circ \varphi_i)\,(\alpha_i \circ \widetilde{\varphi}_i).$$

This definition is correct, consistent with the case of smooth manifolds, and the analogue of Lemma 3.19 holds, i.e., when $F\colon \widetilde{\mathscr{M}} \to \mathscr{N}$ is Lipschitz and λ is a (everywhere defined) bounded Borel $(k+1)$-form on $\mathscr{N}$, we have

$$\int_{\widetilde{\mathscr{M}}} F^*\lambda = \sum_i \int_{V_i \times J} (\xi_i \circ \varphi_i)\,(F \circ \widetilde{\varphi}_i)^*\lambda. \qquad \square \tag{3.39}$$

Remark 3.21. For further use, we note the following identity. Consider the setting in Remark 3.20 and assume that I is bounded. Let $\varphi\colon W_j \to W_i$ be as in Lemma 3.7, and set $\widetilde{\varphi}(v,t) := (\varphi(v), t)$, $\forall\, v \in W_j$, $\forall\, t \in \mathbb{R}$. Let ω be a (everywhere defined) bounded Borel k-form on $\mathscr{N}$. Let $f\colon V_i \times J \to \mathbb{R}$ be a bounded Borel function supported in $W_i \times J$. Let $g\colon V_i \times J \to \mathbb{R}$, respectively $G\colon V_i \times J \to \mathscr{N}$, be Lipschitz maps. Then

$$\int_{V_i \times J} f\,\mathrm{d}g \wedge G^*\omega = \int_{V_j \times J} f \circ \widetilde{\varphi}(\mathrm{d}(g \circ \widetilde{\varphi})) \wedge (G \circ \widetilde{\varphi})^*\omega. \tag{3.40}$$

Formula (3.40) is obtained by repeating the proof of (3.32) and using the exterior differential calculus rules for Lipschitz maps (see, e.g., the proof of (3.37)). $\square$

We next extend the definition of $\int_{\mathscr{M}} f^*\omega$ to $W^{1,k}$ maps. Clearly, the definition of $W^{1,p}(\mathscr{M})$ adapted to our setting is the following: a map $f\colon \mathscr{M} \to \mathbb{R}$ belongs to $W^{1,p}(\mathscr{M})$ whenever $f \circ \varphi_i \in W^{1,p}(V_i)$ for every i. The next definition is also natural.

Definition 3.22. For almost every regular point $x = \varphi_i(v) \in \mathscr{M}$, we let

$$\mathrm{d}_x f\colon T_x\mathscr{M} \to \mathbb{R},\ \mathrm{d}_x f(\mathrm{D}_v \varphi_i(\xi)) := \mathrm{D}_v(f \circ \varphi_i)(\xi),\ \forall\, \xi \in \mathbb{R}^k. \tag{3.41}$$

Similarly when $f\colon \mathscr{M} \to \mathbb{R}^n$.

It is obvious, by the chain rule, that the above definitions do not depend on the choice of the chart.

Remark 3.23. We present a counterpart of Remark 3.8 adapted to Sobolev maps. Let $\mathscr{N}$ be a C^1-submanifold of $\mathbb{R}^n$ and V be an open subset of $\mathbb{R}^k$. Assume that $g \in W^{1,1}_{\mathrm{loc}}(V;\mathbb{R}^n)$ is such that $g(V) \subset \mathscr{N}$. We claim that, for a.e. $v \in V$, $\mathrm{D}_v g(\mathbb{R}^k) \subset T_{g(v)}\mathscr{N}$. To prove this, one may rely, e.g., on the following argument. Let $(g_j) \subset C^\infty(V;\mathbb{R}^n)$ converge to g in $W^{1,1}_{\mathrm{loc}}(V)$. Up to extraction of a subsequence, we may further assume that $g_j \to g$ and

$\mathrm{D}g_j \to \mathrm{D}g$ almost everywhere. Let Π be as in Definition 2.10, and let $\widetilde{\Pi} \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$ be such that

$$\widetilde{\Pi}(z) = \Pi(z), \forall z \in \mathscr{N}_{\delta/2}.$$

By the chain rule, the map $\widetilde{\Pi} \circ g_j$ belongs to $W^{1,p}(V; \mathbb{R}^n)$, and satisfies

$$\mathrm{D}_v(\widetilde{\Pi} \circ g_j)(\xi) = \mathrm{D}_{g_j(v)}\widetilde{\Pi}(\mathrm{D}_v g_j(\xi)), \forall v \in V, \forall \xi \in \mathbb{R}^k.$$

When $j \to \infty$, we have

$$\mathrm{D}_v(\widetilde{\Pi} \circ g_j)(\xi) \to \mathrm{D}_{g(v)}\Pi(\mathrm{D}_v g(\xi)) \in T_{g(v)}\mathscr{N}, \text{ for a.e. } v \in V, \forall \xi \in \mathbb{R}^k,$$

where we have used the fact that $\widetilde{\Pi} = \Pi$ near $\mathscr{N}$.

On the other hand, by the continuity of the superposition operator, and up to a further extraction, we may assume that

$$\mathrm{D}_v(\widetilde{\Pi} \circ g_j)(\xi) \to \mathrm{D}_v(\widetilde{\Pi} \circ g)(\xi) = \mathrm{D}_v g(\xi), \text{ for a.e. } v \in V, \forall \xi \in \mathbb{R}^k,$$

which shows our claim. □

We endow $W^{1,p}(\mathscr{M})$ with the natural norm $f \mapsto \sum_i \|f \circ \varphi_i\|_{W^{1,p}(V_i)}$. It is straightforward that two different chart structures yield equivalent norms.

We can extend the definition of $f^*\omega$ (see Definition 3.11) and Lemma 3.19 to the case where $f \in W^{1,p}(\mathscr{M}; \mathscr{N})$.

Definition 3.24. Let $\mathscr{N}$ be a C^1-submanifold of $\mathbb{R}^n$. Let ω be a p-form on $\mathscr{N}$ (defined everywhere) and $f \in W^{1,1}_{\mathrm{loc}}(\mathscr{M}; \mathscr{N})$. For almost every regular point $x = \varphi_i(v) \in \mathscr{M}$, we let

$$f^*\omega(x)(y_1, \ldots, y_p) := \omega(f(x))(\mathrm{d}_x f(y_1), \ldots, \mathrm{d}_x f(y_p)), \forall y_1, \ldots, y_p \in T_x\mathscr{M}. \tag{3.42}$$

We note that the above definition is consistent with Definition 3.11 (which involves Lipschitz maps), and does not depend on i.

Similar to (3.20), one can define, using (3.42), for a.e. regular point $x = \varphi_i(v) \in \mathscr{M}$,

$$(f \circ \varphi_i)^*\omega(v)(\xi_1, \ldots, \xi_p) := \omega(f(x))(\mathrm{D}_v(f \circ \varphi_i)(\xi_1), \ldots, \mathrm{D}_v(f \circ \varphi_i)(\xi_p)),$$
$$\forall \xi_1, \ldots, \xi_p \in \mathbb{R}^k.$$

Then we have the analogue of (3.21),

$$(\varphi_i)^*(f^*\omega) = (f\circ\varphi_i)^*\omega \quad \mathscr{H}^k\text{-a.e. on } V_i.$$

Lemma 3.25. *For $f \in W^{1,k}(\mathscr{M};\mathscr{N})$ and ω a (everywhere defined) bounded Borel k-form on $\mathscr{N}$, we have*

$$\int_{\mathscr{M}} f^*\omega = \sum_i \int_{V_i} (\xi_i\circ\varphi_i)\,(f\circ\varphi_i)^*\omega.$$

Lemma 3.25 is obtained by repeating the proof of Lemma 3.19.

3.6 An adapted Stokes' formula

Throughout this section: (a) $\mathscr{M}$ is a compact k-dimensional Lipschitz manifold oriented by a finite chart structure $\{(U_i, V_i, \varphi_i)\}_{i\in I}$; (b) $J = (a,b)$ is a bounded interval; (c) $\mathscr{N}$ is a closed manifold; (d) ω is a smooth k-form on $\mathscr{N}$. We state and prove a formula in the spirit of the Stokes formula on $\mathscr{M}\times J$. (See Remark 3.20 for the integration on $\mathscr{M}\times J$.) For the sake of concision, given a map $F\colon X\times Y\to Z$ and $y\in Y$, we write $F_y := F(\cdot, y)$.

Proposition 3.26. *Let $F\colon \mathscr{M}\times[a,b]\to\mathscr{N}$ be a Lipschitz map. Then*

$$\int_{\mathscr{M}\times(a,b)} F^*(\mathrm{d}\omega) = \int_{\mathscr{M}} (F_b)^*\omega - \int_{\mathscr{M}} (F_a)^*\omega. \tag{3.43}$$

Similarly, if $F\colon \mathscr{M}\times[a,b]\to\mathbb{R}^n$ is a Lipschitz map and α is a smooth k-form on $\mathbb{R}^n$ with bounded coefficients, then

$$\int_{\mathscr{M}\times(a,b)} F^*(\mathrm{d}\alpha) = \int_{\mathscr{M}} (F_b)^*\alpha - \int_{\mathscr{M}} (F_a)^*\alpha. \tag{3.44}$$

When both $\mathscr{M}$ and F are smooth, (3.43) is a special case of the Stokes formula on $\mathscr{M}\times(a,b)$.

Proof. We only prove (3.43), since (3.44) follows from a similar argument. In particular, one readily checks that all the ingredients involved in the proof of (3.43) below have a valid counterpart when F is $\mathbb{R}^n$ and α is a smooth k-form on $\mathbb{R}^n$ with bounded coefficients.

Let (ξ_i) be a partition of unity subordinated to the finite chart structure $\{(U_i, V_i, \varphi_i)\}_{i\in I}$ on $\mathscr{M}$. With $J := (a,b)$, we use the notation in (3.38).

By Lemma 3.19, we have

$$\begin{aligned}\int_{\mathscr{M}}(F_b)^*\omega-\int_{\mathscr{M}}(F_a)^*\omega&=\sum_i\int_{V_i}(\xi_i\circ\varphi_i)(F_b\circ\varphi_i)^*\omega-\sum_i\int_{V_i}(\xi_i\circ\varphi_i)(F_a\circ\varphi_i)^*\omega\\&=\sum_i\int_{V_i}(\xi_i\circ\varphi_i)[(F\circ\widetilde{\varphi}_i)_b]^*\omega\\&\quad-\sum_i\int_{V_i}(\xi_i\circ\varphi_i)[(F\circ\widetilde{\varphi}_i)_a]^*\omega.\end{aligned}\tag{3.45}$$

We extend $\xi_i\circ\varphi_i$ and $F\circ\widetilde{\varphi}_i$ to Lipschitz maps G_i and H_i defined on $V_i\times\mathbb{R}$ by letting

$$G_i(v,t):=\xi_i\circ\varphi_i(v),\text{ respectively } H_i(v,t):=\begin{cases}F\circ\widetilde{\varphi}_i(v,t), & \text{if } a\le t\le b\\ F\circ\widetilde{\varphi}_i(v,a), & \text{if } t\le a\\ F\circ\widetilde{\varphi}_i(v,b), & \text{if } t\ge b\end{cases}.$$

We next consider open neighborhoods W_i of the support of $\xi_i\circ\varphi_i$ such that $\overline{W}_i\subset V_i$. Finally, we set $G_{i,\varepsilon}:=G_i*\rho_\varepsilon$ and $H_{i,\varepsilon}:=H_i*\rho_\varepsilon$, where ρ is a standard mollifier in $\mathbb{R}^{k+1}$ and $\varepsilon>0$ is chosen sufficiently small so that: (i) $G_{i,\varepsilon}$ and $H_{i,\varepsilon}$ are well-defined and smooth in $W_i\times\mathbb{R}$; (ii) $G_{i,\varepsilon}$ are supported in $W_i\times\mathbb{R}$.

It is readily seen that: (j) $G_{i,\varepsilon}\to G_i$ and $H_{i,\varepsilon}\to H_i$ uniformly as $\varepsilon\to0$; (jj) $\mathrm{D}_{(v,t)}G_{i,\varepsilon}\to\mathrm{D}_{(v,t)}G_i$ and $\mathrm{D}_{(v,t)}H_{i,\varepsilon}\to\mathrm{D}_{(v,t)}H_i$ for almost every (v,t) as $\varepsilon\to0$; (jjj) for any $t\notin[a,b]$, $\mathrm{D}_vH_{i,\varepsilon}(\cdot,t)\to\mathrm{D}_vH_i(\cdot,t)$ for almost every v as $\varepsilon\to0$; (jjjj) $G_{i,\varepsilon}$, $H_{i,\varepsilon}$, $\mathrm{D}G_{i,\varepsilon}$, and $\mathrm{D}H_{i,\varepsilon}$ are uniformly bounded, independently of ε, on $W_i\times\mathbb{R}$.

Fix $c<a<b<d$. Using: (i) the fact that $\xi_i\circ\varphi_i$ is compactly supported in W_i; (ii) the fact that $G_{i,\varepsilon}$ does not depend on t; (iii) the divergence theorem for smooth forms; (iv) the exterior product rules, we find that

$$\begin{aligned}\int_{W_i}&G_{i,\varepsilon}\cdot[(H_{i,\varepsilon})_d]^*\omega-\int_{W_i}G_{i,\varepsilon}\cdot[(H_{i,\varepsilon})_c]^*\omega\\&=\int_{W_i\times(c,d)}\mathrm{d}[G_{i,\varepsilon}\cdot(H_{i,\varepsilon})^*\omega]\\&=\int_{W_i\times(c,d)}[(\mathrm{d}G_{i,\varepsilon})\wedge(H_{i,\varepsilon})^*\omega+G_{i,\varepsilon}\cdot\mathrm{d}((H_{i,\varepsilon})^*\omega)]\\&=\int_{W_i\times(c,d)}[(\mathrm{d}G_{i,\varepsilon})\wedge(H_{i,\varepsilon})^*\omega+G_{i,\varepsilon}\cdot(H_{i,\varepsilon})^*(\mathrm{d}\omega)].\end{aligned}\tag{3.46}$$

Letting $\varepsilon\to0$ in (3.46) and using (j)–(jjjj) above to justify the use of the dominated convergence theorem, we find

$$\int_{W_i} G_i \cdot [(H_i)_d]^* \omega - \int_{W_i} G_i \cdot [(H_i)_c]^* \omega = \int_{W_i \times (c,d)} [(\mathrm{d}G_i) \wedge (H_i)^* \omega + G_i \cdot (H_i)^* (\mathrm{d}\omega)]. \tag{3.47}$$

Then, we observe that, by construction of G_i and H_i, we have

$$\int_{W_i} G_i \cdot [(H_i)_d]^* \omega - \int_{W_i} G_i \cdot [(H_i)_c]^* \omega = \int_{W_i} G_i \cdot [(H_i)_b]^* \omega - \int_{W_i} G_i \cdot [(H_i)_a]^* \omega. \tag{3.48}$$

Letting $c \to a$ and $d \to b$ in (3.47), summing over i, and using: (i) (3.48); (ii) (3.45); (iii) the fact that $\xi_i \circ \varphi_i$ is compactly supported in W_i, we deduce that

$$\int_{\mathscr{M}} (F_b)^* \omega - \int_{\mathscr{M}} (F_a)^* \omega = \sum_i \int_{V_i \times (a,b)} [(\mathrm{d}G_i) \wedge (H_i)^* \omega + G_i \cdot (H_i)^* (\mathrm{d}\omega)]. \tag{3.49}$$

From (3.49), (3.39) (with $\lambda := \mathrm{d}\omega$), and the fact that, on $V_i \times (a, b)$, we have $G_i = \xi_i \circ \varphi_i$ and $H_i = F \circ \widetilde{\varphi}_i$, we find that (3.43) holds provided we have the identity

$$\sum_i \int_{V_i \times (a,b)} (\mathrm{d}(\xi_i \circ \varphi_i)) \wedge (F \circ \widetilde{\varphi}_i)^* \omega = 0, \tag{3.50}$$

that we next prove. Let S denote the sum in (3.50). Since $\sum_j \xi_j = 1$ on $\mathscr{M}$, we have

$$S = \sum_{i,j} \int_{V_i \times (a,b)} \xi_j \circ \varphi_i (\mathrm{d}(\xi_i \circ \varphi_i)) \wedge (F \circ \widetilde{\varphi}_i)^* \omega. \tag{3.51}$$

We next apply to the integrals in (3.51) the identity (3.40) (with $f(v, t) := \xi_j \circ \varphi_i(v)$, $g(v, t) := \xi_i \circ \varphi_i(v)$, $G := F \circ \widetilde{\varphi}_i$) and obtain

$$S = \sum_{i,j} \int_{V_j \times (a,b)} \xi_j \circ \varphi_j (\mathrm{d}(\xi_i \circ \varphi_j)) \wedge (F \circ \widetilde{\varphi}_j)^* \omega,$$

where we have used the fact that $\varphi_i(\varphi(v)) = \varphi_j(v)$, $\forall\, v \in W_j$. Finally, since $\sum_i \xi_i \circ \varphi_j = 1$, $\forall\, j$, we have $\sum_i \mathrm{d}(\xi_i \circ \varphi_j) = 0$ a.e. in $V_j \times (a, b)$, $\forall\, j$. Therefore, $S = 0$, and thus (3.50) holds, as claimed. □

3.7 Integral invariants for $\mathrm{VMO}(\mathscr{M};\mathscr{N})$ maps

Throughout this section: (a) $\mathscr{M}$ is a compact k-dimensional Lipschitz manifold oriented by a finite chart structure $\{(U_i,V_i,\varphi_i)\}_{i\in I}$; (b) $\mathscr{N}$ is a closed manifold; (c) ω is a smooth *closed* k-form on $\mathscr{N}$. We prove that ω induces a homotopical invariant $\int_{\mathscr{M}} f^*\omega$ on $\mathrm{VMO}(\mathscr{M};\mathscr{N})$.

We first investigate the case of Lipschitz maps. Let $\delta=\delta(\mathscr{N})$ be as is Definition 2.10.

Proposition 3.27. *Consider two Lipschitz maps $f,g\colon\mathscr{M}\to\mathscr{N}$ such that $\|f-g\|_\infty\le\delta/2$. Then*

$$\int_{\mathscr{M}} g^*\omega=\int_{\mathscr{M}} f^*\omega.$$

Proof. We have $tg(x)+(1-t)f(x)\in\mathscr{N}_\delta$, $\forall\,x\in\mathscr{M}$, $\forall\,t\in[0,1]$. Therefore, the map $F(x,t):=\Pi(tg+(1-t)f)$, $x\in\mathscr{M}$, $t\in[0,1]$, with Π as in Definition 2.10, is well-defined and Lipschitz. Moreover, we have $F_1=g$ and $F_0=f$. (Recall the notation $F_y:=F(\cdot,y)$.) Hence, we are in position to apply Proposition 3.26, which yields

$$\int_{\mathscr{M}} g^*\omega-\int_{\mathscr{M}} f^*\omega=\int_{\mathscr{M}}(F_1)^*\omega-\int_{\mathscr{M}}(F_0)^*\omega=\int_{\mathscr{M}\times(0,1)}F^*(\mathrm{d}\omega)=0.\qquad\square$$

Corollary 3.28. *For $f\in C(\mathscr{M};\mathscr{N})$, set $\mathscr{I}(f)=\mathscr{I}_{\mathscr{M},\omega}(f):=\int_{\mathscr{M}} g^*\omega$, where $g\in\mathrm{Lip}(\mathscr{M};\mathscr{N})$ is such that $\|f-g\|_\infty\le\delta/4$. Then the definition is correct (i.e., it does not depend on g) and $\mathscr{I}(f)$ is a homotopical invariant.*

(The existence of such g is straightforward, since $\mathscr{M}$ is a compact subset of $\mathbb{R}^m$. On the other hand, if f happens to be Lipschitz, then $\mathscr{I}(f)$ coincides with $\int_{\mathscr{M}} f^*\omega$.)

Corollary 3.29. *For $f\in\mathrm{VMO}(\mathscr{M};\mathscr{N})$, set $\mathscr{I}(f):=\mathscr{I}(f^\varepsilon)$ for $\varepsilon<\varepsilon_1$, with ε_1 as in (2.31). Then the definition is correct (i.e., it does not depend on ε) and $\mathscr{I}(f)$ is a homotopical invariant and locally constant.*

Moreover, the definition of $\mathscr{I}(f)$ is consistent with the one in Corollary 3.28.

Proof. Combine the discussions preceding and following Definition 2.11 with the previous corollary. $\square$

Remark 3.30. In the special case where $\mathscr{M}$ and f are C^1, $\mathscr{N}=\mathbb{S}^k$, and $\omega=\omega_{\mathbb{S}^k}$ is the standard volume form

$$\omega_{\mathbb{S}^k}:=\sum_{j=1}^{k+1}(-1)^{j-1}\widehat{\mathrm{d}x^j},$$

we have

$$\int_{\mathscr{M}} f^* \omega_{\mathbb{S}^k} = |\mathbb{S}^k| \deg f,$$

and thus $\int_{\mathscr{M}} f^* \omega_{\mathbb{S}^k}$ is, up to a constant, the Brouwer degree of the map f, as explored notably in the monograph [30] by Dincă and Mawhin.

In this special case, Corollary 3.29 implies in particular that the Brouwer degree, initially defined for C^1 maps, can be extended to $\mathrm{VMO}(\mathscr{M}; \mathbb{S}^k)$. This fact is already contained in Brezis and Nirenberg [24]. □

Remark 3.31. If f is Lipschitz, then $\int_{\mathscr{M}} f^* \omega$ makes sense, as an integral, for every smooth k-form ω, not necessarily closed. However, Proposition 3.26 suggests that the closedness assumption is necessary to make this quantity a homotopical invariant. On the other hand, if f is merely VMO, then the assumption that ω is closed is required even to *define* $\mathscr{I}(f)$. □

The following corollary asserts that the integral invariant we have just defined is stable under composition with orientation preserving bi-Lipschitz transformations.

Corollary 3.32. *Let $\widetilde{\mathscr{M}}$ be a Lipschitz manifold and $\Psi \colon \widetilde{\mathscr{M}} \to \mathscr{M}$ be a bi-Lipschitz orientation preserving map. For $f \in \mathrm{VMO}(\mathscr{M}; \mathscr{N})$, we have*

$$\mathscr{I}_{\mathscr{M},\omega}(f) = \mathscr{I}_{\widetilde{\mathscr{M}},\omega}(f \circ \Psi).$$

Proof. This is clear if f is Lipschitz (by the chain rule and Lemma 3.19). The case where f is continuous follows by approximation, *via* Corollary 3.28.

For a "general" map $f \in \mathrm{VMO}(\mathscr{M}; \mathscr{N})$, by Corollary 3.29 we have, for small ε,

$$\mathscr{I}_{\mathscr{M},\omega}(f) \coloneqq \mathscr{I}_{\mathscr{M},\omega}(f^\varepsilon) = \mathscr{I}_{\widetilde{\mathscr{M}},\omega}(f^\varepsilon \circ \Psi) = \mathscr{I}_{\widetilde{\mathscr{M}},\omega}(\Pi \circ f_\varepsilon \circ \Psi).$$

By Corollary 2.6 and Lemma 2.7, we have $f_\varepsilon \to f$ in $\mathrm{BMO} \cap \mathscr{L}^1$. It is then straightforward that $f_\varepsilon \circ \Psi \to f \circ \Psi$ in $\mathrm{BMO} \cap \mathscr{L}^1$ (since Ψ is bi-Lipschitz). In particular, we have $f \circ \Psi \in \mathrm{VMO}$ (see the definition (2.7) of VMO). Next, we use the fact that the superposition with Lipschitz functions is continuous in VMO (see Brezis and Nirenberg [24, Lemma A.8]) to deduce that

$$\Pi \circ f_\varepsilon \circ \Psi \to \Pi \circ f \circ \Psi = f \circ \Psi \text{ in } \mathrm{BMO} \cap \mathscr{L}^1. \tag{3.52}$$

We complete the proof by combining (3.52) with Corollary 3.29. □

Combining Corollary 3.29 with Proposition 2.15, we obtain the following result.

Corollary 3.33. *There exists some finite positive constant* $C = C(\mathscr{M}, \mathscr{N})$ *such that*

$$[f \in \mathrm{VMO}(\mathscr{M}; \mathscr{N}), |f|_{\mathrm{BMO}} \leq C] \implies \mathscr{I}(f) = 0.$$

For pedagogical reasons, we postpone the study of further properties of $f^*\omega$ and $\mathscr{I}(f)$ to Section 5; see, in particular, Sections 5.1 and 5.2.

3.8 The case of $W^{1,k}(\mathscr{M}; \mathscr{N})$ maps

Recall the embedding

$$W^{1,k}(\mathscr{M}) \hookrightarrow (\mathrm{VMO} \cap \mathscr{L}^1)(\mathscr{M}). \tag{3.53}$$

Indeed, this is well-known when $\mathscr{M}$ is smooth. In order to prove (3.53) in the Lipschitz case, it suffices to repeat the argument in Brezis and Nirenberg [24, Example 1]. Consequently, when ω is closed, the invariant $\mathscr{I}(f)$ makes sense (see Corollary 3.29) and it is viewed as an *extension* of $\int_{\mathscr{M}} f^*\omega$. However, we have at hand another natural definition of $\int_{\mathscr{M}} f^*\omega$ as the integral of an $\mathscr{L}^1$ function defined a.e. (see Lemma 3.25). Let us note that $\int_{\mathscr{M}} f^*\omega$ makes sense even if ω is not closed. The following proposition shows that, for $f \in W^{1,k}(\mathscr{M}; \mathscr{N})$ and ω a smooth closed k-form, the two definitions yield the same quantity.

Proposition 3.34. *Let* $f \in W^{1,k}(\mathscr{M}; \mathscr{N})$ *and* ω *a smooth closed* k*-form. Then we have*

$$\mathscr{I}(f) = \int_{\mathscr{M}} f^*\omega. \tag{3.54}$$

Proof. The identity (3.54) is obtained *via*: (i) Lemma 3.35 below; (ii) the fact that $f_j \to f$ in $W^{1,k}(\mathscr{M}; \mathscr{N})$ implies $\int_{\mathscr{M}} f_j^*\omega \to \int_{\mathscr{M}} f^*\omega$; (iii) the embedding (3.53); (iv) Corollary 2.13; (v) Corollary 3.29. □

Lemma 3.35. *The space* $\mathrm{Lip}(\mathscr{M}; \mathscr{N})$ *is dense in* $W^{1,k}(\mathscr{M}; \mathscr{N})$*.*

Proof. We let (U_i, V_i, φ_i), K_2 be as in Sections 3.1 and 3.5. Let $\varepsilon_1 = \varepsilon_1(\mathscr{M}) > 0$ be such that the open sets

$$U_i' := \{x \in U_i : B_{\varepsilon_1}((\varphi_i)^{-1}(x)) \subset V_i\} = \varphi_i(\{v \in V_i : \mathrm{dist}\,(v, (V_i)^c) > \varepsilon_1\}) \tag{3.55}$$

cover $\mathscr{M}$.

For $f \in \mathscr{L}^1(\mathscr{M}; \mathbb{R}^n)$ and $v \in V_i$, set $\overline{f}_i(v) := f \circ \varphi_i(v)$.

Consider a Lipschitz partition of unity $(\xi_i)_{i\in I}$ subordinated to the cover $(U_i')_{i\in I}$ of $\mathscr{M}$. Let $\rho \in C_c^\infty(\mathbb{B}^k)$ be a mollifier. For $0 < \varepsilon \le \varepsilon_1$ and $x \in \mathscr{M}$, set

$$\overline{f}_{i,\varepsilon} \coloneqq \xi_i\,[(\overline{f}_i * \rho_\varepsilon)\circ(\varphi_i)^{-1}] \text{ and } \overline{f}_\varepsilon \coloneqq \sum_i \overline{f}_{i,\varepsilon} \tag{3.56}$$

(with the natural convention that $\overline{f}_{i,\varepsilon}(x) = 0$ if $x \notin U_i'$). Clearly $\overline{f}_{i,\varepsilon}$ is Lipschitz, and thus so is $\overline{f}_\varepsilon$.

Step 1. We have $\overline{f}_\varepsilon \to f$ in $W^{1,k}(\mathscr{M})$ as $\varepsilon \to 0$. In order to see this, it suffices to prove that $\overline{f}_{i,\varepsilon} \to \xi_i f$ in $W^{1,k}(\mathscr{M})$. Clearly, we have

$$\overline{f}_{i,\varepsilon}\circ\varphi_i = (\xi_i\circ\varphi_i)(\overline{f}_i * \rho_\varepsilon) \to (\xi_i\circ\varphi_i)\overline{f}_i = (\xi_i\circ\varphi_i)(f\circ\varphi_i) \text{ in } W^{1,k}(V_i). \tag{3.57}$$

Combining (3.57) with the fact that $\varphi_i^{-1}\circ\varphi_j$ is bi-Lipschitz, we obtain that $\overline{f}_{i,\varepsilon}\circ\varphi_j \to (\xi_i\circ\varphi_j)(f\circ\varphi_j)$ in $W^{1,k}(V_j)$, and therefore $\overline{f}_{i,\varepsilon} \to \xi_i f$ in $W^{1,k}(\mathscr{M})$.

Step 2. We have $\operatorname{dist}(\overline{f}_\varepsilon(x), \mathscr{N}) \to 0$ uniformly as $\varepsilon \to 0$. Indeed, starting from the identity $\sum_i \xi_i(x) f(y) = f(y)$ and using (3.53), we find that

$$\begin{aligned}\operatorname{dist}(\overline{f}_\varepsilon(x), \mathscr{N}) &\le \fint_{B_{K_2\varepsilon}(x)} |\overline{f}_\varepsilon(x) - f(y)|\,\mathrm{d}\mathscr{H}^k(y)\\ &\le \sum_i \xi_i(x) \fint_{B_{K_2\varepsilon}(x)} |[\overline{f}_i - f(y)] * \rho_\varepsilon((\varphi_i)^{-1}(x))|\,\mathrm{d}\mathscr{H}^k(y)\\ &\le C M_{K_2\varepsilon}(f).\end{aligned} \tag{3.58}$$

Step 3. We have $\Pi\circ\overline{f}_\varepsilon \to f$ in $W^{1,k}(\mathscr{M};\mathscr{N})$ as $\varepsilon \to 0$. Indeed, by the previous steps, for sufficiently small ε, $\Pi\circ\overline{f}_\varepsilon$ is well-defined and Lipschitz. By a standard property of superposition operators in $W^{1,p}$, we have $\Pi\circ\overline{f}_\varepsilon \to \Pi\circ f = f$ in $W^{1,k}(\mathscr{M};\mathscr{N})$ as $\varepsilon \to 0$. This completes the proof. □

Remark 3.36. For the record, we note that a variant of the proof of Lemma 3.35 leads to the the following result, that we will not use and whose detailed proof is presented in Appendix B. □

Lemma 3.37. *Assume that* $0 < s \le 1$ *and* $1 \le p < \infty$ *are such that* $sp \ge k$*. Then the space* $\operatorname{Lip}(\mathscr{M};\mathscr{N})$ *is dense in* $W^{s,p}(\mathscr{M};\mathscr{N})$*.*

In the above, when $0 < s < 1$, one naturally defines $W^{s,p}(\mathscr{M})$ as $\{f : \mathscr{M} \to \mathbb{R} : |f|_{W^{s,p}} <$

$\infty\}$, where

$$|f|^p_{W^{s,p}} := \int_{\mathscr{M}} \int_{\mathscr{M}} \frac{|f(x) - f(y)|^p}{\operatorname{dist}(x,y)^{k+sp}} \, d\mathscr{H}^k(x) \, d\mathscr{H}^k(y).$$

4 Estimate of $\mathscr{I}(f)$

Throughout this section: (a) $\mathscr{M}$ is a compact k-dimensional Lipschitz manifold oriented by a finite chart structure $\{(U_i, V_i, \varphi_i)\}_{i \in I}$; (b) $\mathscr{N}$ is a closed manifold; (c) ω is a smooth *closed* k-form on $\mathscr{N}$. We establish an analytical estimate on the integral invariants that we constructed in Section 3.7.

We have

$$W^{s,p}(\mathscr{M}) \hookrightarrow \mathrm{VMO}(\mathscr{M}), \text{ when } 0 < s < 1, 1 < p < \infty, \text{ and } sp = k. \tag{4.1}$$

(To see this, it suffices to repeat the argument in Brezis and Nirenberg [24, Example 2, Case 2] and to rely on (3.4).)

From now on, we assume that $0 < s < 1$ *and* $sp = k$ (and thus the embedding in (4.1) holds). For $f \in W^{s,p}(\mathscr{M}; \mathscr{N})$, our purpose here is to control $|\mathscr{I}(f)|$ (defined in Corollary 3.29) by $|f|_{W^{s,p}}$. This significantly generalizes the corresponding result in Bourgain, Brezis, and Mironescu [11]. (There, $\mathscr{M} = \mathbb{S}^k$, $\mathscr{N} = \mathbb{S}^k$, and ω is the standard volume form on $\mathbb{S}^k$.) The main result of this section is the following.

Theorem 4.1. *There exists a finite constant* $C = C(\mathscr{M}, \mathscr{N}, \omega, s, p)$ *such that*

$$|\mathscr{I}(f)| \leq C|f|^p_{W^{s,p}}, \forall f \in W^{s,p}(\mathscr{M}; \mathscr{N}). \tag{4.2}$$

When f is continuous and $\mathscr{M} = \mathbb{S}^k$, Theorem 4.1 was already contained in Van Schaftingen [63, Theorem 6.1].

Remark 4.2. For a refinement of the estimate (4.2), see Appendix B. □

Proof of Theorem 4.1. In view of Corollary 3.33, it suffices to prove, instead of (4.2), the following seemingly weaker estimate

$$|\mathscr{I}(f)| \leq C(|f|^p_{W^{s,p}} + 1), \forall f \in W^{s,p}(\mathscr{M}; \mathscr{N}). \tag{4.3}$$

In what follows, we fix a map $f \in W^{s,p}(\mathscr{M};\mathscr{N})$. Let, as in (2.9),

$$\begin{aligned} F(x,\varepsilon) := f_\varepsilon(x) = K(x,\varepsilon)\int_{\mathscr{M}} \rho(x,\varepsilon,y) f(y)\,\mathrm{d}\mathscr{H}^k(y) \\ = \int_{\mathscr{M}} \widetilde{\rho}(x,\varepsilon,y) f(y)\,\mathrm{d}\mathscr{H}^k(y), \forall\, x\in\mathscr{M}, \forall\,\varepsilon>0, \end{aligned} \tag{4.4}$$

where we have set

$$\widetilde{\rho}(x,\varepsilon,y) := K(x,\varepsilon)\rho(x,\varepsilon,y), \forall\, x,y\in\mathscr{M}, \forall\,\varepsilon>0. \tag{4.5}$$

(The relevance of considering F in the setting of Theorem 4.1 comes from Corollary 3.29 and the definition (2.32).) Let us note that F makes sense when $f\colon \mathscr{M}\to\mathscr{N}$ is merely a measurable map.

The next result, whose proof is postponed, collects some straightforward properties of $\widetilde{\rho}$. Since it does not rely on $\mathscr{M}$ being a Lipschitz manifold, it is stated in the more general setting of Section 2.

Lemma 4.3. *Assume that $\mathscr{M}$ is a compact doubling metric measure space. Then we have*

$$\int_{\mathscr{M}} \widetilde{\rho}(x,\varepsilon,y)\,\mathrm{d}\mu(y) = 1, \forall\, x\in\mathscr{M}, \forall\,\varepsilon>0, \tag{4.6}$$

$$\begin{aligned} |\widetilde{\rho}(x,\varepsilon,y)-\widetilde{\rho}(x',\varepsilon',y)| \le C g(x,x',\varepsilon,\varepsilon',y)[|\varepsilon-\varepsilon'| + \operatorname{dist}(x,x')], \\ \forall\, x,x',y\in\mathscr{M}, \forall\,\varepsilon,\varepsilon'>0, \end{aligned} \tag{4.7}$$

with $C := 2C_{\mathscr{M}} + 4(C_{\mathscr{M}})^2$ and

$$g(x,x',\varepsilon,\varepsilon',y) := \frac{\chi_{B_\varepsilon(x)\cup B_{\varepsilon'}(x')}(y)}{\varepsilon'\mu(B_{\varepsilon'}(x'))} + \frac{\mu(B_\varepsilon(x)\cup B_{\varepsilon'}(x'))\chi_{B_\varepsilon(x)}(y)}{\varepsilon'\mu(B_\varepsilon(x))\mu(B_{\varepsilon'}(x'))}. \tag{4.8}$$

Moreover, we have (with the same C)

$$\begin{aligned} \int_{\mathscr{M}} |\widetilde{\rho}(x,\varepsilon,y)-\widetilde{\rho}(x',\varepsilon',y)|\,\mathrm{d}\mu(y) \le \frac{4C}{\min(\varepsilon,\varepsilon')}[|\varepsilon-\varepsilon'| + \operatorname{dist}(x,x')], \\ \forall\, x,x'\in\mathscr{M}, \forall\,\varepsilon,\varepsilon'>0. \end{aligned} \tag{4.9}$$

Granted Lemma 4.3, we proceed to the proof of Theorem 4.1.

Step 1. There exists some finite constant $C_1 = C_1(\mathscr{M},\mathscr{N})$ such that

$$\begin{aligned} |F(x,\varepsilon)-F(x',\varepsilon')| \le \frac{C_1}{\min(\varepsilon,\varepsilon')}[|\varepsilon-\varepsilon'| + \operatorname{dist}(x,x')], \\ \forall\, x,x'\in\mathscr{M}, \forall\,\varepsilon,\varepsilon'>0, \forall\, f\colon\mathscr{M}\to\mathscr{N}. \end{aligned} \tag{4.10}$$

Indeed, we have, by Lemma 4.3,

$$\begin{aligned}|F(x,\varepsilon)-F(x',\varepsilon')| &\le \int_{\mathscr{M}} |\widetilde{\rho}(x,\varepsilon,y)-\widetilde{\rho}(x',\varepsilon',y)||f(y)|\,\mathrm{d}\mathscr{H}^k(y)\\ &\le \frac{4C}{\min(\varepsilon,\varepsilon')}[|\varepsilon-\varepsilon'|+\operatorname{dist}(x,x')]\max_{z\in\mathscr{N}}|z|,\end{aligned}$$

whence (4.10).

We next define an "almost projection" on $\mathscr{N}$. For this purpose, we consider Π as in Definition 2.10 and let $\widetilde{\Pi}\in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ be such that

$$\widetilde{\Pi}(z)=\Pi(z),\ \forall\, z\in\mathscr{N}_{\delta/2}. \tag{4.11}$$

Set

$$\widetilde{F}:=\widetilde{\Pi}\circ F. \tag{4.12}$$

It is important to note the following. Let ε_1 be such that, for $\varepsilon<\varepsilon_1$, we have $f_\varepsilon(x)\in\mathscr{N}_{\delta/2}$, $\forall\, x\in\mathscr{M}$ (see (2.31)). Then

$$\widetilde{F}_\varepsilon=f^\varepsilon,\ \forall\,\varepsilon<\varepsilon_1 \tag{4.13}$$

(see (2.32)).

Combining (4.13) and Corollary 3.29, we find that

$$\mathscr{I}(f)=\mathscr{I}(f^\varepsilon)=\int_{\mathscr{M}}(f^\varepsilon)^*\omega=\int_{\mathscr{M}}(\widetilde{F}_\varepsilon)^*\omega,\ \forall\,\varepsilon<\varepsilon_1. \tag{4.14}$$

The following is a straightforward consequence of Step 1 and of the properties of $\widetilde{\Pi}$.

Step 2. There exists some finite constant $C_2=C_2(\mathscr{M},\mathscr{N},\widetilde{\Pi})$ such that

$$\begin{gathered}|\widetilde{F}(x,\varepsilon)-\widetilde{F}(x',\varepsilon')|\le\frac{C_2}{\min(\varepsilon,\varepsilon')}[|\varepsilon-\varepsilon'|+\operatorname{dist}(x,x')],\\ \forall\, x,x'\in\mathscr{M},\forall\,\varepsilon,\varepsilon'>0,\forall\, f\colon\mathscr{M}\to\mathscr{N}.\end{gathered} \tag{4.15}$$

(In particular, $\widetilde{F}_\varepsilon$ is Lipschitz, $\forall\,\varepsilon>0$. Therefore, when $\varepsilon<\varepsilon_1$, the right-hand side of (4.14) is a standard integral of a bounded Borel function.)

We next define a convenient extension of ω. Since $\widetilde{\Pi}$ takes its values in $\mathscr{N}$ on $\mathscr{N}_{\delta/2}$, the form $\widetilde{\Pi}^*\omega$ is well-defined on $\mathscr{N}_{\delta/2}$. (Recall that $\mathscr{N}\subset\mathbb{R}^n$.) Now, let $\psi\colon\mathbb{R}^n\to[0,1]$ be a smooth function, compactly supported in $\mathscr{N}_{\delta/2}$, and such that $\psi=1$ on a neighborhood of $\mathscr{N}$. We may therefore set

$$\alpha := \psi\widetilde{\Pi}^*\omega, \tag{4.16}$$

and this definition makes sense on the whole $\mathbb{R}^n$. We claim that

$$\alpha \text{ is a smooth extension of } \omega \text{ to } \mathbb{R}^n. \tag{4.17}$$

Indeed, on $\mathscr{N}$ we have $\widetilde{\Pi} = \mathrm{Id}$. Therefore, for any $z \in \mathscr{N}$ and any $e_1, \dots, e_k \in T_z\mathscr{N}$, it holds

$$\alpha(z)(e_1, \dots, e_k) = \psi(\widetilde{\Pi}(z))\omega(\widetilde{\Pi}(z))(\mathrm{D}_z\widetilde{\Pi}(e_1), \dots, \mathrm{D}_z\widetilde{\Pi}(e_k)) = \omega(z)(e_1, \dots, e_k); \tag{4.18}$$

this proves the claim.

As a consequence of (4.17) and (4.13), we find that

$$(\widetilde{F}_\varepsilon)^*\alpha = (f^\varepsilon)^*\omega, \forall\, \varepsilon < \varepsilon_1. \tag{4.19}$$

We next combine (4.19), (4.14), (4.15) (which, in particular, implies that $\widetilde{F}$ is Lipschitz on $\mathscr{M} \times [\varepsilon, b]$, $\forall\, 0 < \varepsilon < b \leq \infty$), and Proposition 3.26, and find that

$$\mathscr{I}(f) = \int_{\mathscr{M}} (\widetilde{F}_b)^*\alpha - \int_{\mathscr{M}\times(\varepsilon,b)} \widetilde{F}^*(\mathrm{d}\alpha), \forall\, \varepsilon < \varepsilon_1, \forall\, \varepsilon < b < \infty. \tag{4.20}$$

After these preliminaries, we are at the heart of the proof of Theorem 4.1 (Steps 3–5). It will be of interest to note, for each step, the assumptions on s and p. (The assumptions $0 < s < 1$ and $sp = k$ are a common roof to all these steps.)

In what follows, C_j denotes a finite constant depending only on $\mathscr{M}$, $\mathscr{N}$, $\widetilde{\Pi}$, s, p, and ω.

Set

$$h(x) := \inf\{\varepsilon > 0 \colon \operatorname{dist}(F(x, \varepsilon), \mathscr{N}) \geq \delta/2\}. \tag{4.21}$$

Step 3. If $f \colon \mathscr{M} \to \mathscr{N}$, we have

$$\int_{\mathscr{M}\times(0,\infty)} |\widetilde{F}^*(\mathrm{d}\alpha)| \leq C_3 \int_{\mathscr{M}} \frac{1}{[h(x)]^k}\, \mathrm{d}\mathscr{H}^k(x), \tag{4.22}$$

$$\lim_{b\to\infty} \int_{\mathscr{M}} (\widetilde{F}_b)^*\alpha = 0. \tag{4.23}$$

In particular (in view of (4.20)), when $sp = k$ we have

$$|\mathscr{I}(f)| \le C_3 \int_{\mathscr{M}} \frac{1}{[h(x)]^{sp}} \, \mathrm{d}\mathscr{H}^k(x). \tag{4.24}$$

We next proceed to the proof of (4.23). By (4.15), we have

$$|\mathrm{D}_x \widetilde{F}_\varepsilon| \le \frac{C_4}{\varepsilon} \text{ for } \mathscr{H}^k\text{-a.e. } x \in \mathscr{M}, \forall\, \varepsilon > 0,$$

and thus

$$\left| \int_{\mathscr{M}} (\widetilde{F}_b)^* \alpha \right| \le \frac{C_5}{b^k}, \forall\, b > 0,$$

whence (4.23).

It remains to prove (4.22). For this purpose, we first note that, when $z \in \mathscr{N}$ and $e_1, \ldots, e_{k+1} \in T_z\mathscr{N}$, we have (similarly to the proof of (4.18))

$$\begin{aligned}(\mathrm{d}\alpha)(z)(e_1, \ldots, e_{k+1}) &= (\mathrm{d}(\widetilde{\Pi}^* \omega))(z)(e_1, \ldots, e_{k+1}) \\ &= (\mathrm{d}\omega)(\widetilde{\Pi}(z))(\mathrm{D}_z\widetilde{\Pi}(e_1), \ldots, \mathrm{D}_z\widetilde{\Pi}(e_{k+1})) = 0.\end{aligned} \tag{4.25}$$

Here, we use the fact that the differential commutes with the pullback, along with the fact that $\psi = 1$ on a neighborhood of $\mathscr{N}$.

Consider next the set

$$W := \{(x, \varepsilon) \in \mathscr{M} \times (0, \infty) \colon \mathrm{dist}\,(F(x, \varepsilon), \mathscr{N}) < \delta/2\},$$

which is open (recall that F is continuous). Using: (i) (4.25); (ii) the fact that $\widetilde{F}$ is locally Lipschitz; (iii) the fact that (by definition of W) we have $\widetilde{F}(W) \subset \mathscr{N}$, we find that

$$\widetilde{F}^*(\mathrm{d}\alpha) = 0 \text{ a.e. in } W. \tag{4.26}$$

Combining (4.26) with the definition (4.21) of $h(x)$, we find that

$$\int_{\mathscr{M} \times (0,\infty)} |\widetilde{F}^*(\mathrm{d}\alpha)| = \int_{\mathscr{M}} \int_{h(x)}^{\infty} |\widetilde{F}^*(\mathrm{d}\alpha)(x, \varepsilon)| \, \mathrm{d}\varepsilon \, \mathrm{d}\mathscr{H}^k(x). \tag{4.27}$$

On the other hand, using (4.15), we find that

$$|\widetilde{F}^*(\mathrm{d}\alpha)(x, \varepsilon)| \le \frac{C_6}{\varepsilon^{k+1}} \text{ for a.e. } (x, \varepsilon) \in \mathscr{M} \times (0, \infty). \tag{4.28}$$

Inserting (4.28) into (4.27), we obtain (4.22).

Before proceeding further, let us note that the function h defined in (4.21) is measurable. Indeed, by (2.30), we know that $h(x)$, $x \in \mathscr{M}$, has a uniform lower bound $\widetilde{\varepsilon} > 0$. Therefore, for each $x \in \mathscr{M}$ we have $\operatorname{dist}(F(x, h(x)), \mathscr{N}) = \delta/2$, and $\operatorname{dist}(F(x, \varepsilon), \mathscr{N}) < \delta/2$ if $0 < \varepsilon < h(x)$. Using this fact, it is straightforward that h is l.s.c., and thus Borel measurable.

Step 4. For $0 < s < 1$ and $1 \le p < \infty$, we have, $\forall\, f \in W^{s,p}(\mathscr{M}; \mathscr{N})$,

$$\frac{1}{[h(x)]^{sp}} \le C_7 \int_0^\infty \varepsilon^{p(1-s)-1} |\partial_\varepsilon F(x, \varepsilon)|^p \, \mathrm{d}\varepsilon \text{ for } \mathscr{H}^k\text{-a.e. } x \in \mathscr{M}. \tag{4.29}$$

In the proof of (4.29), it suffices to consider points $x \in \mathscr{M}$ such that: (i) x is a Lebesgue point for f (and thus $F(x, \varepsilon) \to f(x)$ as $\varepsilon \to 0$); (ii) $h(x) < \infty$ (and thus $\operatorname{dist}(F(x, h(x)), \mathscr{N}) = \delta/2$). (For (i), we rely on the Lebesgue differentiation theorem for metric measure spaces satisfying the doubling condition; see, e.g., [33, Theorem 2.9.8].) We also note that (iii) $F(x, \cdot)$ is locally absolutely continuous. (This relies on the locally Lipschitz character of F, which does not require that $sp = k$; see Step 1.) For such x, we have (using (i) and (ii))

$$\begin{aligned} \lim_{a \to 0} |F(x, h(x)) - F(x, a)| &= |F(x, h(x)) - f(x)| \\ &\ge \operatorname{dist}(F(x, h(x)), \mathscr{N}) \ge \delta/2. \end{aligned} \tag{4.30}$$

From (4.30), we deduce (using (iii)) that

$$\int_0^{h(x)} |\partial_\varepsilon F(x, \varepsilon)| \, \mathrm{d}\varepsilon \ge \delta/2. \tag{4.31}$$

Combining (4.31) with Hölder's inequality, we obtain

$$\begin{aligned} (\delta/2)^p &\le \left(\int_0^{h(x)} |\partial_\varepsilon F(x, \varepsilon)| \, \mathrm{d}\varepsilon \right)^p \\ &\le \int_0^{h(x)} \varepsilon^{p(1-s)-1} |\partial_\varepsilon F(x, \varepsilon)|^p \, \mathrm{d}\varepsilon \left(\int_0^{h(x)} \varepsilon^{sp/(p-1)-1} \, \mathrm{d}\varepsilon \right)^{p-1} \\ &= \left(\frac{p-1}{sp} \right)^{p-1} [h(x)]^{sp} \int_0^{h(x)} \varepsilon^{p(1-s)-1} |\partial_\varepsilon F(x, \varepsilon)|^p \, \mathrm{d}\varepsilon \end{aligned}$$

(with the obvious modification when $p = 1$), whence (4.29).

Step 5. When $0 < s < 1$ and $1 \le p < \infty$, we have, $\forall f \in W^{s,p}(\mathscr{M};\mathbb{R}^n)$,

$$\int_{\mathscr{M}} \int_0^\infty \varepsilon^{p(1-s)-1} |\partial_\varepsilon F(x,\varepsilon)|^p \, d\varepsilon \, d\mathscr{H}^k(x) \le C_8 |f|^p_{W^{s,p}} + C_9. \tag{4.32}$$

(This is well-known (with $C_9 = 0$) in the Euclidean case, see, e.g., the account of the theory of weighted Sobolev spaces in [50].)

The starting point in the proof of (4.32) is the following. Using (4.6)–(4.8), we find that, for $x \in \mathscr{M}$ and $-\varepsilon/2 < h < \varepsilon$, we have

$$\begin{aligned} \left| \frac{F(x,\varepsilon+h) - F(x,\varepsilon)}{h} \right| &= \left| \int_{\mathscr{M}} \frac{\widetilde{\rho}(x,\varepsilon+h,y) - \widetilde{\rho}(x,\varepsilon,y)}{h} f(y) \, d\mathscr{H}^k(y) \right| \\ &= \left| \int_{\mathscr{M}} \frac{\widetilde{\rho}(x,\varepsilon+h,y) - \widetilde{\rho}(x,\varepsilon,y)}{h} (f(y) - f(x)) \, d\mathscr{H}^k(y) \right| \\ &\le \frac{C_{10}}{\varepsilon} \fint_{B_{2\varepsilon}(x)} |f(y) - f(x)| \, d\mathscr{H}^k(y). \end{aligned} \tag{4.33}$$

Combining (4.33) with (3.4), we find that, for some appropriate $r_0 > 0$, we have

$$|\partial_\varepsilon F(x,\varepsilon)| \le \frac{C_{11}}{\varepsilon^{k+1}} \int_{B_{2\varepsilon}(x)} |f(x) - f(y)| \, d\mathscr{H}^k(y), \forall x \in \mathscr{M}, \text{for a.e. } 0 < \varepsilon \le r_0. \tag{4.34}$$

Using (4.34), (3.4), and Hölder's inequality, we find that

$$\begin{aligned} |\partial_\varepsilon F(x,\varepsilon)|^p \le \frac{C_{12}}{\varepsilon^{k+p}} \int_{B_{2\varepsilon}(x)} |f(x) - f(y)|^p \, d\mathscr{H}^k(y), \\ \forall x \in \mathscr{M}, \text{for a.e. } 0 < \varepsilon \le r_0. \end{aligned} \tag{4.35}$$

On the other hand, (4.33) yields

$$|\partial_\varepsilon F(x,\varepsilon)| \le \frac{C_{13}}{\varepsilon}, \forall x \in \mathscr{M}, \text{for a.e. } \varepsilon \ge r_0. \tag{4.36}$$

Using (4.35) and (4.36), we obtain

$$\begin{aligned} &\int_{\mathscr{M}} \int_0^\infty \varepsilon^{p(1-s)-1} |\partial_\varepsilon F(x,\varepsilon)|^p \, d\varepsilon \, d\mathscr{H}^k(x) \\ &\le C_{12} \int_{\mathscr{M}} \int_0^\infty \varepsilon^{-k-sp-1} \int_{B_{2\varepsilon}(x)} |f(x) - f(y)|^p \, d\mathscr{H}^k(y) \, d\varepsilon \, d\mathscr{H}^k(x) + C_{14} \\ &\le C_{12} \int_{\mathscr{M}} \int_{\mathscr{M}} \int_{\operatorname{dist}(x,y)/2}^\infty \varepsilon^{-k-sp-1} |f(x) - f(y)|^p \, d\varepsilon \, d\mathscr{H}^k(y) \, d\mathscr{H}^k(x) + C_{14} \end{aligned}$$

$$= C_{15}\int_{\mathcal{M}}\int_{\mathcal{M}}\frac{|f(x)-f(y)|^p}{[\operatorname{dist}(x,y)]^{k+sp}}\,\mathrm{d}\mathcal{H}^k(y)\,\mathrm{d}\mathcal{H}^k(x)+C_{14}=C|f|^p_{W^{s,p}}+C_{14}.$$

Estimate (4.3) (and thus Theorem 4.1) follows from Steps 3–5. We note that the only place where we use the assumption $sp = k$ in the proof is to connect Steps 3 and 4 through (4.24). □

Proof of Lemma 4.3. By definition of K (see (2.8)), (4.6) is obvious.

We now proceed to the proof of (4.7). Set $B := B_\varepsilon(x)$ and $B' := B_{\varepsilon'}(x')$. By (2.8), we have

$$\begin{aligned}|\rho(x,\varepsilon,y)-\rho(x',\varepsilon',y)| &\le |\varepsilon-\varepsilon'-\operatorname{dist}(x,y)+\operatorname{dist}(x',y)|\chi_{B\cup B'}(y)\\ &\le [|\varepsilon-\varepsilon'|+\operatorname{dist}(x,x')]\chi_{B\cup B'}(y).\end{aligned}\tag{4.37}$$

On the other hand, (2.8), (4.37), and (2.12) yield

$$\begin{aligned}|K(x,\varepsilon)-K(x',\varepsilon')| &\le K(x,\varepsilon)K(x',\varepsilon')\int_{\mathcal{M}}|\rho(x,\varepsilon,y)-\rho(x',\varepsilon',y)|\,\mathrm{d}\mu(y)\\ &\le 4(C_{\mathcal{M}})^2\frac{\mu(B\cup B')}{\varepsilon\varepsilon'\mu(B)\mu(B')}[|\varepsilon-\varepsilon'|+\operatorname{dist}(x,x')].\end{aligned}\tag{4.38}$$

Combining (4.37) and (4.38) with (2.10) and (2.12), we find that

$$\begin{aligned}&|\widetilde{\rho}(x,\varepsilon,y)-\widetilde{\rho}(x',\varepsilon',y)|\\ &\quad\le |\rho(x,\varepsilon,y)-\rho(x',\varepsilon',y)|K(x',\varepsilon')+\rho(x,\varepsilon,y)|K(x,\varepsilon)-K(x',\varepsilon')|\\ &\quad\le \left(2C_{\mathcal{M}}\frac{\chi_{B\cup B'}(y)}{\varepsilon'\mu(B')}+4(C_{\mathcal{M}})^2\frac{\mu(B\cup B')\chi_B(y)}{\varepsilon'\mu(B)\mu(B')}\right)[|\varepsilon-\varepsilon'|+\operatorname{dist}(x,x')],\end{aligned}$$

whence (4.7) with g as in (4.8).

Finally, we prove (4.9). Without loss of generality, we may assume that $\mu(B)\le\mu(B')$. Integrating (4.7) in y and using (4.8), we find that

$$\begin{aligned}\int_{\mathcal{M}}|\widetilde{\rho}(x,\varepsilon,y)-\widetilde{\rho}(x',\varepsilon',y)|\,\mathrm{d}\mu(y) &\le C\frac{2\mu(B\cup B')}{\varepsilon'\mu(B')}[|\varepsilon-\varepsilon'|+\operatorname{dist}(x,x')]\\ &\le C\frac{4\mu(B')}{\varepsilon'\mu(B')}[|\varepsilon-\varepsilon'|+\operatorname{dist}(x,x')],\end{aligned}$$

whence (4.9). □

Remark 4.4. Let $0<s<1$ and $1<p<\infty$ be such that $sp=k$. Let ω be a smooth closed k-form on $\mathcal{N}$. Combining (4.20), (4.22), (4.23), (4.29) and (4.32), we have the following explicit formula:

$$\mathcal{I}(f) = -\int_{\mathcal{M}\times(0,\infty)} \widetilde{F}^*(\mathrm{d}\alpha), \forall f \in W^{s,p}(\mathcal{M};\mathcal{N}). \qquad \square \quad (4.39)$$

5 Additional properties of $f^*\omega$

Throughout this section: (a) $\mathcal{M}$ is a compact k-dimensional Lipschitz manifold oriented by a finite chart structure $\{(U_i, V_i, \varphi_i)\}_{i\in I}$; (b) $\mathcal{N}$ is a closed manifold; (c) ω is a smooth closed k-form on $\mathcal{N}$.

5.1 An explicit formula for $\mathcal{I}(f)$ when $f \in \mathrm{VMO}$

As a continuation of our excursion into the land of Sobolev maps, we prove that (4.39) still holds for VMO maps.

Proposition 5.1. *Let α be a smooth compactly supported extension of ω to $\mathbb{R}^n$. Let $f \in \mathrm{VMO}(\mathcal{M};\mathcal{N})$. Let F, respectively $\widetilde{F}$, be as in (4.4), respectively (4.12). Then*

$$\mathcal{I}(f) = -\int_{\mathcal{M}\times(0,\infty)} \widetilde{F}^*(\mathrm{d}\alpha). \tag{5.1}$$

Proof. An inspection of the proof of Theorem 4.1 shows that the specific extension α of ω we take plays no role in the proof, and thus (4.39) holds for any such α. Moreover, (4.39) holds for any $f \in \mathrm{Lip}(\mathcal{M};\mathcal{N})$.

Let now $f \in \mathrm{VMO}(\mathcal{M};\mathcal{N})$. Let ε_1 be as in (2.31). Using: (i) Corollary 3.29; (ii) the fact that f^ε is Lipschitz when $0 < \varepsilon \le \varepsilon_1$ (see Step 1 in the proof of Theorem 4.1); (iii) the proof of Theorem 4.1, we find that

$$\mathcal{I}(f) = \mathcal{I}(f^\varepsilon) = \int_{\mathcal{M}} (f^\varepsilon)^*\omega = -\int_{\mathcal{M}\times(\varepsilon,\infty)} \widetilde{F}^*(\mathrm{d}\alpha), \forall\, 0 < \varepsilon \le \varepsilon_1. \tag{5.2}$$

In order to obtain (5.1) from (5.2), it suffices to prove that $F^*(\mathrm{d}\alpha)$ is integrable on $\mathcal{M}\times(0,\infty)$. For this purpose, we note that, clearly, the number $h(x)$ introduced in (4.21) satisfies

$$h(x) \ge \varepsilon_1, \forall\, x \in \mathcal{M}. \tag{5.3}$$

Combining (5.3) with (4.26) and (4.28), we obtain, with some finite constant C, the domination

$$|\widetilde{F}^*(\mathrm{d}\alpha)(x,\varepsilon)| \le \frac{C}{\varepsilon^{k+1}}\chi_{(\varepsilon_1,\infty)}(\varepsilon) \text{ for a.e. } (x,\varepsilon) \in \mathcal{M}\times(0,\infty),$$

which implies the integrability of $\widetilde{F}^*(\mathrm{d}\alpha)$ on $\mathscr{M} \times (0, \infty)$. □

Remark 5.2. There is a lot of freedom in the choice of the extension F yielding $\widetilde{F}$. For example, one can prove that (5.1) still holds for F defined as in Lemma 5.8 below. □

5.2 Action on the de Rham cohomology classes

An immediate consequence of Proposition 5.1 is the following.

Corollary 5.3. *If ω is exact, then*

$$\mathscr{I}(f) = 0, \forall f \in \mathrm{VMO}(\mathscr{M}; \mathscr{N}).$$

Proof. Let η be a $(k-1)$-form such that $\mathrm{d}\eta = \omega$. With the notation after Step 2 in the proof of Theorem 4.1, set $\alpha := \mathrm{d}(\psi \widetilde{\Pi}^* \eta)$. Since $\psi = 1$ in an open neighborhood Y of $\mathscr{N}$, we have, in Y,

$$\alpha = \psi\, \mathrm{d}(\widetilde{\Pi}^* \eta) = \psi\, \widetilde{\Pi}^* (\mathrm{d}\eta) = \psi\, \widetilde{\Pi}^* \omega,$$

and thus (4.17) holds.

Using (5.1) and the definition of α, we find that

$$\mathscr{I}(f) = -\int_{\mathscr{M} \times (0,\infty)} \widetilde{F}^*(\mathrm{d}\alpha) = -\int_{\mathscr{M} \times (0,\infty)} \widetilde{F}^*(\mathrm{d}^2(\psi \widetilde{\Pi}^* \eta)) = 0, \forall f \in \mathrm{VMO}(\mathscr{M}; \mathscr{N}).$$ □

Combining Corollary 3.29 and Corollary 5.3, we obtain the following

Corollary 5.4. *The quantity $\mathscr{I}(f)$, with $f \in \mathrm{VMO}(\mathscr{M}; \mathscr{N})$, depends only on the homotopy class of f and the (de Rham) cohomology class of ω.*

Remark 5.5. In the special case where $\mathscr{N} = \mathbb{S}^k$, the de Rham cohomology group $H^k_{\mathrm{dR}}(\mathbb{S}^k)$ satisfies $H^k_{\mathrm{dR}}(\mathbb{S}^k) = \mathbb{R}$ and is generated by the standard volume form $\omega_{\mathbb{S}^k}$ on $\mathbb{S}^k$, whose expression has been recalled in Example 3.30. Therefore, the information given by all the homotopical invariants $\mathscr{I}_{\mathscr{M},\omega}(f)$ is entirely contained in the single invariant $\mathscr{I}_{\mathscr{M},\omega_{\mathbb{S}^k}}(f)$.

If, moreover, $\mathscr{M}$ is C^1, connected, and closed, then actually the invariant $\mathscr{I}_{\mathscr{M},\omega_{\mathbb{S}^k}}(f)$ *completely characterizes* homotopy classes of maps $f\colon \mathscr{M} \to \mathbb{S}^k$, that is, if $\mathscr{I}_{\mathscr{M},\omega_{\mathbb{S}^k}}(f) = \mathscr{I}_{\mathscr{M},\omega_{\mathbb{S}^k}}(g)$, then $f \sim g$. When f and g are is continuous, this is Hopf's theorem, see, e.g., Milnor [48, § 7]. The case of VMO maps follows from the general theory developed above. When $\mathscr{M} = \mathbb{S}^k$, this characterization is a special case of Proposition A.2, which features a more general criterion on $\mathscr{N}$ for the invariants $\mathscr{I}$ to characterize homotopy classes of $\mathscr{N}$-valued maps, and will be discussed more thoroughly in Appendix A. □

5.3 A digression: the distribution $f^*\omega$

This section is in the spirit of Brezis and Nguyen [23]. When $f : \mathscr{M} \to \mathscr{N}$ is Lipschitz, $f^*\omega$ can be identified with a bounded Borel function ($\mathscr{H}^k$-a.e. defined on $\mathscr{M}$), and then $\int_{\mathscr{M}} f^*\omega$ is merely the integral of this function with respect to $\mathscr{H}^k$. Therefore, if ξ is a real Borel integrable "test" function on $\mathscr{M}$, then one may consider the integral $\int_{\mathscr{M}} \xi\, f^*\omega$. (Similar considerations apply to the case where $f \in W^{1,k}(\mathscr{M};\mathscr{N})$ and ξ is bounded; see Section 3.8.) We discuss here the possibility of giving a robust meaning to the latter integral, possibly under more restrictive assumptions on ξ. This is a generalization of the case where $\mathscr{M} = \mathscr{N} = \mathbb{S}^k$ and ω is the standard volume form on $\mathbb{S}^k$, investigated in Brezis and Nguyen [23]. (However, strictly speaking the results below are not generalizations of the results in [23].) Our purpose here is to illustrate how the ideas used in the proof of Theorem 4.1 can be adapted to this context, and also to provide heuristics for Section 6. The results we present below are otherwise off topic, and therefore the proofs are rather sketchy.

For simplicity, in addition to the assumptions (a)–(c) at the beginning of Section 5, we make here the extra assumption (d) $\mathscr{M}$ is connected. Also, in order to slightly simplify the statement of Lemma 5.8 below, we make the assumption (e) the constant K_2 in (3.1) equals 1. (The latter assumption can be achieved by a scale change.)

Remark 5.6. A preliminary observation is that, even in the smooth case, $\int_{\mathscr{M}} \xi\, f^*\omega$ is *not* a homotopical invariant. To illustrate this assertion, assume, e.g., that $\mathscr{M} = \mathscr{N}$ contains a flat ball, that we identify with $\mathbb{B}^k$, the unit ball in $\mathbb{R}^k$. Consider a k-form ω that coincides, on $\mathbb{B}^k$, with the standard volume form. Let $\xi \in C^\infty_c(\mathbb{B}^k;[0,1]) \setminus \{0\}$. If f, $g \in C^\infty(\mathscr{M};\mathbb{B}^k)$, then clearly f and g are homotopic. Choose now $f := \psi\,\mathrm{Id}$, where $\psi \in C^\infty_c(\mathbb{B}^k)$ and $\psi = 1$ on $\operatorname{supp}\xi$, and $g := 0$. By the above, f and g are homotopic. However, we have

$$\int_{\mathscr{M}} \xi\, f^*\omega = \int_{\mathscr{M}} \xi > 0,$$

while $\int_{\mathscr{M}} \xi\, g^*\omega = 0$. □

We next present two results in the spirit of Theorem 4.1. We start with the easier case where $k \geq 2$.

Theorem 5.7. *Assume $k \geq 2$. Let $0 < s < 1$ and $1 < p < \infty$ be such that $sp = k$. Let $\xi \in \mathrm{Lip}(\mathscr{M};\mathbb{R})$. Then the mapping*

$$\mathrm{Lip}(\mathscr{M};\mathscr{N}) \ni f \mapsto \int_{\mathscr{M}} \xi\, f^*\omega$$

can be extended by density to $(W^{s,p} \cap W^{1-1/k,k})(\mathscr{M};\mathscr{N})$.

The extension, still denoted $f \mapsto \int_{\mathscr{M}} \xi\, f^*\omega$*, satisfies*

$$\begin{aligned} \left|\int_{\mathscr{M}} \xi\, f^*\omega\right| \le C_1 |f|^p_{W^{s,p}} \|\xi\|_\infty + C_2 |f|^k_{W^{1-1/k,k}} |\xi|_{\mathrm{Lip}}, \\ \forall\, f \in (W^{s,p} \cap W^{1-1/k,k})(\mathscr{M};\mathscr{N}), \forall\, \xi \in \mathrm{Lip}(\mathscr{M};\mathbb{R}), \end{aligned} \tag{5.4}$$

for some finite constants C_1 *and* C_2 *independent of* f *and* ξ.

We note the extra assumption $f \in W^{1-1/k,k}(\mathscr{M};\mathscr{N})$, which was not needed in Theorem 4.1.

Sketch of proof. Let us start by guessing the analogue of (4.39) in this context. We use notation similar to the one in the proof of Theorem 4.1. Let F be an extension of f to be defined later and set $\widetilde{F} := \widetilde{\Pi} \circ F$. Set $\widetilde{\xi}(x,\varepsilon) := \xi(x)$, $\forall\, x \in \mathscr{M}$, $\forall\, \varepsilon > 0$. With (4.23) in mind, we formally have the following chain of equalities:

$$\int_{\mathscr{M}} \xi\, f^*\omega = -\int_{\mathscr{M}\times(0,\infty)} \mathrm{d}[\widetilde{\xi}\, \widetilde{F}^*\alpha] = -\int_{\mathscr{M}\times(0,\infty)} \mathrm{d}\widetilde{\xi} \wedge \widetilde{F}^*\alpha - \int_{\mathscr{M}\times(0,\infty)} \widetilde{\xi}\, \widetilde{F}^*(\mathrm{d}\alpha). \tag{5.5}$$

The strategy of the rigorous proof of (5.5) is now clear; see Steps 1–5 below.

Step 1. Definition of an appropriate extension operator $f \mapsto F$. This is the content of the following auxiliary result, inspired by the proof of Lemma 3.35. We consider the notation in (3.55)–(3.56).

Lemma 5.8. *Let* $f \in \mathscr{L}^1(\mathscr{M};\mathbb{R}^n)$. *For* $0 < \varepsilon \le \varepsilon_1 = \varepsilon_1(\mathscr{M})$ *and* $x \in \mathscr{M}$, *set*

$$F(x,\varepsilon) := \overline{f}_\varepsilon(x) = \sum_i \xi_i(x)\,[(\overline{f}_i * \rho_\varepsilon)((\varphi_i)^{-1}(x))]. \tag{5.6}$$

For $x \in \mathscr{M}$ *and the other non-negative values of* ε, *we set*

$$F(x,\varepsilon) := \begin{cases} f(x), & \text{if } \varepsilon = 0 \\ \displaystyle\fint_{\mathscr{M}} f, & \text{if } \varepsilon \ge 2\varepsilon_1 \\ \displaystyle\left(2 - \frac{\varepsilon}{\varepsilon_1}\right) F(x,\varepsilon_1) + \left(\frac{\varepsilon}{\varepsilon_1} - 1\right) \fint_{\mathscr{M}} f, & \text{if } \varepsilon_1 < \varepsilon < 2\varepsilon_1 \end{cases}.$$

The linear operator $\mathscr{L}^1(\mathscr{M};\mathbb{R}^n) \ni f \mapsto F$ *has the following properties (with finite constants independent of* f*).*

(1) *If* f *is Lipschitz, then so is* F.

(2) *For* $\varepsilon_1 \le \varepsilon \le 2\varepsilon_1$, *we have*

$$\partial_\varepsilon F(x,\varepsilon) = \frac{1}{\varepsilon_1}\left(\fint_{\mathscr{M}} f - F(x,\varepsilon_1)\right)$$
$$= -\frac{1}{\varepsilon_1}\sum_i \xi_i(x)\left[\left(f - \fint_{\mathscr{M}} f\right)\circ\varphi_i\right] * \rho_{\varepsilon_1}((\varphi_i)^{-1}(x))$$

and therefore

$$|\partial_\varepsilon F(x,\varepsilon)| \le C_3 \int_{\mathscr{M}}\int_{\mathscr{M}} |f(y)-f(z)|\,\mathrm{d}\mathscr{H}^k(y)\,\mathrm{d}\mathscr{H}^k(z), \quad \forall\, x\in\mathscr{M}, \forall\, \varepsilon_1 \le \varepsilon \le 2\varepsilon_1. \tag{5.7}$$

(3) *For* $0 < \varepsilon \le \varepsilon_1$ *and* $\mathscr{H}^k$*-a.e.* $x \in \mathscr{M}$, *we have*

$$|\nabla F(x,\varepsilon)| \le \frac{C_4}{\varepsilon}\fint_{B_\varepsilon(x)} |f(y)-f(x)|\,\mathrm{d}\mathscr{H}^k(y). \tag{5.8}$$

(4) *If* $f\colon \mathscr{M} \to \mathscr{N}$, *then, for every* $x \in \mathscr{M}$ *and* $0 < \varepsilon \le \varepsilon_1$,

$$\operatorname{dist}(F(x,\varepsilon),\mathscr{N}) \le C_5 M_\varepsilon(f).$$

Sketch of proof. The proof of (1) follows readily from the fact that $\mathrm{Lip}\cap\mathscr{L}^\infty$ is an algebra. The proof of (2) is a straightforward computation. Property (4) is proved in (3.58).

To prove (3), we calculate the gradient of $F(x,\varepsilon)$ *via* the Leibniz rule. For the term involving $\nabla\xi_i$, we rely on the fact that $\sum_i \nabla\xi_i = 0$ a.e. to obtain that, for a.e. $x \in \mathscr{M}$, it holds

$$\begin{aligned}\left|\sum_i [\nabla\xi_i(x)]\,[(\overline{f}_i * \rho_\varepsilon)((\varphi_i)^{-1}(x))]\right| &= \left|\sum_i [\nabla\xi_i(x)][(\overline{f}_i * \rho_\varepsilon)((\varphi_i)^{-1}(x)) - f(x)]\right| \\ &\le C_6 \sum_i \fint_{B_\varepsilon((\varphi_i)^{-1}(x))} |\overline{f}_i(v) - f(x)|\,\mathrm{d}v \\ &\le C_7 \fint_{B_\varepsilon(x)} |f(y)-f(x)|\,\mathrm{d}\mathscr{H}^k(y),\end{aligned} \tag{5.9}$$

where we have used the extra assumption (e) in the last inequality.

For the term involving $f_i * \nabla\rho_\varepsilon$ (where ∇ stands for the gradient in both x and ε of

$(x, \varepsilon) \mapsto \rho_\varepsilon(x)$), we rely on the fact that the integral of $\nabla\rho_\varepsilon$ is zero to deduce that

$$(\overline{f}_i * \nabla\rho_\varepsilon)((\varphi_i)^{-1}(x)) = \int_{B_\varepsilon((\varphi_i)^{-1}(x))} \nabla\rho_\varepsilon((\varphi_i)^{-1}(x) - v)(f_i(v) - f(x))\,\mathrm{d}v. \tag{5.10}$$

We obtain (5.8) from (5.9) and (5.10) combined with: (i) the fact that (ξ_i) is a partition of unity; (ii) the fact that φ_i is bi-Lipschitz; (iii) the estimate $|\nabla\rho_\varepsilon| \le C_8\varepsilon^{-k-1}$. □

Step 2. Justification of (5.5) when $f \in \mathrm{Lip}(\mathscr{M}; \mathscr{N})$. The starting point is the following result.

Proposition 5.9. *Let $F\colon \mathscr{M} \times [a, b] \to \mathscr{N}$ and $Z\colon \mathscr{M} \times [a, b] \to \mathbb{R}$ be Lipschitz maps. Then*

$$\int_{\mathscr{M}\times(a,b)} \mathrm{d}Z \wedge F^*\omega + \int_{\mathscr{M}\times(a,b)} Z\,F^*(\mathrm{d}\omega) = \int_{\mathscr{M}} Z_b\,(F_b)^*\omega - \int_{\mathscr{M}} Z_a\,(F_a)^*\omega.$$

Similarly, if $F\colon \mathscr{M} \times [a, b] \to \mathbb{R}^n$ and $Z\colon \mathscr{M} \times [a, b] \to \mathbb{R}$ are Lipschitz maps, and if α is a smooth k-form on $\mathbb{R}^n$ with bounded coefficients, then

$$\int_{\mathscr{M}\times(a,b)} \mathrm{d}Z \wedge F^*\alpha + \int_{\mathscr{M}\times(a,b)} Z\,F^*(\mathrm{d}\alpha) = \int_{\mathscr{M}} Z_b\,(F_b)^*\alpha - \int_{\mathscr{M}} Z_a\,(F_a)^*\alpha.$$

This is a cousin of Proposition 3.26, and its proof is a straightforward variant of the one of Proposition 3.26.

By Lemma 5.8 (1), when f is Lipschitz, so is F (and thus $\widetilde{F}$). We are therefore in position to apply Proposition 5.9 to $\widetilde{\xi}$, $\widetilde{F}$, and α. Using (4.17) and the fact that, by definition of F, $\widetilde{F}(x, \varepsilon)$ is constant for $x \in \mathscr{M}$ and $\varepsilon \ge 2\varepsilon_1$, we find that

$$\begin{aligned}\int_{\mathscr{M}} \xi\, f^*\omega &= -\int_{\mathscr{M}\times(0,2\varepsilon_1)} \mathrm{d}\widetilde{\xi} \wedge \widetilde{F}^*\alpha - \int_{\mathscr{M}\times(0,2\varepsilon_1)} \widetilde{\xi}\,\widetilde{F}^*(\mathrm{d}\alpha)\\ &= -\int_{\mathscr{M}\times(0,\infty)} \mathrm{d}\widetilde{\xi} \wedge \widetilde{F}^*\alpha - \int_{\mathscr{M}\times(0,\infty)} \widetilde{\xi}\,\widetilde{F}^*(\mathrm{d}\alpha).\end{aligned}$$

This completes Step 2.

Step 3. Justification of (5.4) when $f \in \mathrm{Lip}(\mathscr{M}; \mathscr{N})$. By (5.7), (5.8), and the definitions of F and $\widetilde{F}$, we have, for $\mathscr{H}^k$-a.e. $x \in \mathscr{M}$,

$$|\nabla\widetilde{F}(x, \varepsilon)| \le \frac{C_9}{\varepsilon} \fint_{B_\varepsilon(x)} |f(y) - f(x)|\,\mathrm{d}\mathscr{H}^k(y) \text{ if } 0 < \varepsilon < \varepsilon_1, \tag{5.11}$$

$$|\nabla\widetilde{F}(x,\varepsilon)| \le C_{10}\int_{\mathscr{M}}|f(y)-f(x)|\,\mathrm{d}\mathscr{H}^k(y)$$
$$+\,C_{11}\int_{\mathscr{M}}\int_{\mathscr{M}}|f(y)-f(z)|\,\mathrm{d}\mathscr{H}^k(y)\,\mathrm{d}\mathscr{H}^k(z) \text{ if } \varepsilon_1\le\varepsilon<2\varepsilon_1, \tag{5.12}$$
$$\nabla\widetilde{F}(x,\varepsilon)=0 \text{ if } \varepsilon\ge 2\varepsilon_1. \tag{5.13}$$

Combining (5.11)–(5.13) with the proof of (4.32), we obtain the following estimate.

Lemma 5.10. *Let* $0<r<1$ *and* $1\le q<\infty$*. Then*

$$\int_{\mathscr{M}\times(0,\infty)}\varepsilon^{q(1-r)-1}|\nabla\widetilde{F}(x,\varepsilon)|^q \le C_{12}\int_{\mathscr{M}}\int_{\mathscr{M}}\frac{|f(x)-f(y)|^q}{[\operatorname{dist}(x,y)]^{k+rq}}\,\mathrm{d}\mathscr{H}^k(y)\,\mathrm{d}\mathscr{H}^k(x).$$

Applying Lemma 5.10 with $r := 1-1/k$ and $q := k$, we obtain the estimate

$$\left|\int_{\mathscr{M}\times(0,\infty)}\mathrm{d}\widetilde{\xi}\wedge\widetilde{F}^*\alpha\right| \le C_{13}|f|^k_{W^{1-1/k,k}}|\xi|_{\mathrm{Lip}}. \tag{5.14}$$

On the other hand, using Lemma 5.8 (4) and repeating the proofs of (4.22) and (4.29) (proofs that are "robust" with respect to the definition of F), we find that

$$\left|\int_{\mathscr{M}\times(0,\infty)}\widetilde{\xi}\,\widetilde{F}^*(\mathrm{d}\alpha)\right| \le C_{14}\int_{\mathscr{M}\times(0,\infty)}\varepsilon^{p(1-s)-1}|\partial_\varepsilon\widetilde{F}(x,\varepsilon)|^p\,\|\xi\|_\infty. \tag{5.15}$$

From (5.15) and Lemma 5.10 with $r := s$ and $q := p$, we obtain

$$\left|\int_{\mathscr{M}\times(0,\infty)}\widetilde{\xi}\,\widetilde{F}^*(\mathrm{d}\alpha)\right| \le C_{15}|f|^p_{W^{s,p}}\|\xi\|_\infty. \tag{5.16}$$

We complete Step 3 *via* (5.14) and (5.16).

Step 4. Continuity of the right-hand side of (5.5) in $(W^{s,p}\cap W^{1-1/k,k})(\mathscr{M};\mathscr{N})$. We essentially rely on the converse to the dominated convergence theorem. Consider f_j, $f\colon\mathscr{M}\to\mathscr{N}$ such that $f_j\to f$ in $W^{s,p}\cap W^{1-1/k,k}$ as $j\to\infty$.

There exists some maps $G, H\colon\mathscr{M}\times\mathscr{M}\to[0,\infty]$ such that

$$\int_{\mathscr{M}}\int_{\mathscr{M}}\frac{|G(x,y)|^k}{[\operatorname{dist}(x,y)]^{2k-1}}\,\mathrm{d}\mathscr{H}^k(y)\,\mathrm{d}\mathscr{H}^k(x)<\infty, \tag{5.17}$$
$$\int_{\mathscr{M}}\int_{\mathscr{M}}\frac{|H(x,y)|^p}{[\operatorname{dist}(x,y)]^{2k}}\,\mathrm{d}\mathscr{H}^k(y)\,\mathrm{d}\mathscr{H}^k(x)<\infty, \tag{5.18}$$

and, up to a subsequence,

$$|f_j(x) - f_j(y)| \le G(x,y), \tag{5.19}$$
$$|f_j(x) - f_j(y)| \le H(x,y). \tag{5.20}$$

Let F_j, $\widetilde{F}_j$ be the corresponding maps associated with f_j. It is straightforward that, for $\mathscr{H}^k$-a.e. $x \in \mathscr{M}$ and every $\varepsilon > 0$, we have

$$\nabla \widetilde{F}_j(x,\varepsilon) \to \nabla \widetilde{F}(x,\varepsilon). \tag{5.21}$$

Combining dominated convergence with (5.17), (5.19), (5.21), (5.11)–(5.13), and the proof of Lemma 5.10 with $r := 1 - 1/k$ and $q := k$, we find that

$$\int_{\mathscr{M}\times(0,\infty)} \mathrm{d}\widetilde{\xi} \wedge \widetilde{F}_j^* \alpha \to \int_{\mathscr{M}\times(0,\infty)} \mathrm{d}\widetilde{\xi} \wedge \widetilde{F}^* \alpha, \forall\, \xi \in \mathrm{Lip}(\mathscr{M};\mathbb{R}). \tag{5.22}$$

On the other hand, with h_j associated with f_j as in (4.21), we have, by (4.26), the proof of (4.29), and (5.11)–(5.13),

$$|\widetilde{F}_j^*(\mathrm{d}\alpha)(x,\varepsilon)| \le \frac{C_{16}}{\varepsilon^{k+1}} \chi_{(h_j(x),\infty)}(\varepsilon), \text{ for a.e. } x, \varepsilon, \tag{5.23}$$
$$\frac{1}{[h_j(x)]^{sp}} \le C_{17} \int_0^\infty \varepsilon^{p(1-s)-1} |\partial_\varepsilon \widetilde{F}_j(x,\varepsilon)|^p \,\mathrm{d}\varepsilon, \text{ for } \mathscr{H}^k\text{-a.e. } x \in \mathscr{M}. \tag{5.24}$$

Combining dominated convergence with (5.23), (5.24), (5.18), (5.20), and the proof of Lemma 5.10 with $r := s$ and $q := p$, we find that

$$\int_{\mathscr{M}\times(0,\infty)} \widetilde{\xi}\, \widetilde{F}_j^*(\mathrm{d}\alpha) \to \int_{\mathscr{M}\times(0,\infty)} \widetilde{\xi}\, \widetilde{F}^*(\mathrm{d}\alpha), \forall\, \xi \in \mathscr{L}^\infty(\mathscr{M};\mathbb{R}). \tag{5.25}$$

We complete Step 4 *via* (5.22) and (5.25).

Step 5. Density of $\mathrm{Lip}(\mathscr{M};\mathscr{N})$ in $(W^{s,p} \cap W^{1-1/k,k})(\mathscr{M};\mathscr{N})$. Thanks to Lemma 5.8 (4) and the embedding $W^{s,p} \hookrightarrow \mathrm{VMO}$, for small ε, we have $\widetilde{\Pi} \circ F(\cdot,\varepsilon)\colon \mathscr{M} \to \mathscr{N}$. We also have $\widetilde{\Pi} \circ f = f$. We complete Step 5 by combining the next two results. (The first one is straightforward, and the second one is an easy consequence of the converse to the dominated convergence theorem.)

Lemma 5.11. *Let $0 < r < 1$ and $1 \le q < \infty$. Let $f \in W^{r,q}(\mathscr{M};\mathbb{R}^n)$. Let F be as in (5.6). Then $F(\cdot,\varepsilon) \to f$ in $W^{r,q}$ as $\varepsilon \to 0$.*

Lemma 5.12. *Let $0 < r < 1$ and $1 \le q < \infty$. Let $\Phi \in \mathrm{Lip}(\mathbb{R}^n;\mathbb{R}^\ell)$. Then the mapping $f \mapsto \Phi \circ f$ is continuous from $W^{r,q}(\mathscr{M};\mathbb{R}^n)$ to $W^{r,q}(\mathscr{M};\mathbb{R}^\ell)$.* □

Formally, when $k = 1$, the assumption on f in Theorem 5.7 becomes $f \in (W^{s,p} \cap \mathscr{L}^1)(\mathscr{M};\mathscr{N})$. However, for such f the proof of Theorem 5.7 does not work anymore, since, already in the case where $\mathscr{M}$ is flat, the above extension F of an $\mathscr{L}^1$ function need not have a gradient in $\mathscr{L}^1$. (This is a well-known phenomenon, see, e.g., Peetre [58].)

The educated guess in the next statement comes from the estimate (5.11). (For more insight, see [50, Theorem 1.15].) Set

$$|f|_X := \int_{\mathscr{M}}\int_{\mathscr{M}} \frac{|f(x)-f(y)|}{\operatorname{dist}(x,y)}\, d\mathscr{H}^k(x)\, d\mathscr{H}^k(y),$$
$$X := \{f\colon \mathscr{M} \to \mathbb{R}^n\colon |f|_X < \infty\}.$$

Theorem 5.13. *Assume $k = 1$. Let $0 < s < 1$ and $1 < p < \infty$ be such that $sp = 1$. Let $\xi \in \operatorname{Lip}(\mathscr{M};\mathbb{R})$. Then the mapping*

$$\operatorname{Lip}(\mathscr{M};\mathscr{N}) \ni f \mapsto \int_{\mathscr{M}} \xi\, f^*\omega$$

can be extended by density to $(W^{s,p} \cap X)(\mathscr{M};\mathscr{N})$.

The extension, still denoted $f \mapsto \int_{\mathscr{M}} \xi\, f^*\omega$, satisfies

$$\left|\int_{\mathscr{M}} \xi\, f^*\omega\right| \le C_1 |f|^p_{W^{s,p}} \|\xi\|_\infty + C_2 |f|_X |\xi|_{\mathrm{Lip}},$$
$$\forall\, f \in (W^{s,p} \cap X)(\mathscr{M};\mathscr{N}), \forall\, \xi \in \operatorname{Lip}(\mathscr{M};\mathbb{R}),$$

for some finite constants C_1 and C_2 independent of f and ξ.

Theorem 5.13 follows by repeating the proof of Theorem 5.7.

5.4 A further digression: help from topology and lifting

We next discuss, mostly at a formal level, an alternative, and potentially more powerful, approach to the existence of the distribution $f^*\omega$. We also present one specific instance where this approach is successful, see Theorem 5.14 below. In great generality, a careful analysis of some cases where this approach can be rigorously implemented will be presented in Detaille and Xiao [29].

As in the previous section, our objective is to give a robust meaning to $\int_{\mathscr{M}} \xi\, f^*\omega$. In order to simplify the presentation of the main idea, we assume that $\mathscr{M}$ is a ball $B \subset \mathbb{R}^k$ and ξ is compactly supported in B. (The general case can be reduced to this one, *via* a partition of unity and working in chart domains.) Consider an embedded manifold $\mathscr{E}$ and a smooth map $\Theta\colon \mathscr{E} \to \mathscr{N}$ with the two following crucial properties: (a) ("killing" property) the closed form $\Theta^*\omega$ is exact: there exists some $(k-1)$-form γ such that $\Theta^*\omega = \mathrm{d}\gamma$; (b) (lifting property) every "sufficiently smooth" map $f\colon B \to \mathscr{N}$ has a "sufficiently smooth" lifting $\widetilde{f}\colon B \to \mathscr{E}$ (i.e., $\widetilde{f}$ satisfies $\Theta \circ \widetilde{f} = f$).

A typical example occurs when $k = 1$, $\mathscr{E}$ is the universal cover of $\mathscr{N}$, and Θ is the corresponding covering map. Indeed, since $\mathscr{E}$ is simply connected, the smooth closed 1-form $\Theta^*\omega$ is automatically exact. On the other hand, if $f \in W^{s,p}(B;\mathscr{N})$ (with B an interval), where $0 < s < 1$ and $sp \geq 1$, then f has a lifting $\widetilde{f} \in W^{s,p}(B;\mathscr{E})$ (see Bourgain, Brezis, and Mironescu [10] and Bethuel and Chiron [7]).

The following formal calculation shows the help one can expect from the existence of $\mathscr{E}$ and Θ:

$$\int_B \xi\, f^*\omega = \int_B \xi\, (\Theta \circ \widetilde{f})^*\omega = \int_B \xi\, \widetilde{f}^*(\Theta^*\omega) = \int_B \xi\, \widetilde{f}^*(\mathrm{d}\gamma) = (-1)^k \int_B \widetilde{f}^*\gamma \wedge \mathrm{d}\xi. \tag{5.26}$$

We are now in a situation similar to the one in the proof of Theorem 5.7: we can *define* the left-hand side of (5.26) as the right-hand side of (5.26), provided the latter integral makes sense. We note that, in principle, we are now in a better position than initially, since γ is a $(k-1)$-form and thus the right-hand side of (5.26) is defined when, e.g., $\widetilde{f} \in W^{1,k-1}(B;\mathscr{E})$ and

$$\text{there exists a compact set } K \subset \mathscr{E} \text{ such that } \widetilde{f}(B) \subset K \tag{5.27}$$

– this is to be compared with the natural condition for the existence of the left-hand side of (5.26), which is $f \in W^{1,k}(B;\mathscr{N})$.

Actually, when $k \geq 3$, one can even go beyond $W^{1,k-1}(B;\mathscr{E})$, by adapting the main idea of the proof of Theorem 5.7, as follows. Let $\widetilde{f} \in W^{1-1/k,k}(B;\mathscr{E}))$ satisfy (5.27) – by the Gagliardo–Nirenberg inequalities, this condition is weaker than $\widetilde{f} \in (W^{1,k-1} \cap \mathscr{L}^\infty)(B;\mathscr{E})$. Take an extension $\widetilde{F} \in W^{1,k} \cap \mathscr{L}^\infty$ of $\widetilde{f}$, and let $\widetilde{\xi}(x,\varepsilon) := \xi(x)$, $\forall\, x \in B$, $\forall\, \varepsilon > 0$. Assuming that $\mathscr{E}$ is embedded in $\mathbb{R}^\iota$, consider a smooth compactly supported $(k-1)$-form $\widetilde{\gamma}$ on $\mathbb{R}^\iota$ that coincides with γ on K. Then, at least formally,

$$\begin{aligned} \int_B \xi\, f^*\omega = (-1)^k \int_B \widetilde{f}^*\gamma \wedge \mathrm{d}\xi &= (-1)^{k+1} \int_{\partial(B\times(0,\infty))} (\widetilde{F}^*\widetilde{\gamma}) \wedge \mathrm{d}\widetilde{\xi} \\ &= (-1)^{k+1} \int_{B\times(0,\infty)} [\widetilde{F}^*(\mathrm{d}\widetilde{\gamma})] \wedge \mathrm{d}\widetilde{\xi}, \end{aligned} \tag{5.28}$$

and, as above, we may *define* the left-hand side of (5.26) or (5.28) as the right-hand side of (5.28), potentially obtaining in this way the existence of the distribution $\xi \mapsto \int_B \xi\, f^*\omega$ for maps f of lower regularity than expected. For more insight, see [29].

When $k = 1$, the general philosophy presented above yields a 0-form γ, i.e., a function, and thus $\widetilde{f}^*\gamma = \gamma(\widetilde{f})$ is a function. This suggests that natural function spaces leading to a robust distribution $\xi \mapsto \int_B \xi\, f^*\omega$ *involve no derivatives of* f.

We next illustrate the effectiveness of this approach when $k = 1$, $\mathscr{N} = \mathbb{S}^1$, $\mathscr{E} = \mathbb{R}$, and $\Theta(t) = e^{\imath t}$, $\forall\, t \in \mathbb{R}$. In this case, any 1-form ω on $\mathscr{N}$ is automatically closed, and its pullback $\Theta^*\omega$ is automatically exact. Since, in this setting, $\mathscr{M}$ is a Lipschitz closed curve, we assume, for simplicity, that $\mathscr{M} = \mathbb{S}^1$. (The general case may be easily reduced to this one.) In this case, we have the following result, suggested by the above discussion.

Theorem 5.14. *Let ω be a smooth 1-form on $\mathbb{S}^1$.*

(1) *Let $\xi \in W^{1,1}(\mathbb{S}^1;\mathbb{R})$. Then the mapping*

$$C^1(\mathbb{S}^1;\mathbb{S}^1) \ni f \mapsto \int_{\mathbb{S}^1} \xi\, f^*\omega \tag{5.29}$$

has a unique extension by continuity + density to $C(\mathbb{S}^1;\mathbb{S}^1)$ (with the uniform convergence metric).

(2) *Let $\xi : \mathbb{S}^1 \to \mathbb{R}$ be such that ξ' belongs to the Hardy space $\mathscr{H}^1(\mathbb{S}^1)$. Then the mapping (5.29) has a unique extension by continuity + density to $\mathrm{VMO}(\mathbb{S}^1;\mathbb{S}^1)$ (with the metric induced by the $\mathrm{BMO} \cap \mathscr{L}^1$ convergence).*

Remark 5.15. 1. Let us note that, in this very special situation, there is no need for a partition of unity and we do not make any support assumption on ξ.

2. When ω is the canonical volume form on $\mathbb{S}^1$, Theorem 5.14 is due to Brezis and Nguyen [23, Definition 2], and (5.31) below coincides with formula (7.2) presented in Brezis and Nguyen [23, Remark 14]. □

Proof. We denote by $z = e^{\imath\theta}$ a generic point on $\mathbb{S}^1$. Let $\omega = \alpha(z)\,\omega_{\mathbb{S}^1}$ be a smooth form on $\mathbb{S}^1$, with $\alpha : \mathbb{S}^1 \to \mathbb{R}$ smooth and $\omega_{\mathbb{S}^1}(e^{\imath\theta}) := \mathrm{d}\theta$ the standard volume form of $\mathbb{S}^1$. Let $\beta(\theta) := \alpha(e^{\imath\theta})$ and let B be a (fixed) primitive of β. Clearly, we have

$$B(\theta + 2\ell\pi) = B(\theta) + \ell \int_{\mathbb{S}^1} \omega,\ \forall\, \theta \in \mathbb{R}, \forall\, \ell \in \mathbb{Z}. \tag{5.30}$$

Let $\psi(\theta) := \xi(e^{\imath\theta})$.

We first assume that $f \in C^1(\mathbb{S}^1;\mathbb{S}^1)$. Let $\varphi \in C^1(\mathbb{R};\mathbb{R})$ be such that $f(e^{\imath\theta}) = e^{\imath\varphi(\theta)}$,

$\forall\, \theta \in \mathbb{R}$. Then, for each $\theta_0 \in \mathbb{R}$, we have

$$\begin{aligned}\int_{\mathbb{S}^1} \xi\, f^*\omega &= \int_{\theta_0}^{2\pi+\theta_0} \psi(\theta)[\alpha(\varphi(\theta))]\, \varphi'(\theta)\, \mathrm{d}\theta = \int_{\theta_0}^{2\pi+\theta_0} \psi(\theta)[B(\varphi(\theta))]'\, \mathrm{d}\theta \\ &= \Big[\psi(\theta)\, B(\varphi(\theta))\Big]_{\theta_0}^{2\pi+\theta_0} - \int_{\theta_0}^{2\pi+\theta_0} \psi'(\theta)\, [B(\varphi(\theta))]\, \mathrm{d}\theta \\ &= \deg(f)\, \psi(\theta_0) \int_{\mathbb{S}^1} \omega - \int_{\theta_0}^{2\pi+\theta_0} \psi'(\theta)\, [B(\varphi(\theta))]\, \mathrm{d}\theta,\end{aligned} \tag{5.31}$$

where we have used (5.30) and the fact that $\varphi(2\pi + \theta_0) - \varphi(\theta_0) = 2\pi \deg(f)$. It is clear, from (5.31), that the last line in (5.31) does not depend on φ. This can also be derived from the fact that, for two possible choices φ_1 and φ_2 of φ, $B(\varphi_1) - B(\varphi_2)$ is constant (by (5.30)), combined with the fact that $\int_{\theta_0}^{2\pi+\theta_0} \xi' = 0$.

Proof of item (1). Given $f \in C(\mathbb{S}^1; \mathbb{S}^1)$, let $\varphi \in C(\mathbb{R}; \mathbb{R})$ be such that $f(e^{\imath\theta}) = e^{\imath\varphi(\theta)}$, $\forall\, \theta \in \mathbb{R}$. If $(f_j) \subset C^1(\mathbb{S}^1; \mathbb{S}^1)$ is such that $f_j \to f$ uniformly, then there exist $\varphi_j \in C^1(\mathbb{R}; \mathbb{R})$ such that $f_j(e^{\imath\theta}) = e^{\imath\varphi_j(\theta)}$, $\forall\, \theta \in \mathbb{R}$, $\forall\, j$, and $\varphi_j \to \varphi$ uniformly. Clearly, (5.31) implies that

$$\int_{\mathbb{S}^1} \xi\, (f_j)^*\omega \to \deg(f)\, \psi(\theta_0) \int_{\mathbb{S}^1} \omega - \int_{\theta_0}^{2\pi+\theta_0} \psi'(\theta)\, [B(\varphi(\theta))]\, \mathrm{d}\theta.$$

This proves item (1). Moreover, since the right-hand side of (5.31) is well-defined for $f \in C(\mathbb{S}^1; \mathbb{S}^1)$ and does not depend on the choice of a continuous lifting φ or of a point θ_0, we can take it as the *definition* of $\int_{\mathbb{S}^1} \xi\, f^*\omega$ for continuous f.

Proof of item (2). This is slightly more involved. To start with, it will be convenient to rewrite, for $f \in C^1(\mathbb{S}^1; \mathbb{S}^1)$, the identity (5.31) in a form involving only maps well-defined on $\mathbb{S}^1$ (which is not the case for φ and $B \circ \varphi$).

Step 1. An alternative form of (5.31). Assume that f is C^1. We write

$$f(z) = z^{\deg(f)} e^{\imath\overline{\varphi}(z)},\ \forall\, z \in \mathbb{S}^1, \tag{5.32}$$

where $\overline{\varphi} \in C^1(\mathbb{S}^1; \mathbb{R})$. The connection between $\overline{\varphi}$ and φ above is that, up to a constant integer multiple of 2π, we have

$$\varphi(\theta) = \deg(f)\, \theta + \overline{\varphi}(e^{\imath\theta}),\ \forall\, \theta \in \mathbb{R}.$$

We also write $\alpha = \alpha_0 + \overline{\alpha}$, where $\alpha_0 := \fint_{\mathbb{S}^1} \omega$ and $\overline{\alpha} := \alpha - \alpha_0$. Let C denote a primitive of $\theta \mapsto \overline{\alpha}(e^{\imath\theta})$. By contrast with B, the function C is 2π-periodic, and thus the map

$$e^{\imath\theta} \mapsto D(e^{\imath\theta}) := C(\theta)$$

is well-defined and smooth.

Applying (5.31) with

$$B(\theta) := C(\theta) + \alpha_0\theta = D(e^{\imath\theta}) + \alpha_0\theta, \forall\, \theta \in \mathbb{R},$$

and

$$\varphi(\theta) := \deg(f)\theta + \overline{\varphi}(e^{\imath\theta}), \forall\, \theta \in \mathbb{R},$$

we obtain

$$\begin{aligned}\int_{\mathbb{S}^1} \xi\, f^*\omega &= \deg(f)\,\psi(\theta_0)\int_{\mathbb{S}^1}\omega - \int_{\theta_0}^{2\pi+\theta_0} \psi'(\theta)\,[B(\varphi(\theta))]\,\mathrm{d}\theta \\ &= 2\pi\deg(f)\psi(\theta_0)\,\alpha_0 \\ &\quad - \int_{\theta_0}^{2\pi+\theta_0} \psi'(\theta)\,[D(f(e^{\imath\theta})) + \alpha_0\times(\deg(f)\,\theta + \overline{\varphi}(e^{\imath\theta}))]\,\mathrm{d}\theta \\ &= \alpha_0\deg(f)\int_{\mathbb{S}^1}\xi - \int_{\mathbb{S}^1}\xi' D(f) - \alpha_0\int_{\mathbb{S}^1}\xi'\,\overline{\varphi} \\ &= \frac{1}{2\pi}\deg(f)\int_{\mathbb{S}^1}\omega\int_{\mathbb{S}^1}\xi - \int_{\mathbb{S}^1}\xi' D(f) - \frac{1}{2\pi}\int_{\mathbb{S}^1}\omega\int_{\mathbb{S}^1}\xi'\,\overline{\varphi}.\end{aligned} \tag{5.33}$$

Step 2. Density. The space $C^1(\mathbb{S}^1;\mathbb{S}^1)$ is dense in $\mathrm{VMO}(\mathbb{S}^1;\mathbb{S}^1)$ (with the $\mathrm{BMO}\cap\mathscr{L}^1$ convergence). Indeed, $\mathrm{Lip}(\mathbb{S}^1;\mathbb{S}^1)$ is dense in $\mathrm{VMO}(\mathbb{S}^1;\mathbb{S}^1)$ (see, e.g., the construction in the proof of [22, Corollary 15.5]). By a standard smoothing argument, this implies that $C^1(\mathbb{S}^1;\mathbb{S}^1)$ is dense in $\mathrm{VMO}(\mathbb{S}^1;\mathbb{S}^1)$. (See also [23, Lemma 4].)

Step 3. Existence of lifting. If $f \in \mathrm{VMO}(\mathbb{S}^1;\mathbb{S}^1)$, then f has a well-defined winding number. This follows from the considerations in Section 2.2, using the fact that the winding number accounts for the homotopy class of continuous maps from $\mathbb{S}^1$ to $\mathbb{S}^1$. Moreover, there exists some lifting $\overline{\varphi} \in \mathrm{VMO}(\mathbb{S}^1;\mathbb{R})$, unique up to a constant integer multiple of 2π, such that (5.32) holds (see Brezis and Nirenberg [24, Theorem 3, Remark 10 (iii)]). In particular, if $f \in C^k$, then $\overline{\varphi} \in C^k$ and thus $\overline{\varphi}$ is a classical lifting of $z \mapsto f(z)/z^{\deg(f)}$. (This follows by uniqueness.) In addition, if $f_j, f \in \mathrm{VMO}(\mathbb{S}^1;\mathbb{S}^1)$ and $f_j \to f$ in $\mathrm{BMO}\cap\mathscr{L}^1$, then, for large j we have $\deg(f_j) = \deg(f)$ (by Corollary 2.13) and we may choose the corresponding liftings $\overline{\varphi}_j$ such that $\overline{\varphi}_j \to \overline{\varphi}$ in $\mathrm{BMO}\cap\mathscr{L}^1$. (For the latter fact, see Lemma 5.16 below.)

Step 4. Conclusion. Let $f \in \mathrm{VMO}(\mathbb{S}^1;\mathbb{S}^1)$. Consider a sequence $(f_j) \subset C^1(\mathbb{S}^1;\mathbb{S}^1)$ such that $f_j \to f$ in $\mathrm{BMO}\cap\mathscr{L}^1$ and the same holds for corresponding liftings $\overline{\varphi}_j$ and $\overline{\varphi}$. Using: (i) Corollary 2.13; (ii) the fact that $D(f_j) \to D(f)$ in BMO [24, Lemma A.8]; (iii) the fact that BMO and $\mathscr{H}^1$ are in duality, we find that

$$\begin{aligned}
&\frac{1}{2\pi}\deg(f_j)\int_{\mathbb{S}^1}\omega\int_{\mathbb{S}^1}\xi-\int_{\mathbb{S}^1}\xi' D(f_j)-\frac{1}{2\pi}\int_{\mathbb{S}^1}\omega\int_{\mathbb{S}^1}\xi'\overline{\varphi}_j\\
&\to\frac{1}{2\pi}\deg(f)\int_{\mathbb{S}^1}\omega\int_{\mathbb{S}^1}\xi-\langle\xi',D(f)\rangle-\frac{1}{2\pi}\left(\int_{\mathbb{S}^1}\omega\right)\times\langle\xi',\overline{\varphi}\rangle,
\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ stands for the duality pairing between $\mathscr{H}^1$ and BMO.

Therefore, the last line in (5.33): (j) is well-defined for $f\in \mathrm{VMO}(\mathbb{S}^1;\mathbb{S}^1)$ (if we interpret the second and the third integral as duality pairings); (jj) is continuous with respect to the BMO $\cap\,\mathscr{L}^1$ convergence; (jjj) can be taken as *definition* of $\int_{\mathbb{S}^1}\xi\, f^*\omega$ for $f\in\mathrm{VMO}$.

□

We next complete Step 3 in the proof of Theorem 5.14.

Lemma 5.16. *Let f_j, $f\in\mathrm{VMO}(\mathbb{S}^1;\mathbb{S}^1)$ be such that $f_j\to f$ in BMO $\cap\,\mathscr{L}^1$. Then, for sufficiently large j, there exist $\overline{\varphi}_j$, $\overline{\varphi}\in\mathrm{VMO}(\mathbb{S}^1;\mathbb{R})$ such that*

$$f_j(z)=z^{\deg(f)}e^{\imath\overline{\varphi}_j(z)},\ f(z)=z^{\deg(f)}e^{\imath\overline{\varphi}(z)},\ \forall\, z\in\mathbb{S}^1,\ \forall\, j,$$

and

$$\overline{\varphi}_j\to\overline{\varphi}\text{ in BMO}\cap\mathscr{L}^1.$$

Proof. Let $g_j:=f_j/f:\mathbb{S}^1\to\mathbb{S}^1$. Since $f_j\to f$ in BMO$\cap\mathscr{L}^1$, we have $g_j\to 1$ in BMO$\cap\mathscr{L}^1$. (Apply [24, Lemma A.8] to the map $(z,w)\mapsto z\overline{w}$.) By Corollary 2.13, for large j we have $\deg(g_j)=0$, and thus we may write $g_j=e^{\imath\widetilde{\varphi}_j}$, with $\widetilde{\varphi}_j\in\mathrm{VMO}(\mathbb{S}^1;\mathbb{R})$ [24, Theorem 3]. Moreover, for large j, we may choose $\widetilde{\varphi}_j$ such that

$$|\widetilde{\varphi}_j|_{\mathrm{BMO}}\le 4|g_j|_{\mathrm{BMO}} \tag{5.34}$$

([24, Theorem 4]).

Set $c_j:=\fint_{\mathbb{S}^1}\widetilde{\varphi}_j$. Combining (2.5) with (5.34), we find that

$$\|g_j-e^{\imath c_j}\|_1\le\|e^{\imath\widetilde{\varphi}_j}-e^{\imath c_j}\|_1\le\|\widetilde{\varphi}_j-c_j\|_1\le 4C|g_j|_{\mathrm{BMO}}. \tag{5.35}$$

Therefore, we have $e^{\imath c_j}\to 1$ as $j\to\infty$, and, after adding to each $\overline{\varphi}_j$ (and c_j) a suitable integer multiple of 2π, we may assume that $c_j\to 0$. Going back to (5.35), we find that $\widetilde{\varphi}_j\to 0$ in BMO $\cap\,\mathscr{L}^1$. Finally, the conclusions of the lemma hold with $\overline{\varphi}_j:=\overline{\varphi}+\widetilde{\varphi}_j$, where $\overline{\varphi}\in\mathrm{VMO}(\mathbb{S}^1;\mathbb{R})$ is such that $f(z)=z^{\deg(f)}e^{\imath\overline{\varphi}(z)}$, $\forall\, z\in\mathbb{S}^1$. □

6 A higher dimensional case

6.1 Heuristics

In Sections 3 and 4, we have considered a situation where $\dim \mathscr{M}$ and $k := \deg \omega$ coincide. A typical more general situation consists of considering maps $f\colon \mathscr{M}\times W \to \mathscr{N}$, where $W \subset \mathbb{R}^\ell$ is an (open) set of parameters. Let $0 < s < 1$ and $1 < p < \infty$ be such that $sp = k$. Assuming that $f(\cdot, w) \in W^{s,p}$ for a.e. w, one may consider the map $w \mapsto \mathscr{I}(f(\cdot, w))$, establish its properties, and estimate its "size". In view of the applications we have in mind, we investigate here a similar, but slightly different, situation.

In what follows, we consider: (a) a smooth *closed* k-form ω on $\mathscr{N}$; (b) $0 < s < 1$ and $1 < p < \infty$ such that $sp = k$; (c) an integer $N > k$.

In order to simplify the presentation, we consider only maps "that live in a compact set". We could consider for example the space $W^{s,p}(\Omega; \mathscr{N})$, with $\Omega \subset \mathbb{R}^N$ a smooth bounded open set. Our actual choice is to work instead in the space

$$W^{s,p}_1(\mathbb{R}^N; \mathscr{N}) := \{f \in \dot{W}^{s,p}(\mathbb{R}^N; \mathscr{N})\colon f \text{ is constant outside } K = K(f) \subset \mathbb{B}^N\},$$

where K is a compact set. (However, all the results below have counterparts for the space $W^{s,p}(\Omega; \mathscr{N})$.)

The purpose of this section is to give a robust meaning to the action of the k-form $f^*\omega$ on appropriate "test forms", with $f \in W^{s,p}_1(\mathbb{R}^N; \mathscr{N})$, and to exhibit the homotopical information encoded by this action, at least for "nice" f's.

An initial remark is that $f^*\omega$, which is formally a k-form, may act, up to the action of the Hodge $*$-operator, either on k-forms, or on $(N-k)$-forms. For convenience matters, it is customary to choose the latter perspective.

We now present some heuristics, provided by the next formal calculation, inspired by (5.5). If $\xi \in C^\infty_c(\mathbb{R}^N; \Lambda^{N-k})$, then we formally have

$$\begin{aligned}\int_{\mathbb{R}^N} f^*\omega \wedge \xi = -\int_{\mathbb{R}^N\times(0,\infty)} \mathrm{d}[\widetilde{F}^*\alpha \wedge \widetilde{\xi}] = -\int_{\mathbb{R}^N\times(0,\infty)} \widetilde{F}^*(\mathrm{d}\alpha) \wedge \widetilde{\xi} \\ + (-1)^{k+1}\int_{\mathbb{R}^N\times(0,\infty)} \widetilde{F}^*\alpha \wedge \mathrm{d}\widetilde{\xi}.\end{aligned} \tag{6.1}$$

As explained in Section 5.3, in order to treat the latter integral in (6.1), the assumption $f \in W^{s,p}_1(\mathbb{R}^N; \mathscr{N})$ is not sufficient. For example, when $k \geq 2$, we have to make the extra assumption $f \in W^{1-1/k,k}_1(\mathbb{R}^N; \mathscr{N})$ (see Theorem 5.7). In order to work in the minimal space $f \in W^{s,p}_1(\mathbb{R}^N; \mathscr{N})$, it is natural to require that the latter integral in (6.1) vanishes.

This is the case if $\mathrm{d}\widetilde{\xi} = 0$, and can be achieved if $\mathrm{d}\xi = 0$ (take $\widetilde{\xi}(x,\varepsilon) := \xi(x)$, $\forall\, x \in \mathbb{R}^N$, $\forall\, \varepsilon > 0$). In $\mathbb{R}^N$, the assumption $\mathrm{d}\xi = 0$ is equivalent to ξ being exact. With this in mind, we do not investigate below the action of $f^*\omega$ on general $(N-k)$-forms, but only on exterior differentials of $(N-k-1)$-forms.

In view of the above discussion, it is natural to consider the operator (at least formally) given by

$$\langle Tf, \zeta\rangle := \int_{\mathbb{R}^N} f^*\omega \wedge \mathrm{d}\zeta,\ \forall\, \zeta \in \mathrm{Lip}(\mathbb{R}^N; \Lambda^{N-k-1}). \tag{6.2}$$

To connect the definition (6.2) with the informal exposition in the introduction, we mention more specifically that our definition of T amounts to

$$Tf = (-1)^{k+1} * \mathrm{d}(f^*\omega) \quad \text{in the sense of distributions (or rather currents)}, \tag{6.3}$$

so that the assumption $Tf = 0$ is indeed the same as $\mathrm{d}[f^*\omega] = 0$.

To justify the above, writing $\alpha = f^*\omega$ and using standard identities from exterior calculus (see, e.g., [33, 1.7.8]), we find that

$$(\mathrm{d}\alpha) \wedge \zeta = (**\,\mathrm{d}\alpha) \wedge (**\,\zeta) = \langle **\,\mathrm{d}\alpha, *\zeta\rangle = \langle *\mathrm{d}\alpha, \zeta\rangle.$$

On the other hand, we have

$$0 = \int_{\mathbb{R}^N} \mathrm{d}(\alpha \wedge \zeta) = \int_{\mathbb{R}^N} (\mathrm{d}\alpha) \wedge \zeta + (-1)^k \int_{\mathbb{R}^N} \alpha \wedge \mathrm{d}\zeta,$$

whence the claimed identity.

A crucial property in what follows is the density of $W^{1,k}_1(\mathbb{R}^N; \mathscr{N})$ into $W^{s,p}_1(\mathbb{R}^N; \mathscr{N})$. Although not stated in these terms, this property was implicitly obtained in Brezis and Mironescu [21]. Indeed, the proof of Theorem 3 in [21] explicitly exhibits, for a given $f \in W^{s,p}_1(\mathbb{R}^N; \mathscr{N})$, a sequence $(f_j) \subset W^{1,k}_1(\mathbb{R}^N; \mathscr{N})$ such that $f_j - f \to 0$ in $W^{s,p}(\mathbb{R}^N)$. Moreover, this sequence satisfies the additional properties: (j) $(f_j) \subset W^{1,q}_1(\mathbb{R}^N; \mathscr{N})$, $\forall 1 \le q < k+1$; (jj) $f_j = f$ in $\mathbb{R}^N \setminus \mathbb{B}^N$. For further use, we note that, by the Gagliardo–Nirenberg inequalities, if $k \ge 2$, then $W^{1,k}_1(\mathbb{R}^N; \mathscr{N}) \subset W^{s,p}_1(\mathbb{R}^N; \mathscr{N})$. This fails when $k = 1$, but we have $W^{1,q}_1(\mathbb{R}^N; \mathscr{N}) \subset W^{s,p}_1(\mathbb{R}^N; \mathscr{N})$, $\forall\, q > 1$.

Clearly, when $f \in W^{1,k}_1(\mathbb{R}^N; \mathscr{N})$: (j)$f^*\omega$ is naturally defined a.e. as the pullback of ω through $\mathrm{d}f$; (jj) $f^*\omega \in \mathscr{L}^1(\mathbb{R}^N; \Lambda^k)$; (jjj) $\langle Tf, \zeta\rangle$ is well-defined and satisfies the obvious bound

$$|\langle Tf, \zeta\rangle| \le C\|\nabla f\|_k^k |\zeta|_{\mathrm{Lip}},$$

where the finite constant C depends only on ω. For such f, Tf was considered by Bethuel, Coron, Demengel, and Hélein [8], with the purpose of characterizing the closure of smooth maps in the space of $W^{1,k}$ mappings. In the same functional setting, some of the properties of T were investigated by Giaquinta and collaborators – see for example Giaquinta, Modica, and Souček [36, Chapter 5.4], Giaquinta and Mucci [39], Giaquinta and Modica [34] – and in a different direction by Alberti, Baldo, and Orlandi [1]. These ideas have their roots in the work notably by Bethuel [4], Almgren, Browder, and Lieb [2], and Brezis, Coron, and Lieb [19].

Our main purpose in this section is first to give a robust meaning to Tf when $f \in W_1^{s,p}(\mathbb{R}^N;\mathscr{N})$ and to obtain the corresponding estimate, and then to exploit this object to obtain a characterization of the closure of smooth maps in $W_1^{s,p}(\mathbb{R}^N;\mathscr{N})$ for a large class of target manifolds $\mathscr{N}$. When $\mathscr{N} = \mathbb{S}^k$ and ω is the standard volume form, the first part of this program was completed by Bourgain, Brezis, and Mironescu [11] when $N = k+1$, and for a general $N \geq k+1$ by Bousquet and Mironescu [14]. The second part of this program was addressed by Mucci [54] in the special case where $\mathscr{N} = \mathbb{S}^k$.

6.2 Existence of a robust map T

Recall that we consider: (a) a smooth *closed* k-form ω on $\mathscr{N}$; (b) $0 < s < 1$ and $1 < p < \infty$ such that $sp = k$; (c) an integer $N > k$.

The first main result in this section is

Theorem 6.1. *Let $k \geq 2$.*

(1) *The map T, defined in (6.2) for $f \in W_1^{1,k}(\mathbb{R}^N;\mathscr{N})$, has an (unique) extension by continuity to $W_1^{s,p}(\mathbb{R}^N;\mathscr{N})$.*

(2) *The extension, still denoted T, satisfies*

$$|\langle Tf,\zeta\rangle| \leq C|f|^p_{W^{s,p}}|\zeta|_{\mathrm{Lip}}, \forall\, f \in W_1^{s,p}(\mathbb{R}^N;\mathscr{N}), \forall\, \zeta \in \mathrm{Lip}(\mathbb{R}^N;\Lambda^{N-k-1}), \tag{6.4}$$

where the finite constant C depends only on s, p, N, and ω.

(3) *Let $\widetilde{\Pi}$ be as in (4.11) and $\alpha \in C_c^\infty(\mathbb{R}^n;\Lambda^k)$ be an extension of ω. We have the following formula:*

$$\langle Tf,\zeta\rangle = -\int_{\mathbb{R}^N\times(0,\infty)} \widetilde{F}^*(\mathrm{d}\alpha)\wedge \mathrm{d}\widetilde{\zeta},\ \forall\, f \in W_1^{s,p}(\mathbb{R}^N;\mathscr{N}), \quad \forall\, \zeta \in \mathrm{Lip}(\mathbb{R}^N;\Lambda^{N-k-1}). \tag{6.5}$$

Here, $\widetilde{F} := \widetilde{\Pi}\circ F$, with F defined by (6.6) below, and $\widetilde{\zeta}(x,t) := \zeta(x)$, $\forall x \in \mathbb{R}^N$, $\forall t \geq 0$.

Remark 6.2. When $\mathscr{N} = \mathbb{S}^k$ and $\omega = \omega_{\mathbb{S}^k}$ is the standard volume form on $\mathbb{S}^k$, then

$$T_{\omega_{\mathbb{S}^k}} f = (-1)^{k+1}(k+1) * \operatorname{Jac} f$$

(see (6.3)), where $\operatorname{Jac} f$ is the distributional Jacobian (or more precisely, the Jacobian in the sense of currents) as defined in [11, 14]. □

The proof of Theorem 6.1 relies on the following cousin of Proposition 5.9.

Proposition 6.3. *For $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$, let $i_t(x) := (x, t)$. Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded open set. For $F \in \operatorname{Lip}(\Omega \times [a, b]; \mathbb{R}^n)$ and $\xi \in \operatorname{Lip}_c(\Omega \times [a, b]; \Lambda^{N-k-1})$, we have*

$$\int_{\Omega\times(a,b)} F^*(\mathrm{d}\alpha) \wedge \mathrm{d}\xi = \int_{\Omega} (F_b)^*\alpha \wedge \mathrm{d}(i_b^*\xi) - \int_{\Omega} (F_a)^*\alpha \wedge \mathrm{d}(i_b^*\xi).$$

In particular, if $\zeta \in \operatorname{Lip}_c(\Omega; \Lambda^{N-k-1})$, then

$$\int_{\Omega\times(a,b)} F^*(\mathrm{d}\alpha) \wedge \mathrm{d}\widetilde{\zeta} = \int_{\Omega} (F_b)^*\alpha \wedge \mathrm{d}\zeta - \int_{\Omega} (F_a)^*\alpha \wedge \mathrm{d}\zeta.$$

The proof of Proposition 6.3 is a straightforward variant of the one of Proposition 3.26.

Proof of Theorem 6.1. In what follows, C_j denotes a finite constant independent of f.

Let $\rho \in C_c^\infty(\mathbb{B}^N)$ be a mollifier (in $\mathbb{R}^N$). For $\varepsilon > 0$, set

$$F(x, \varepsilon) := f * \rho_\varepsilon(x). \tag{6.6}$$

For every $f \in \mathscr{L}^1_{\mathrm{loc}}(\mathbb{R}^N)$, the map F is smooth and

$$|\nabla F(x, \varepsilon)| \leq \frac{C_1}{\varepsilon} \fint_{B_\varepsilon(x)} |f(y) - f(x)| \, \mathrm{d}y, \forall x \in \mathbb{R}^N, \varepsilon > 0. \tag{6.7}$$

We define the associated map $\widetilde{F}$ by $\widetilde{F} := \widetilde{\Pi} \circ F$.

The existence of an extension of T to $W_1^{s,p}(\mathbb{R}^N; \mathscr{N})$ relies on Steps 1 and 2 below.

Step 1. For $f \in W_1^{1,k}(\mathbb{R}^N; \mathscr{N})$, we have

$$\int_{\mathbb{R}^N\times(0,\infty)} \left|\widetilde{F}^*(\mathrm{d}\alpha)\right| < \infty. \tag{6.8}$$

Indeed, since $f \in \mathscr{L}^\infty \cap W_1^{1,k}$, by the Gagliardo–Nirenberg inequality, we have

$$f \in W_1^{1-1/(k+1),k+1}(\mathbb{R}^N; \mathscr{N}). \tag{6.9}$$

By standard trace theory, (6.9) implies that

$$F \in \dot{W}^{1,k+1}(\mathbb{R}^N \times (0,\infty)). \tag{6.10}$$

We obtain (6.8) from (6.10) and the fact that $|\widetilde{F}^*(\mathrm{d}\alpha)(x,\varepsilon)| \le C_2|\nabla F(x,\varepsilon)|^{k+1}$.

Step 2. For $f \in W_1^{1,k}(\mathbb{R}^N;\mathscr{N})$, we have

$$\int_{\mathbb{R}^N} f^*\omega \wedge \mathrm{d}\zeta = -\int_{\mathbb{R}^N\times(0,\infty)} \widetilde{F}^*(\mathrm{d}\alpha) \wedge \mathrm{d}\widetilde{\zeta}. \tag{6.11}$$

Indeed, let $0 < a < b < \infty$. Let $\phi = \phi(x) \in C_c^\infty(B_{b+2}(0))$, $x \in \mathbb{R}^N$, be such that $\phi = 1$ in $\overline{B}_{b+1}(0)$. Since $F(\cdot,t)$ is constant in $(\mathbb{R}^N\backslash\overline{B}_{b+1}(0)) \times [a,b]$, we have, by Proposition 6.3 (applied with $\Omega := B_{b+2}(0)$),

$$\begin{aligned}
\int_{\mathbb{R}^N\times(a,b)} \widetilde{F}^*(\mathrm{d}\alpha) \wedge \mathrm{d}\widetilde{\zeta} &= \int_{B_{b+1}(0)\times(a,b)} \widetilde{F}^*(\mathrm{d}\alpha) \wedge \mathrm{d}\widetilde{\zeta} \\
&= \int_{B_{b+1}(0)\times(a,b)} \widetilde{F}^*(\mathrm{d}\alpha) \wedge \mathrm{d}(\widetilde{\phi}\widetilde{\zeta}) \\
&= \int_{B_{b+2}(0)\times(a,b)} \widetilde{F}^*(\mathrm{d}\alpha) \wedge \mathrm{d}(\widetilde{\phi}\widetilde{\zeta}) \\
&= \int_{B_{b+2}(0)} (\widetilde{F}_b)^*\alpha \wedge \mathrm{d}(\phi\zeta) - \int_{B_{b+2}(0)} (\widetilde{F}_a)^*\alpha \wedge \mathrm{d}(\phi\zeta) \\
&= \int_{B_{b+1}(0)} (\widetilde{F}_b)^*\alpha \wedge \mathrm{d}\zeta - \int_{B_{b+1}(0)} (\widetilde{F}_a)^*\alpha \wedge \mathrm{d}\zeta \\
&= \int_{\mathbb{R}^N} (\widetilde{F}_b)^*\alpha \wedge \mathrm{d}\zeta - \int_{\mathbb{R}^N} (\widetilde{F}_a)^*\alpha \wedge \mathrm{d}\zeta.
\end{aligned} \tag{6.12}$$

We notice that

$$\left|\int_{\mathbb{R}^N} (\widetilde{F}_b)^*\alpha \wedge \mathrm{d}\zeta\right| \le C_3|\zeta|_{\mathrm{Lip}} \int_{\mathbb{R}^N} |\nabla F(x,b)|^k \,\mathrm{d}x. \tag{6.13}$$

By (6.7) and the fact that $f = c_f$ outside $\mathbb{B}^N$ for some constant c_f, for $b > 1$, we have

$$|\nabla F(x,b)| \le \begin{cases} 0, & \text{if } |x| \ge 1+b \\ C_4/b^{N+1}, & \text{if } 1 < |x| < 1+b \\ C_5/b, & \text{if } |x| \le 1 \end{cases}. \tag{6.14}$$

We justify, e.g., the second estimate. If $1 < |x| < 1 + b$, we have, by (6.7):

$$|\nabla F(x,b)| \le \frac{C_6}{b^{N+1}} \int_{B_b(x)} |f(x) - f(y)| \, dy = \frac{C_6}{b^{N+1}} \int_{B_b(x)} |c_f - f(y)| \, dy$$
$$\le \frac{C_6}{b^{N+1}} \int_{\mathbb{B}^N} |c_f - f(y)| \, dy \le \frac{C_7}{b^{N+1}}.$$

Combining (6.13) and (6.14), one gets

$$\left| \int_{\mathbb{R}^N} (\widetilde{F}_b)^* \alpha \wedge d\zeta \right| \le C_8 \left(\frac{b^N}{b^{k+Nk}} + \frac{1}{b^k} \right) |\zeta|_{\mathrm{Lip}} \to 0 \text{ as } b \to \infty. \tag{6.15}$$

We next note that

$$F_\varepsilon \to f \text{ in } W^{1,k}(\mathbb{R}^N) \text{ as } \varepsilon \to 0. \tag{6.16}$$

Combining (6.16) with the $W^{1,k}$-continuity of the superposition with Lipschitz functions (see, e.g., [22, Theorem 15.6]), we find that

$$\widetilde{\Pi} \circ F_\varepsilon \to \widetilde{\Pi} \circ f \text{ in } W^{1,k}(\mathbb{R}^N) \text{ as } \varepsilon \to 0,$$

i.e.,

$$\widetilde{F}(\cdot, \varepsilon) \to f \text{ in } W^{1,k}(\mathbb{R}^N) \text{ as } \varepsilon \to 0. \tag{6.17}$$

From (6.17) and the $\mathscr{L}^p$-continuity of the superposition with Carathéodory functions (see, e.g., Rindler [59, Theorem 2.13]), we have

$$(\widetilde{F}_\varepsilon)^* \alpha \to f^* \alpha \text{ in } \mathscr{L}^1 \text{ as } \varepsilon \to 0. \tag{6.18}$$

Finally, (6.11) follows from Step 1, (6.12), (6.15), (6.18), and the fact that, for every extension α of ω, we have

$$f^* \alpha = f^* \omega \text{ a.e.}$$

(This last property follows from the chain rule for the superposition of a smooth map and a Sobolev map.) Step 2 is completed.

In view of (6.11), it is natural to define, for $f \in W_1^{s,p}(\mathbb{R}^N; \mathscr{N})$, $\langle Tf, \zeta \rangle$ as the quantity on the right hand side of (6.11). This requires first to prove that this quantity makes sense and is finite. The proof of these facts is reminiscent of the one of Theorem 4.1.

For this purpose, we first settle a measurability issue by introducing $\widetilde{h}(x)$, a convenient substitute of $h(x)$ defined as in (4.21). To motivate the definition of $\widetilde{h}(x)$ below, we note that, if $x \in \mathbb{R}^N$ is a Lebesgue point of f, then $h(x) > 0$. (Since we are no longer in the setting where $W^{s,p}$ is embedded into VMO, we cannot use (2.30) anymore to conclude that h has a uniform lower bound.) Assuming further that $h(x) < \infty$, we therefore have

$$\frac{\delta}{2} = \operatorname{dist}(F(x,h(x)),\mathcal{N}) \leq |F(x,h(x)) - f(x)| \leq \int_0^{h(x)} |\partial_\varepsilon F(x,\varepsilon)| \,\mathrm{d}\varepsilon. \tag{6.19}$$

With (6.19) in mind, we set

$$G(x,\varepsilon) := |\partial_\varepsilon F(x,\varepsilon)|, \forall\, x \in \mathbb{R}^N, \forall\, \varepsilon > 0,\, c := \delta/2, \tag{6.20}$$

$$\widetilde{h}(x) := \begin{cases} 0, & \text{if } \int_0^1 G(x,\varepsilon)\,\mathrm{d}\varepsilon = \infty \\ \infty, & \text{if } \int_0^t G(x,\varepsilon)\,\mathrm{d}\varepsilon < c, \forall t > 0 \\ \inf\left\{t > 0 \colon \int_0^t G(x,\varepsilon)\,\mathrm{d}\varepsilon \geq c\right\}, & \text{otherwise} \end{cases}. \tag{6.21}$$

By the above, we have

$$\widetilde{h}(x) \leq h(x) \text{ for a.e. } x \in \mathbb{R}^N. \tag{6.22}$$

The measurability of $\widetilde{h}(x)$ is an easy consequence of the following result.

Lemma 6.4. *Let X be a metric space. Let $g \colon X \times (0,\infty) \to [0,\infty)$ be continuous. For $0 < c < \infty$, define*

$$\widetilde{g}(x) := \begin{cases} 0, & \textit{if } \int_0^1 g(x,\varepsilon)\,\mathrm{d}\varepsilon = \infty \\ \infty, & \textit{if } \int_0^t g(x,\varepsilon)\,\mathrm{d}\varepsilon < c, \forall t > 0 \\ \inf\left\{t > 0 \colon \int_0^t g(x,\varepsilon)\,\mathrm{d}\varepsilon \geq c\right\}, & \textit{otherwise} \end{cases}. \tag{6.23}$$

Then $\widetilde{g}(x)$ is a Borel function.

Granted Lemma 6.4, the function $\widetilde{h}$ is Borel.

Step 3. For $f \in W_1^{s,p}(\mathbb{R}^N;\mathcal{N})$, the form $\widetilde{F}^*(\mathrm{d}\alpha)$ is integrable over $\mathbb{R}^N \times (0,\infty)$, and

$$\int_{\mathbb{R}^N \times (0,\infty)} \left|\widetilde{F}^*(\mathrm{d}\alpha)\right| \leq C_9 |f|^p_{W^{s,p}}. \tag{6.24}$$

Repeating the proof of (4.28) (relying on (6.7) instead of (4.15)) and using (6.22), we have, for a.e. $x \in \mathbb{R}^N$,

$$\begin{aligned}\int_0^\infty |\widetilde{F}^*(d\alpha)(x,\varepsilon)|\, d\varepsilon &\le \int_{h(x)}^\infty |\widetilde{F}^*(d\alpha)(x,\varepsilon)|\, d\varepsilon \\ &\le \int_{\widetilde{h}(x)}^\infty |\widetilde{F}^*(d\alpha)(x,\varepsilon)|\, d\varepsilon \\ &\le C_{10}\frac{1}{[\widetilde{h}(x)]^{sp}}.\end{aligned} \tag{6.25}$$

On the other hand, by Hölder's inequality, we have

$$\left(\int_0^t |\nabla F(x,\varepsilon)|\, d\varepsilon\right)^p \le t^{sp}\left(\frac{p-1}{sp}\right)^{p-1} \int_0^t \varepsilon^{p(1-s)-1}|\nabla F(x,\varepsilon)|^p\, d\varepsilon. \tag{6.26}$$

By the standard theory of weighted Sobolev spaces (see, e.g., [50, Theorem 1.2]), we have

$$\int_{\mathbb{R}^N\times(0,\infty)} \varepsilon^{p(1-s)-1}|\nabla F(x,\varepsilon)|^p\, dx\, d\varepsilon \le C_{11}|f|^p_{W^{s,p}}. \tag{6.27}$$

Combining (6.26), (6.27), and the definition of $\widetilde{h}$, we find that $\widetilde{h}(x) > 0$ for a.e. $x \in \mathbb{R}^N$. On the other hand, if $0 < \widetilde{h}(x) < \infty$, then clearly

$$\int_0^{\widetilde{h}(x)} |\partial_\varepsilon F(x,\varepsilon)|\, d\varepsilon = \frac{\delta}{2}. \tag{6.28}$$

Combining (6.28) and (6.26), we find that

$$\frac{1}{[\widetilde{h}(x)]^{sp}} \le C_{12}\int_0^\infty \varepsilon^{p(1-s)-1}|\partial_\varepsilon F(x,\varepsilon)|^p\, d\varepsilon \text{ for a.e. } x \in \mathbb{R}^N. \tag{6.29}$$

The estimate (6.24) follows from (6.25), (6.27), and (6.29), hence completing Step 3.

Finally, the existence and uniqueness of the extension of T rely on the density property of $W^{1,k}_1(\mathbb{R}^N;\mathcal{N})$ in $W^{s,p}_1(\mathbb{R}^N;\mathcal{N})$ (see the discussion in Section 6.1) and the next step.

Step 4. The map $W^{s,p}_1(\mathbb{R}^N;\mathcal{N}) \ni f \mapsto \widetilde{F}^*(d\alpha) \in \mathscr{L}^1(\mathbb{R}^N\times(0,\infty);\Lambda^{k+1})$ is continuous in the following sense: if $f_j, f \in W^{s,p}_1(\mathbb{R}^N\times(0,\infty);\mathcal{N})$ are such that $|f_j - f|_{W^{s,p}} \to 0$ and $c_{f_j} \to c_f$, then

$$\widetilde{F}_j^*(d\alpha) \to \widetilde{F}^*(d\alpha) \text{ in } \mathscr{L}^1(\mathbb{R}^N\times(0,\infty)). \tag{6.30}$$

In order to prove (6.30), we first note that

$$(f_j - c_{f_j}) - (f - c_f) \to 0 \text{ in } \mathscr{L}^p. \tag{6.31}$$

This follows using: (i) $|f_j - f|_{W^{s,p}} \to 0$; (ii) the fact that $(f_j - c_{f_j}) - (f - c_f)$ is supported in $\mathbb{B}^N$; (iii) the Poincaré type inequality $\|g\|_p \le C_{13}|g|_{W^{s,p}}$, valid for g supported in $\mathbb{B}^N$.

Using (6.31) we find that, up to a subsequence, $f_j \to f$ a.e. and then, by dominated convergence,

$$\mathrm{D}^\ell \widetilde{F}_j \to \mathrm{D}^\ell \widetilde{F} \text{ uniformly on compacts of } \mathbb{R}^N \times (0,\infty),\ \forall\, \ell. \tag{6.32}$$

Using (6.32) with $\ell = 1$ yields

$$\widetilde{F}_j^*(\mathrm{d}\alpha) \to \widetilde{F}^*(\mathrm{d}\alpha) \text{ pointwise}. \tag{6.33}$$

On the other hand, let $\widetilde{h}_j$ be associated with f_j as in (6.21). By the proof of (4.28), we have

$$|\widetilde{F}_j^*(\mathrm{d}\alpha)(x,\varepsilon)| \le \frac{C_{14}}{\varepsilon^{k+1}} \chi_{\{\varepsilon > \widetilde{h}_j(x)\}},\ \forall\, x \in \mathbb{R}^N, \forall\, \varepsilon > 0.$$

We next claim that there exists an $\mathscr{L}^1(\mathbb{R}^N)$ function $H = H(x)$ such that, up to a subsequence,

$$\frac{1}{[\widetilde{h}_j(x)]^{sp}} \le H(x). \tag{6.34}$$

Indeed, by (6.27), we have

$$\varepsilon^{1-s-1/p}\nabla F_j(x,\varepsilon) \to \varepsilon^{1-s-1/p}\nabla F(x,\varepsilon) \text{ in } \mathscr{L}^p(\mathbb{R}^N \times (0,\infty)).$$

By the converse to the dominated convergence theorem, up to a subsequence, there exists some $J = J(x,\varepsilon) \in \mathscr{L}^p(\mathbb{R}^N \times (0,\infty))$ such that

$$\varepsilon^{1-s-1/p}|\nabla F_j(x,\varepsilon)| \le J(x,\varepsilon),\ \forall\, j, \forall\, x \in \mathbb{R}^N, \forall\, \varepsilon > 0.$$

For this subsequence, (6.29) yields

$$\frac{1}{[\widetilde{h}_j(x)]^{sp}} \le C_{15} \int_0^\infty [J(x,\varepsilon)]^p \, \mathrm{d}\varepsilon =: H(x),$$

so that (6.34) holds, as claimed.

Combining (6.33)–(6.34), we obtain (6.30), possibly up to a subsequence. However, the uniqueness of the limit in (6.30) implies that (6.30) holds for the full sequence.

The conclusions of the theorem follow from Steps 1–4. □

We next consider the case $k = 1$. As explained in Section 6.1, some care is needed to define initially Tf, because of the non-embedding $W^{1,1}_1(\mathbb{R}^N;\mathscr{N}) \not\subset W^{s,p}_1(\mathbb{R}^N;\mathscr{N})$. With this in mind, we have the following version of Theorem 6.1.

Theorem 6.5. *Let $k = 1$. Let $1 < q < 2$.*

(1) *The map T, defined in (6.2) for $f \in W^{1,q}_1(\mathbb{R}^N;\mathscr{N})$, has an (unique) extension by continuity to $W^{s,p}_1(\mathbb{R}^N;\mathscr{N})$.*

(2) *The extension, still denoted T, satisfies*

$$|\langle Tf,\zeta\rangle| \le C|f|^p_{W^{s,p}}|\zeta|_{\mathrm{Lip}}, \forall\, f \in W^{s,p}_1(\mathbb{R}^N;\mathscr{N}), \forall\, \zeta \in \mathrm{Lip}(\mathbb{R}^N;\Lambda^{N-2}),$$

where the finite constant C depends only on s, p, N, and ω.

(3) *Let $\widetilde{\Pi}$ be as in Section 4 and $\alpha \in C^\infty_c(\mathbb{R}^n;\Lambda^1)$ be an extension of ω. We have the following formula:*

$$\langle Tf,\zeta\rangle = -\int_{\mathbb{R}^N\times(0,\infty)} \widetilde{F}^*(\mathrm{d}\alpha)\wedge \mathrm{d}\widetilde{\zeta},\ \ \forall\, f \in W^{s,p}_1(\mathbb{R}^N;\mathscr{N}), \forall\, \zeta \in \mathrm{Lip}(\mathbb{R}^N;\Lambda^{N-2}).$$

Here, $\widetilde{F} := \widetilde{\Pi}\circ F$, with F defined by (6.6), and $\widetilde{\zeta}(x,t) := \zeta(x)$, $\forall x \in \mathbb{R}^N$, $\forall t \ge 0$.

Proof. The proof is essentially the same as the one of Theorem 6.1. The only difference occurs in Steps 1 and 2, where we rely on the Gagliardo–Nirenberg embedding $W^{1,q}_1(\mathbb{R}^N)\cap \mathscr{L}^\infty(\mathbb{R}^N) \subset W^{1/2,2}(\mathbb{R}^N)$, valid when $q > 1$ (but wrong when $q = 1$). □

We continue with the

Proof of Lemma 6.4. We only need to prove the lemma in the case where g is positive. Indeed, if the lemma holds for positive maps and g is only assumed to be nonnegative, we let $g_n(x,\varepsilon) := g(x,\varepsilon) + 1/n$, $n \ge 1$. Then, $\widetilde{g}_n \nearrow \widetilde{g}$, and by the lemma for positive functions, the $\widetilde{g}_n$'s are Borel functions. Therefore, $\widetilde{g}$ is a Borel function as well.

Hence, we assume that g is positive, and we will prove the lemma by constructing a sequence $(\widetilde{g}_n)$ of continuous functions such that $\widetilde{g}_n \to \widetilde{g}$ pointwise.

Step 1. Define

$$g_n(x,\varepsilon) := \begin{cases} 1, & \text{if } \varepsilon < 1/n \\ n, & \text{if } \varepsilon > n \\ g(x,\varepsilon), & \text{otherwise} \end{cases},$$

and let $\widetilde{g}_n$ be associated with g_n as in (6.23). Since $g(\cdot,\varepsilon)$ is continuous, we clearly have

$$\int_0^1 g_n(x,\varepsilon)\,\mathrm{d}\varepsilon < \infty \text{ and } \lim_{t\to\infty}\int_0^t g_n(x,\varepsilon)\,\mathrm{d}\varepsilon = \infty.$$

By (6.23) and the fact that $g > 0$, this implies that $0 < \widetilde{g}_n(x) < \infty$ is the only number such that

$$\int_0^{\widetilde{g}_n(x)} g_n(x,\varepsilon)\,\mathrm{d}\varepsilon = c. \tag{6.35}$$

Combining (6.35) with the continuity of g, one easily obtains that $\widetilde{g}_n$ is continuous.

Step 2. We prove that $\lim_{n\to\infty}\widetilde{g}_n(x) = \widetilde{g}(x)$, which implies that $\widetilde{g}$ is a Borel function. To prove this, we have to consider the three cases occurring in the definition (6.23).

Assume first that $\int_0^1 g(x,t)\,\mathrm{d}\varepsilon = \infty$ (and thus $\widetilde{g}(x) = 0$). For large n, we have $\widetilde{g}_n(x) > 1/n$ and

$$\int_{1/n}^{\widetilde{g}_n(x)} g_n(x,\varepsilon)\,\mathrm{d}\varepsilon = c - \frac{1}{n}. \tag{6.36}$$

On the other hand, for any given $t > 0$ and large n (depending on t), we have

$$\int_{1/n}^{t} g_n(x,\varepsilon)\,\mathrm{d}\varepsilon = \int_{1/n}^{t} g(x,\varepsilon)\,\mathrm{d}\varepsilon > c. \tag{6.37}$$

For such n, we have $\widetilde{g}_n(x) < t$. (This follows from (6.36) and (6.37).) Therefore, in this case we have $\widetilde{g}_n(x) \to 0 = \widetilde{g}(x)$.

The case where $\widetilde{g}(x) = \infty$ is similar, since for any fixed $0 < M < \infty$ and large n (depending on M), we have

$$\int_{1/n}^{M} g_n(x,\varepsilon)\,\mathrm{d}\varepsilon = \int_{1/n}^{M} g(x,\varepsilon)\,\mathrm{d}\varepsilon < c - \frac{1}{n},$$

and thus, for such n, we have $\widetilde{g}_n(x) > M$.

Finally, assume that $0 < \widetilde{g}(x) < \infty$, and thus $\int_0^{\widetilde{g}(x)} g(x,\varepsilon)\,\mathrm{d}\varepsilon = c$. If $t < \widetilde{g}(x)$, then, for large n, we have

$$\int_{1/n}^{t} g_n(x,\varepsilon)\,\mathrm{d}\varepsilon = \int_{1/n}^{t} g(x,\varepsilon)\,\mathrm{d}\varepsilon < \int_{0}^{t} g(x,\varepsilon)\,\mathrm{d}\varepsilon < c - \frac{1}{n},$$

and thus, for such n, we have $\widetilde{g}_n(x) > t$. Similarly, if $M > \widetilde{g}(x)$ then, for large n, we have $\widetilde{g}_n(x) < M$. □

By analogy with Corollary 5.4, we have the following Corollary.

Corollary 6.6. *Two cohomologous forms yield the same T.*

Proof. Let T_ω be the operator T associated with ω. Let $\omega_1 = \omega + \mathrm{d}\eta$, with $\eta \in C^\infty(\mathcal{N};\Lambda^{k-1})$, be an element of de Rham cohomology class $[\omega]$. If α is a extension of ω, we claim that $\alpha_1 := \alpha + \mathrm{d}(\psi\,\widetilde{\Pi}^*\eta)$ (with ψ as in (4.16)) is an extension of ω_1. Indeed, this amounts to proving that $\mathrm{d}(\psi\,\widetilde{\Pi}^*\eta)$ is an extension of $\mathrm{d}\eta$. In turn, this property is obtained as follows. We have

$$\mathrm{d}(\psi\,\widetilde{\Pi}^*\eta) = \mathrm{d}\psi \wedge \widetilde{\Pi}^*\eta + \psi\,\mathrm{d}\widetilde{\Pi}^*\eta. \tag{6.38}$$

By the proof of (4.17) and the facts that $\psi = 1$ and $\mathrm{d}\psi = 0$ near $\mathcal{N}$, we find that the right-hand side of (6.38) is indeed an extension of $\mathrm{d}\eta$.

By Theorem 6.1 and the fact that clearly $\mathrm{d}\alpha_1 = \mathrm{d}\alpha$, we have

$$\langle T_{\omega_1} f, \zeta\rangle = -\int_{\mathbb{R}^N\times(0,\infty)} \widetilde{F}^*(\mathrm{d}\alpha_1)\wedge \mathrm{d}\widetilde{\zeta} = -\int_{\mathbb{R}^N\times(0,\infty)} \widetilde{F}^*(\mathrm{d}\alpha)\wedge \mathrm{d}\widetilde{\zeta} = \langle T_\omega f, \zeta\rangle.$$ □

6.3 T "hears" singularities

In this section, we consider: (a) a smooth *closed* k-form ω on $\mathcal{N}$; (b) an integer $N > k$. In this setting, we provide, for "nice" f's, an explicit formula for $\langle Tf, \zeta\rangle$ in terms of the homotopy classes "carried" by the singular set of f. We start by defining adapted nice f's. Consider the class

$$\begin{aligned}\mathscr{R}_1 := \{f\colon \mathbb{R}^N \to \mathcal{N}\colon\ & f \text{ is constant in } \mathbb{R}^N\setminus\mathbb{B}^N,\ f\in C^\infty(\overline{\mathbb{B}^N}\setminus\mathcal{S}(f)),\\ & \mathcal{S}(f) \text{ is a } (N-k-1)\text{-closed, oriented submanifold}\\ & \text{of } \mathbb{B}^N,\ |\nabla f(x)| \le C(f)/\operatorname{dist}(x,\mathcal{S}(f)),\ \forall\, x\in\mathbb{R}^N\}.\end{aligned}$$

(It is important to note that both the manifold $\mathcal{S}(f)$ and the finite constant $C(f)$ depend on the nice map f.) Adapting the arguments in Detaille [28], one may prove that $\mathscr{R}_1$ is dense in $W^{s,p}_1(\mathbb{R}^N;\mathscr{N})$ when $0 < s \leq 1$ and $k \leq sp < k+1$. (See [28, Theorem 1.4] for a similar statement in the function space $W^{s,p}((-1,1)^N;\mathscr{N})$.) However, since density is not relevant for the main result of this section, we overlook this property and focus on the calculation of Tf for $f \in \mathscr{R}_1$.

Let $f \in \mathscr{R}_1$ and let $\mathcal{S}_1, \ldots, \mathcal{S}_\ell$ be the connected components of $\mathcal{S} = \mathcal{S}(f)$. Consider, for $z \in \mathcal{S}_i$, the affine normal space $N_z\mathcal{S}_i$ to $\mathcal{S}_i$ (passing through z), with the "natural" orientation induced by the one of $\mathcal{S}_i$, i.e., we ask that a direct basis of $T_z\mathcal{S}_i$, completed with a direct basis of $N_z\mathcal{S}_i$, forms a direct basis of $\mathbb{R}^N$. Let $S_\varepsilon(z)$ be the sphere of radius ε of $N_z\mathcal{S}_i$ centered at z, with the orientation induced by the one of $N_z\mathcal{S}_i$. It is straightforward (using the fact that, on a sphere S, $f \mapsto \int_S f^*\omega$ is a homotopical invariant; see Corollary 3.29) that the quantity $\int_{S_\varepsilon(z)} f^*\omega$ does not depend on small ε (smallness depending only on $\mathcal{S}$) or on z. With this in mind, we may set

$$c_i := \int_{S_\varepsilon(z)} f^*\omega, \forall\, z \in \mathcal{S}_i, \forall\, 0 < \varepsilon < \overline{\varepsilon} = \overline{\varepsilon}(\mathcal{S}).$$

Our result is the following.

Theorem 6.7. *Let $f \in \mathscr{R}_1$ and define c_i as above. Then*

$$\langle Tf, \zeta\rangle = (-1)^{k(N+1)+1} \sum_{i=1}^{\ell} c_i \int_{\mathcal{S}_i} \zeta, \forall\, \zeta \in \mathrm{Lip}(\mathbb{R}^N; \Lambda^{N-k-1}). \tag{6.39}$$

The above result was obtained for a slightly different, "less nice", dense class of maps, by Giaquinta, Modica, and Souček [36, Section 4.2, Theorem 1]; see also Jerrard and Soner [45, Theorem 1.2], Alberti, Baldo, and Orlandi [1, Theorem 3.8], and Bousquet [13, Proposition 1]. Their proofs require more advanced geometric measure theory arguments than the proof we present below, which merely relies on an iterated application of the Stokes formula.

To prove the theorem, we first consider a special case.

Lemma 6.8. *Let (x,y), with $x \in \mathbb{R}^{N-k-1}$ and $y \in \mathbb{R}^{k+1}$, denote a point in $\mathbb{R}^N$. Let $f = f(x,y)$, with $x \in \Omega$ and $y \in B_r(0) \setminus \{0\}$, be a smooth map such that $f(x,y) \in \mathscr{N}$ and*

$$|\nabla f(x,y)| \leq C(f)|y|^{-1}, \forall\, x \in \Omega, \forall\, y \in B_r(0) \setminus \{0\}. \tag{6.40}$$

Let $\zeta \in \mathrm{Lip}_c(\Omega \times B_r(0); \Lambda^{N-k-1})$. *Then*

$$\begin{aligned}\int_{\Omega\times B_r(0)} f^*\omega \wedge \mathrm{d}\zeta &= (-1)^{k(N+1)+1} \int_{S_\varepsilon} f(x_0,\cdot)^*\omega \int_\Omega \zeta_x(\cdot,0)\\ &= (-1)^{k(N+1)+1} \int_{S_\varepsilon} f(x_0,\cdot)^*\omega \int_{\Omega\times\{0\}} \zeta,\\ &\qquad\qquad \forall\, x_0 \in \Omega, \forall\, 0<\varepsilon<r.\end{aligned} \tag{6.41}$$

Here: (*i*) S_ε *is the sphere of radius* ε *of* $\mathbb{R}^{k+1}$ *centered at* 0; (*ii*) $\zeta_x(x,y)$ *is the coefficient of* $\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{N-k-1}$ *in* ζ.

Proof. We first note that it suffices to prove (6.41) when $\zeta \in C_c^\infty$. (The general case is then obtained by smoothing, using dominated convergence in the first and the third integral.)

Using: (i) the estimate (6.40); (ii) the fact that the degree of ω is $< k+1$; (iii) Stokes' formula; (iv) the fact that ζ has compact support in $\Omega \times B_r(0)$; we find that

$$\begin{aligned}\int_{\Omega\times B_r(0)} f^*\omega \wedge \mathrm{d}\zeta &= (-1)^k \lim_{\varepsilon\to 0} \int_{\Omega\times\{\varepsilon<y<r\}} \mathrm{d}(f^*\omega\wedge\zeta)\\ &= (-1)^{k+1} \lim_{\varepsilon\to 0} \int_{\Omega\times S_\varepsilon} f^*\omega \wedge \zeta.\end{aligned} \tag{6.42}$$

Next, we write, with $\zeta_{\alpha,\beta} = \zeta_{\alpha,\beta}(x,y)$,

$$\zeta = \sum_{\substack{\alpha\subset[\![1,N-k-1]\!],\,\beta\subset[\![1,k+1]\!]\\ \#\alpha+\#\beta=N-k-1}} \zeta_{\alpha,\beta}\, \mathrm{d}x^\alpha \wedge \mathrm{d}y^\beta \tag{6.43}$$

(with the convention that the indices in α and β are taken in the natural order).

Let us note that $\zeta_x = \zeta_{\alpha,\beta}$, with $\alpha := \{1,\ldots,N-k-1\}$ and $\beta := \emptyset$.

The decomposition (6.43) is suggesting that we have to show that the contribution in (6.42) of the coefficient $\zeta_{\alpha,\beta}$ converges to 0 as $\varepsilon \to 0$ when $\beta \neq \emptyset$. Without loss of generality, we may assume that $\alpha = \{1,2,\ldots,a\}$ (possibly with $a=0$) and $\beta = \{1,2,\ldots,b\}$, with $b \geq 1$. Set $\beta' := \{1,2,\ldots,b-1\}$ and let

$$\eta := y^b \cdot f^*\omega \wedge (\zeta_{\alpha,\beta}\, \mathrm{d}x^\alpha \wedge \mathrm{d}y^{\beta'}).$$

Then η is an $(N-2)$-form in $\mathbb{R}^N$, smooth in $\mathbb{R}^{N-k-1} \times (\mathbb{R}^{k+1} \setminus \{0\})$, and such that $\eta = 0$

near $\partial(\Omega \times S_\varepsilon)$. Using Stokes' formula again, we find that

$$\begin{aligned} 0 = \int_{\Omega\times S_\varepsilon} \mathrm{d}\eta = (-1)^k \int_{\Omega\times S_\varepsilon} y^b \cdot f^*\omega \wedge \mathrm{d}\zeta_{\alpha,\beta} \wedge \mathrm{d}x^\alpha \wedge \mathrm{d}y^{\beta'} \\ + (-1)^{N-2} \int_{\Omega\times S_\varepsilon} f^*\omega \wedge (\zeta_{\alpha,\beta}\,\mathrm{d}x^\alpha \wedge \mathrm{d}y^\beta). \end{aligned} \tag{6.44}$$

Using the fact that $|\nabla f(x,y)| \le C|y|^{-1}$ and $\zeta \in C_c^\infty(\Omega \times \{|y| < r\}; \Lambda^{N-k-1})$, we have

$$\left| \int_{\Omega\times S_\varepsilon} y^b \cdot f^*\omega \wedge \mathrm{d}\zeta_{\alpha,\beta} \wedge \mathrm{d}x^\alpha \wedge \mathrm{d}y^{\beta'} \right| \le C \int_{\Omega\times S_\varepsilon} \varepsilon \cdot \frac{1}{\varepsilon^k} = O(\varepsilon). \tag{6.45}$$

From (6.44) and (6.45), we find that, with $\alpha := \{1, \ldots, N-k-1\}$, we have

$$\int_{\Omega\times S_\varepsilon} f^*\omega \wedge \zeta = \int_{\Omega\times S_\varepsilon} f^*\omega \wedge (\zeta_x\,\mathrm{d}x^\alpha) + O(\varepsilon). \tag{6.46}$$

Since $|\zeta_x(x,y) - \zeta_x(x,0)| \le \varepsilon \|\nabla\zeta_x\|_\infty$ for any $x \in \Omega$ and $y \in S_\varepsilon$, (6.46) implies that

$$\begin{aligned} \int_{\Omega\times S_\varepsilon} f^*\omega \wedge \zeta &= \int_{\Omega\times S_\varepsilon} f^*\omega \wedge (\zeta_x(x,0)\,\mathrm{d}x^\alpha) + O(\varepsilon) \\ &= \int_{\Omega\times S_\varepsilon} (f(x,\cdot)^*\omega) \wedge (\zeta_x(x,0)\,\mathrm{d}x^\alpha) + O(\varepsilon) \\ &= (-1)^{k(N-k-1)} \int_\Omega \zeta_x(x,0) \left(\int_{S_\varepsilon} f(x,\cdot)^*\omega \right) \mathrm{d}x + O(\varepsilon), \end{aligned} \tag{6.47}$$

where the last equality follows from the Fubini theorem.

Combining (6.42) and (6.47), we obtain (6.41), since, by standard (smooth) homotopy arguments, the integral $\int_{S_\varepsilon} f(x,\cdot)^*\omega$ does not depend on $x \in \Omega$ and $\varepsilon < r$. □

Proof of Theorem 6.7. As in the proof of Lemma 6.8, we may assume that ζ is smooth. Without loss of generality, we may also assume that $\operatorname{supp}\zeta \subset \mathbb{B}^N$.

Consider a finite cover of $\overline{\mathbb{B}}^N$ with open sets U_j such that, for each j, either $U_j \cap \mathcal{S} = \varnothing$ or there exists an orientation preserving diffeomorphism $\Phi_j \colon \{|x| < 1\} \times \{|y| < 1\} \to U_j$ such that $\Phi_j^{-1}(\mathcal{S} \cap U_j) = \{(x,0) \colon |x| < 1\}$. (Here, as in Lemma 6.8, we have $x \in \mathbb{R}^{N-k-1}$ and $y \in \mathbb{R}^{k+1}$.) Using a partition of unity subordinated to the cover (U_j) and the linearity of (6.39) with respect to ζ, we may assume that ζ is compactly supported in some U_j.

If $\mathcal{S} \cap U_j = \varnothing$, then

$$\int_{\mathbb{R}^N} f^*\omega \wedge \mathrm{d}\zeta = (-1)^k \int_{U_j} \mathrm{d}(f^*\omega \wedge \zeta) = 0.$$

If $\mathcal{S} \cap U_j \neq \emptyset$, let i be such that $\mathcal{S} \cap U_j \subset \mathcal{S}_i$. Using: (i) the estimate $|\nabla f(x)| \leq C(f)/\operatorname{dist}(x, \mathcal{S}(f))$; (ii) the fact that the degree of ω is $< k+1$; (iii) standard properties of the exterior differential calculus, we find that

$$\int_{\mathbb{R}^N} f^*\omega \wedge \mathrm{d}\zeta = \int_{U_j} f^*\omega \wedge \mathrm{d}\zeta$$
$$= \int_{\{|x|<1\}\times\{|y|<1\}} [(f \circ \Phi_j)^*\omega] \wedge \mathrm{d}((\Phi_j)^*\zeta).$$

We deduce from Lemma 6.8 that, for $|x_0| < 1$ and $\varepsilon < 1$,

$$\int_{\{|x|<1\}\times\{|y|<1\}} [(f \circ \Phi_j)^*\omega] \wedge \mathrm{d}((\Phi_j)^*\zeta) = (-1)^{k(N+1)+1} \int_{S_\varepsilon} (f \circ \Phi_j(x_0, \cdot))^*\omega \int_{\{|x|<1\}\times\{0\}} (\Phi_j)^*\zeta. \tag{6.48}$$

By change of variables, the latter integral in (6.48) equals

$$\int_{\mathcal{S}\cap U_j} \zeta = \int_{\mathcal{S}_i} \zeta. \tag{6.49}$$

Therefore, (6.39) follows from (6.48) (with $x_0 = 0$) and (6.49) provided

$$\int_{S_\varepsilon} (f \circ \Phi_j(0, \cdot))^*\omega = c_i. \tag{6.50}$$

For this purpose, consider, for $z \in \mathcal{S}$, an orientation preserving isometry $T = T_z$ of $\mathbb{R}^{k+1}$ onto $N_z\mathcal{S}$ such that $T(0) = z$. We now obtain (6.50) from

$$\int_{S_\varepsilon} (f \circ \Phi_j(0, \cdot))^*\omega = \int_{\Phi_j(\{0\}\times S_\varepsilon)} f^*\omega$$

and standard (smooth) homotopy arguments, using the fact that, for small ε, the embeddings

$$S_\varepsilon \ni y \mapsto \Phi_j(0, y), \text{ respectively } S_\varepsilon \ni y \mapsto T_{\Phi_j(0,0)}(y)$$

of $\Phi_j(\{0\} \times S_\varepsilon)$, respectively $S_\varepsilon(\Phi_j(0,0))$ (viewed as a positively oriented sphere on $N_{\Phi_j(0,0)}\mathcal{S}$) are isotopical in $\mathbb{R}^N \setminus \mathcal{S}$. □

6.4 Slicing

In this section, we consider: (a) a smooth *closed* k-form ω on $\mathscr{N}$; (b) $0 < s < 1$ and $1 < p < \infty$ such that $sp = k$; (c) an integer $N \geq k+1$ (most often, $N > k+1$).

We start with a formal calculation that will provide insight for the main results in this section. Let $N > k+1$ and write $N = \ell + \nu$, with $\ell \geq k+1$ and $\nu \geq 1$. Let $f \in W^{1,k}_1(\mathbb{R}^\ell; \mathscr{N})$ and consider a Lipschitz form of the type $\zeta = \eta \, \mathrm{d}x^\alpha = \eta(x) \, \mathrm{d}x^\alpha$, with $x \in \mathbb{R}^\ell$ and $\alpha \subset [\![1, \ell]\!]$, $\#\alpha = \ell - k - 1$. Then

$$\langle Tf, \zeta \rangle = \int_{\mathbb{R}^\ell} f^*\omega \wedge \mathrm{d}\zeta = \int_{\mathbb{R}^\ell} f^*\omega \wedge \mathrm{d}\eta \wedge \mathrm{d}x^\alpha.$$

Consider next $f = f(x,y) \in W^{1,k}_1(\mathbb{R}^N; \mathscr{N})$, with $x \in \mathbb{R}^\ell$ and $y \in \mathbb{R}^\nu$, and a Lipschitz form of the type

$$\zeta = \eta \, \mathrm{d}x^\alpha \wedge \mathrm{d}y = \eta(x,y) \, \mathrm{d}x^\alpha \wedge \mathrm{d}y, \tag{6.51}$$

with $\#\alpha = \ell - k - 1$ and $\mathrm{d}y := \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^\nu$. Using the identity

$$f^*\omega \wedge \mathrm{d}[\eta \, \mathrm{d}x^\alpha \wedge dy] = f(\cdot, y)^*\omega \wedge \mathrm{d}(\eta(\cdot, y) \, \mathrm{d}x^\alpha) \wedge \mathrm{d}y,$$

and the Fubini theorem, we find that, for ζ as in (6.51), we have

$$\langle Tf, \zeta \rangle = \int_{\mathbb{R}^\nu} \langle Tf(\cdot, y), \eta(\cdot, y) \, \mathrm{d}x^\alpha \rangle \, \mathrm{d}y. \tag{6.52}$$

Our first purpose in this section is to extend the validity of (6.52) to $f \in W^{s,p}_1(\mathbb{R}^N; \mathscr{N})$, allowing also permutations of the coordinates x^i and y^j.

Consider a partition $[\![1, N]\!] = A \sqcup B$, with $A = \{i_1 < i_2 < \cdots < i_\ell\}$, $B = \{j_1 < j_2 < \cdots < j_\nu\}$. (The above calculations correspond to the choice $A = [\![1, \ell]\!]$, $B = [\![\ell + 1, N]\!]$.) Given a point $z = (z^1, \ldots, z^N) \in \mathbb{R}^N$, let $x = (z^{i_1}, \ldots, z^{i_\ell}) \sim (x^1, \ldots, x^\ell) \in \mathbb{R}^\ell$, $y = (z^{j_1}, \ldots, z^{j_\nu}) \sim (y^1, \ldots, y^\nu) \in \mathbb{R}^\nu$, and identify z with (x,y). We associate with each partition (A, B) a *signature* $\sigma = \sigma(B) \in \{-1, 1\}$ through the formula

$$\mathrm{d}x^1 \wedge \cdots \mathrm{d}x^\ell \wedge \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^\nu = \sigma(B) \, \mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^N. \tag{6.53}$$

Let $\alpha \subset [\![1, \ell]\!]$ be such that $\#\alpha = \ell - k - 1$ and consider an "elementary" Lipschitz form of the type

$$\zeta = \eta \, \mathrm{d}x^\alpha \wedge \mathrm{d}y = \eta(x,y) \, \mathrm{d}x^\alpha \wedge \mathrm{d}y. \tag{6.54}$$

It is important to note that every Lipschitz form is the sum of at most $\binom{N}{N-k-1}$ Lipschitz forms as in (6.54), and thus Proposition 6.9 below provides a "slicing" or "disintegration" formula for $\langle Tf, \zeta\rangle$ for *any* ζ.

We next note that, if $f \in W_1^{s,p}(\mathbb{R}^N; \mathscr{N})$, then, for a.e. $y = (y^1, \dots, y^\nu) \in \mathbb{R}^\nu$, the partial function $f(\cdot, y)$ belongs to $W_1^{s,p}(\mathbb{R}^\ell; \mathscr{N})$, and thus the distribution $Tf(\cdot, y)$ makes sense and acts on forms $\xi \in \mathrm{Lip}(\mathbb{R}^\ell; \Lambda^{\ell-k-1})$.

Proposition 6.9. *Let $N > k+1$. Let $f \in W_1^{s,p}(\mathbb{R}^N; \mathscr{N})$ and let ζ be as in (6.54). Then the map*

$$\mathbb{R}^\nu \ni y \mapsto G_f(y) := \langle Tf(\cdot, y), \eta(\cdot, y)\, \mathrm{d}x^\alpha\rangle$$

is defined a.e. and is (Lebesgue) integrable.

Moreover, we have, with $\sigma = \sigma(B)$ as in (6.53),

$$\begin{aligned} \langle Tf, \eta\, \mathrm{d}x^\alpha \wedge \mathrm{d}y\rangle &= \sigma \int_{\mathbb{R}^\nu} \langle Tf(\cdot, y), \eta(\cdot, y)\, \mathrm{d}x^\alpha\rangle\, \mathrm{d}y \\ &:= \sigma \int_{\mathbb{R}^\nu} \langle Tf(\cdot, y), \eta(\cdot, y)\, \mathrm{d}x^\alpha\rangle\, \mathrm{d}\mathscr{H}^\nu(y). \end{aligned} \tag{6.55}$$

In the special case where $\mathscr{N} = \mathbb{S}^1$ and ω is the standard volume form, formula (6.55) was proved by Mironescu, Russ, and Sire [51, Section 3.4, (3.64)].

Proof. We present the proof when $k > 1$. The case $k = 1$ is similar; we start from $f \in W_1^{1,q}$, with $1 < q < 2$, instead of $f \in W_1^{1,1}$. We divide the proof into two steps.

Step 1. Formula (6.55) holds for $f \in W_1^{1,k}(\mathbb{R}^N; \mathscr{N})$. Indeed, arguing as in the proof of (6.52) and using: (i) Theorem 6.1; (ii) the identity

$$f^*\omega \wedge \mathrm{d}\eta \wedge \mathrm{d}x^\alpha \wedge \mathrm{d}y = f(\cdot, y)^*\omega \wedge \mathrm{d}[\eta(\cdot, y)\, \mathrm{d}x^\alpha] \wedge \mathrm{d}y;$$

(iii) the definition of σ in (6.53); (iv) the Fubini theorem, we find that

$$\begin{aligned} \langle Tf, \eta\, \mathrm{d}x^\alpha \wedge \mathrm{d}y\rangle &= \int_{\mathbb{R}^N} f^*\omega \wedge \mathrm{d}\eta \wedge \mathrm{d}x^\alpha \wedge \mathrm{d}y \\ &= \int_{\mathbb{R}^N} f(\cdot, y)^*\omega \wedge \mathrm{d}[\eta(\cdot, y)\, \mathrm{d}x^\alpha] \wedge \mathrm{d}y \\ &= \sigma \int_{\mathbb{R}^\nu} \left(\int_{\mathbb{R}^\ell} f(\cdot, y)^*\omega \wedge \mathrm{d}[\eta(\cdot, y)\, \mathrm{d}x^\alpha] \right) \mathrm{d}\mathscr{H}^\nu(y) \\ &= \sigma \int_{\mathbb{R}^\nu} \langle Tf(\cdot, y), \eta(\cdot, y)\, \mathrm{d}x^\alpha\rangle\, \mathrm{d}\mathscr{H}^\nu(y). \end{aligned}$$

Incidentally, the Fubini theorem implies that G_f is Lebesgue integrable.

Step 2. If $(f_j) \subset W_1^{1,k}(\mathbb{R}^N;\mathscr{N})$, $f \in W_1^{s,p}(\mathbb{R}^N;\mathscr{N})$, and $f_j \to f$ in $W_1^{s,p}$, then $G_{f_j} \to G_f$ in $\mathscr{L}^1(\mathbb{R}^\nu)$. Indeed, it suffices to obtain the conclusion up to a subsequence (then use Theorem 6.1 on the left-hand side of (6.55)). The argument is similar to the one used in Step 4 in the proof of Theorem 6.1. There exists a null set $A \subset \mathbb{R}^\nu$ and a function $F \in \mathscr{L}^p(\mathbb{R}^\nu)$ such that, possibly up to a subsequence, we have

$$f_j(\cdot, y) \to f(\cdot, y) \text{ in } W_1^{s,p}(\mathbb{R}^\ell), \text{ for each } y \in \mathbb{R}^\nu \setminus A, \tag{6.56}$$

$$|f_j(\cdot, y)|_{W^{s,p}} \le F(y), \forall\, y, \forall\, j. \tag{6.57}$$

Combining (6.56) and Theorem 6.1, we have $G_{f_j}(y) \to G_f(y)$, $\forall\, y \in \mathbb{R}^\nu \setminus A$. On the other hand, (6.57) and (6.4) imply that $|G_{f_j}(y)| \le C[F(y)]^p$, $\forall\, y$, $\forall\, j$, whence the conclusion of Step 2. □

Consider next a general $\zeta \in \mathrm{Lip}(\mathbb{R}^N;\Lambda^{N-k-1})$. Then we may write $\zeta = \sum_\gamma \zeta_\gamma\, \mathrm{d}z^\gamma = \sum_\gamma \zeta_\gamma(z)\, \mathrm{d}z^\gamma$. Here the sum is taken over $\gamma \subset [\![1, N]\!]$ such that $\#\gamma = N - k - 1$. We may rewrite

$$\zeta = \sum_\alpha \sum_\beta \eta_{\alpha,\beta}\, \mathrm{d}z^\alpha \wedge \mathrm{d}z^\beta = \sum_\alpha \sum_\beta \eta_{\alpha,\beta}(z)\, \mathrm{d}z^\alpha \wedge \mathrm{d}z^\beta, \tag{6.58}$$

where: (i) the sums are over $\alpha \subset [\![1, N]\!]$ such that $\#\alpha = l - k - 1$, respectively $\beta \subset [\![1, N]\!]$ such that $\#\beta = \nu$; (ii)

$$\eta_{\alpha,\beta} := \begin{cases} (C_\nu)^{-1}\sigma(\alpha,\beta)\zeta_{\alpha\sqcup\beta}, & \text{if } \alpha \cap \beta = \varnothing \\ 0, & \text{if } \alpha \cap \beta \ne \varnothing \end{cases},$$

where $C_\nu := \binom{N-k-1}{\nu}$ and $\sigma(\alpha,\beta) \in \{-1, 1\}$ is the sign such that $\mathrm{d}z^\alpha \wedge \mathrm{d}z^\beta = \sigma(\alpha,\beta)\, \mathrm{d}z^{\alpha\sqcup\beta}$.

The following Corollary is a direct consequence of Proposition 6.9 and the identity (6.58).

Corollary 6.10. *With the notation above, we have*

$$\begin{aligned} \langle Tf, \zeta\rangle &= \sum_\beta \sigma(\beta) \int_{\mathbb{R}^\nu} \left\langle Tf(\cdot, z^\beta), \sum_\alpha \eta_{\alpha,\beta}(\cdot, z^\beta)\, \mathrm{d}z^\alpha \right\rangle \mathrm{d}z^\beta \\ &= \frac{1}{\binom{N-k-1}{\nu}} \sum_\alpha \sum_\beta \sigma(\alpha,\beta)\sigma(\beta) \int_{\mathbb{R}^\nu} \langle Tf(\cdot, z^\beta), \zeta_{\alpha\sqcup\beta}(\cdot, z^\beta)\, \mathrm{d}z^\alpha\rangle\, \mathrm{d}z^\beta. \end{aligned}$$

With the help of the slicing property, one can prove the following dimensional reduction property. *In the next result, we consider the setting of Proposition 6.9.*

Proposition 6.11. *Let $\rho = \rho(y)$ be a standard mollifier and set $\rho_{\varepsilon,y_0}(y) := \rho_\varepsilon(y - y_0)$, $\forall\, y$, $y_0 \in \mathbb{R}^\nu$. Then, for a.e. $y_0 \in \mathbb{R}^\nu$, we have, with $\sigma = \sigma(B)$ as in (6.53),*

$$\langle Tf(\cdot, y_0), \xi\rangle = \sigma \lim_{\varepsilon\to 0}\langle Tf, \xi \wedge (\rho_{\varepsilon,y_0}\, \mathrm{d}y)\rangle,\ \forall\, \xi = \xi(x) \in \mathrm{Lip}(\mathbb{R}^\ell; \Lambda^{\ell-k-1}). \tag{6.59}$$

Proof. Using Proposition 6.9 and the fact that, once B is fixed, the signature $\sigma(B)$ does not depend on the choice of $\alpha \subset A$, we find that

$$\langle Tf, \xi \wedge (\rho_{\varepsilon,y_0}\, \mathrm{d}y)\rangle = \sigma \int_{\mathbb{R}^\nu} \langle Tf(\cdot, y), \xi\rangle \rho_{\varepsilon,y_0}(y)\, \mathrm{d}y. \tag{6.60}$$

Let $\widetilde{\rho} = \widetilde{\rho}(x)$ be a standard mollifier in $\mathbb{R}^\ell$. Using the notation in Section 6.2, set (with α as in (4.16)):

$$f_y(x) := f(x, y),\ F_y(x, \widetilde{\varepsilon}) := f_y * \widetilde{\rho}_{\widetilde{\varepsilon}},\ \widetilde{F}_y := \widetilde{\Pi} \circ F_y,\ H_y := \widetilde{F}_y(\mathrm{d}\alpha),$$
$$x \in \mathbb{R}^\ell,\ y \in \mathbb{R}^\nu,\ \widetilde{\varepsilon} > 0.$$

Then there exists a null set $A \subset \mathbb{R}^\nu$ such that $f(\cdot, y) \in W^{s,p}_1(\mathbb{R}^\ell)$, $\forall\, y \in \mathbb{R}^\nu \setminus A$. Formula (6.24) in Step 3 in the proof of Theorem 6.1 implies that, for every $y \in \mathbb{R}^\nu \setminus A$, we have

$$\int_{\mathbb{R}^\ell\times(0,\infty)} |H_y|\, \mathrm{d}x\, \mathrm{d}\widetilde{\varepsilon} \le C_1 |f(\cdot, y)|^p_{W^{s,p}}. \tag{6.61}$$

Combining (6.61) and the Besov type inequality

$$\int_{\mathbb{R}^\nu} |f(\cdot, y)|^p_{W^{s,p}}\, \mathrm{d}y \le C_2 |f|^p_{W^{s,p}} \tag{6.62}$$

(see, e.g., Brezis and Mironescu [22, Corollary 15.1] or Leoni [47, Theorem 6.35]), we obtain

$$\mathbb{R}^\nu \ni y \mapsto H_y \in \mathscr{L}^1(\mathbb{R}^\nu; \mathscr{L}^1(\mathbb{R}^\ell \times (0, \infty))).$$

We next note that, by (6.60) and (6.5), we have

$$
\begin{aligned}
|\langle Tf, \xi \wedge (\rho_{\varepsilon,y_0}\,\mathrm{d}y)\rangle - \sigma\langle Tf(\cdot, y_0), \xi\rangle| \\
&\leq C_3 \fint_{B_\varepsilon(y_0)} |\langle Tf(\cdot, y), \xi\rangle - \langle Tf(\cdot, y_0), \xi\rangle|\,\mathrm{d}y \\
&\leq C_4 |\xi|_{\mathrm{Lip}} \fint_{B_\varepsilon(y_0)} \int_{\mathbb{R}^\ell \times (0,\infty)} |H_y - H_{y_0}|\,\mathrm{d}x\,\mathrm{d}\varepsilon'\,\mathrm{d}y \\
&= C_4 |\xi|_{\mathrm{Lip}} \fint_{B_\varepsilon(y_0)} \|H_y - H_{y_0}\|_1\,\mathrm{d}y.
\end{aligned}
\tag{6.63}
$$

We finally invoke the vector-valued Lebesgue differentiation theorem (see, e.g., Heinonen, Koskela, Shanmugalingam, and Tyson [44, Section 3.4]), which implies that, for a.e. $y_0 \in \mathbb{R}^\nu$, we have

$$
\lim_{\varepsilon \to 0} \fint_{B_\varepsilon(y_0)} \|H_y - H_{y_0}\|_1\,\mathrm{d}y = 0. \tag{6.64}
$$

Combining (6.64) with (6.63), one obtains (6.59). □

Remark 6.12. In the above proof, the exceptional null set A depends on the choice of the closed k-form ω. We claim that, actually, we may pick *the same* null set for *every* ω. Indeed, by Corollary 6.6, the exceptional set A depends only on the de Rham cohomology class $[\omega]$. On the other hand, it is clear that the mapping $\omega \mapsto T_\omega$ (with ω closed k-form) is linear. The claim follows by combining these considerations with the fact that the k-th de Rham cohomology group of $\mathcal{N}$ is of finite dimension (since $\mathcal{N}$ is compact). □

We next present a version of slicing in the case where $N = k+1$. As discussed in Section 5.3, this requires considering "test forms" ζ whose restriction to k-dimensional slices are constant. Proposition 6.14 below is such a possible result (others could be considered) and is fitted to our main result, Theorem 6.23.

To start with, we note the following

Lemma 6.13. *Let $f \in W^{s,p}(\mathbb{R}^{k+1})$ and set $Q_r := (-r, r)^{k+1}$. Then, for every $P \in \mathbb{R}^{k+1}$, we have*

$$
f_{|P+\partial Q_r} \in W^{s,p}(P + \partial Q_r) \text{ for a.e. } r > 0. \tag{6.65}
$$

Moreover, for a.e. $r > 0$, we have $f_{|P+\partial Q_r} \in W^{s,p}(P + \partial Q_r)$ for a.e. $P \in \mathbb{R}^{k+1}$.

Proof. Let $0 < a < b < \infty$. Using the fact that the set $\{x \in \mathbb{R}^{k+1} \colon a \leq |x - P| \leq b\}$ is bi-Lipschitz homeomorphic to $\partial Q_1 \times [0,1]$ and a standard cousin of (6.62), we find that

$$\int_a^b |f_{|P+\partial Q_r}|^p_{W^{s,p}(P+\partial Q_r)}\,\mathrm{d}r \le C(a,b)|f|^p_{W^{s,p}}, \tag{6.66}$$

whence (6.65).

The second part follows from the Tonelli theorem. □

Proposition 6.14. *Let $f \in W^{s,p}_1(\mathbb{R}^{k+1};\mathscr{N})$ and $\psi\colon (0,\infty) \to \mathbb{R}$ be Lipschitz, with* $\operatorname{supp}\psi' \subset (0,\infty)$*. Then, for every $P \in \mathbb{R}^{k+1}$, we have*

$$\langle Tf, \psi(|\cdot - P|_\infty)\rangle = (-1)^k \int_0^\infty \psi'(r)\mathscr{I}(f_{|P+\partial Q_r})\,\mathrm{d}r. \tag{6.67}$$

Here: (a) $\mathscr{I}(f_{|P+\partial Q_r}) = \mathscr{I}_{P+\partial Q_r,\omega}(f_{|P+\partial Q_r})$ is defined in Corollary 3.29 (with $\mathscr{M} = P+\partial Q_r$); (b) the orientation on $P+\partial Q_r$ is as in Example 3.16.

Proof. With no loss of generality, we may assume that $P=0$. Assume that $\operatorname{supp}\psi' \subset (a,b)$ for some $a, b > 0$.

Step 1. Proof of (6.67) when $f \in W^{1,k}_1(\mathbb{R}^{k+1};\mathscr{N})$. In this case, $f^*\omega$ can be written as $\sum_\ell \beta_\ell\,\widehat{\mathrm{d}x^\ell}$, with $\beta_\ell = \beta_\ell(x) \in \mathscr{L}^1_c(\mathbb{R}^{k+1})$ and $\widehat{\mathrm{d}x^\ell}$ as in (3.35). Let

$$\Omega_{\ell,\pm} = \{x = (x^1,\ldots,x^{k+1}) \in \mathbb{R}^{k+1}\colon \max_{j\neq\ell}|x^j| < \pm x^\ell\},\ 1 \le \ell \le k+1.$$

Then,

$$\begin{aligned}\langle Tf, \psi(|\cdot|_\infty)\rangle &= \sum_\ell \int_{\mathbb{R}^{k+1}} \beta_\ell\,\widehat{\mathrm{d}x^\ell} \wedge \mathrm{d}[\psi(|\cdot|_\infty)]\\ &= \sum_\ell \Big(\int_{\Omega_{\ell,+}} \beta_\ell\psi'(x^\ell)\,\widehat{\mathrm{d}x^\ell}\wedge \mathrm{d}x^\ell - \int_{\Omega_{\ell,-}} \beta_\ell\psi'(-x^\ell)\,\widehat{\mathrm{d}x^\ell}\wedge\mathrm{d}x^\ell\Big).\end{aligned}$$

By the Fubini theorem,

$$\begin{aligned}\int_{\Omega_{\ell,+}} \beta_i\psi'(x^\ell)\,\widehat{\mathrm{d}x^\ell}\wedge\mathrm{d}x^\ell &= (-1)^{k-\ell+1}\int_{\Omega_{\ell,+}}\beta_\ell\psi'(x^\ell)\\ &= (-1)^{k-\ell+1}\int_a^b \psi'(x^\ell)\Big(\int_{F_{\ell,x^\ell,+}}\beta_\ell(\cdot,x^\ell)\Big)\,\mathrm{d}x^\ell,\end{aligned}$$

where

$$F_{\ell,x^\ell,\pm} := \{\widehat{x^\ell} = (x^1,\ldots,x^{\ell-1},x^{\ell+1},\ldots,x^{k+1}) \in \mathbb{R}^k\colon (x^1,\ldots,x^\ell,\ldots,x^{k+1}) \in \Omega_{\ell,\pm}\}.$$

A similar identity holds on $\Omega_{\ell,-}$. Taking the sum over ℓ and using: (i) Definition 3.24; (ii) equation (3.36) in Example 3.18 (with $C = P + Q_r$ and $\alpha_{\ell,\pm}(x) = \beta_\ell(x^1, \ldots, x^{\ell-1}, \pm x^\ell, x^{\ell+1}, \ldots, x^{k+1})$); (iii) Proposition 3.34, we find that

$$\begin{aligned}\langle Tf, \psi(|\cdot|_\infty)\rangle &= \sum_\ell (-1)^{k-\ell+1} \int_a^b \psi'(x^\ell)\left(\int_{F_{\ell,x^\ell,+}} \beta_\ell(\cdot, x^\ell)\right) \mathrm{d}x^\ell \\ &\quad - \sum_\ell (-1)^{k-\ell+1} \int_{-b}^{-a} \psi'(-x^\ell)\left(\int_{F_{\ell,x^\ell,-}} \beta_\ell(\cdot, -x^\ell)\right) \mathrm{d}x^\ell \\ &= (-1)^k \int_a^b \psi'(r)\left(\int_{\partial Q_r} (f_{|\partial Q_r})^*\omega\right) \mathrm{d}r \\ &= (-1)^k \int_a^b \psi'(r)\mathscr{I}(f_{|\partial Q_r})\, \mathrm{d}r.\end{aligned}$$

Step 2. Proof of (6.67) for a general $f \in W^{s,p}_1(\mathbb{R}^{k+1}; \mathscr{N})$. Consider some $k < q < k+1$ and a sequence $(f_j) \subset W^{1,q}_1(\mathbb{R}^{k+1}; \mathscr{N})$ such that

$$f_j \to f \text{ in } W^{s,p}_1(\mathbb{R}^{k+1}; \mathscr{N}).$$

By Lemma 6.13 and a standard argument, possibly up to a subsequence, we have, for a.e. $r > 0$,

$$f_{j|\partial Q_r} \to f_{|\partial Q_r} \text{ in } W^{s,p}(\partial Q_r; \mathscr{N}). \tag{6.68}$$

Set

$$F_j(r) := \int_{\partial Q_r} (f_{j|\partial Q_r})^*\omega \text{ and } F(r) := \mathscr{I}(f_{|\partial Q_r})$$

(which are well-defined for a.e. $r > 0$).

Using: (i) (6.68); (ii) the embedding $W^{s,p} \hookrightarrow \mathrm{VMO}$; (iii) Proposition 3.34; (iv) Corollary 3.29, we find that

$$F_j(r) = \mathscr{I}(f_{j|\partial Q_r}) \to F(r) \text{ for a.e. } r > 0. \tag{6.69}$$

In view of Theorem 6.1 and Step 1, in order to obtain (6.67), it suffices to prove that $F_j \to F$ in $\mathscr{L}^1((a,b))$. For this purpose, consider $\Phi_r \colon \partial Q_1 \to \partial Q_r$, $\Phi_r(x) := rx$. By Corollary 3.32, we have

$$F_j(r) = \mathscr{I}(f_{j|\partial Q_r} \circ \Phi_r).$$

Combining this with Theorem 4.1 with $\mathscr{M} = \partial Q_1$, we obtain

$$|F_j(r)| \le C\big|f_{j|\partial Q_r} \circ \Phi_r\big|^p_{W^{s,p}(\partial Q_1)} = C\big|f_{j|\partial Q_r}\big|^p_{W^{s,p}(\partial Q_r)}. \tag{6.70}$$

Combining (6.66), (6.69), (6.70), and the converse to the dominated convergence theorem, we obtain the desired conclusion $F_j \to F$ in $\mathscr{L}^1((a, b))$. □

Using a special choice of ψ, we obtain the following variant of Proposition 6.14 adapted to boundaries of cubes.

Proposition 6.15. *Let $f \in W^{s,p}_1(\mathbb{R}^{k+1}; \mathscr{N})$. For $\varepsilon > 0$ and $0 < \eta \le \varepsilon/2$, let $\psi_\eta = \psi_{\varepsilon,\eta}$ be defined by*

$$\psi_\eta(r) = \psi_{\eta,\varepsilon}(r) := \begin{cases} 1, & \text{if } r \le \varepsilon - \eta \\ (\varepsilon - r)/\eta, & \text{if } \varepsilon - \eta \le r \le \varepsilon \\ 0, & \text{if } r \ge \varepsilon \end{cases}.$$

Let $\eta_\ell \to 0$. Then, for a.e. $\varepsilon > 0$, we have

$$\mathscr{I}(f_{|P+\partial Q_\varepsilon}) = \lim_{\ell\to\infty} (-1)^{k+1}\langle Tf, \psi_{\eta_\ell}(|\cdot - P|_\infty)\rangle, \text{ for a.e. } P \in \mathbb{R}^{k+1}. \tag{6.71}$$

Proof. Let $G(P, \varepsilon) := \mathscr{I}(f_{|P+\partial Q_\varepsilon})$ and

$$G_\ell(P, \varepsilon) := \frac{1}{\eta_\ell}\int_{\varepsilon-\eta_\ell}^{\varepsilon} \mathscr{I}(f_{|P+\partial Q_r})\,\mathrm{d}r = (-1)^{k+1}\langle Tf, \psi_{\eta_\ell}(|\cdot - P|_\infty)\rangle,$$

where the equality follow from Proposition 6.14.

Therefore, (6.71) amounts to

$$\lim_{\ell\to\infty} G_\ell(P, \varepsilon) = G(P, \varepsilon), \tag{6.72}$$

for a.e. $\varepsilon > 0$ and, once ε is fixed, for a.e. $P \in \mathbb{R}^{k+1}$.

Fix $P \in \mathbb{R}^{k+1}$. By the proof of Proposition 6.14, $\varepsilon \mapsto \mathscr{I}(f_{|P+\partial Q_\varepsilon})$ is in $\mathscr{L}^1_{\mathrm{loc}}((0, \infty))$. The Lebesgue differentiation theorem then implies that, for a.e. $\varepsilon > 0$, we have

$$\lim_{\ell\to\infty} G_\ell(P, \varepsilon) = G(P, \varepsilon). \tag{6.73}$$

Next, we set $\widetilde{G}(P, \varepsilon) := \liminf_{\ell\to\infty} G_\ell(P, \varepsilon)$ and let

$$A := \{(P, \varepsilon) \in \mathbb{R}^{k+1} \times (0, \infty) : G(P, \varepsilon) \ne \widetilde{G}(P, \varepsilon)\}.$$

We note that, for a.e. $\varepsilon > 0$, we have $f \in \mathrm{VMO}(P + \partial Q_\varepsilon; \mathscr{N})$, and in this case (by Proposition 5.1)

$$G(P, \varepsilon) = \mathscr{I}(f_{|P+\partial Q_\varepsilon}) = -\int_{(P+\partial Q_\varepsilon)\times(0,\infty)} (\widetilde{\Pi} \circ F_{P,\varepsilon})^*(\mathrm{d}\alpha),$$

where, as in (4.4), we let

$$F_{P,\varepsilon}(x, t) := \int_{P+\partial Q_\varepsilon} \widetilde{\rho}(x, t, y) f(y)\, \mathrm{d}\mathscr{H}^k(y).$$

Using (2.8) (with $\mathscr{M} := P + \partial Q_\varepsilon$), we find that $F_{P,\varepsilon}(x, t)$ is measurable with respect to (P, ε, x, t), and that G is measurable with respect to (x, ε). Similarly, G_ℓ and $\widetilde{G}(P, \varepsilon)$ are measurable with respect to (P, ε), and $A := (G - \widetilde{G})^{-1}\{0\}$ is a Borel set. For fixed $P \in \mathbb{R}^{k+1}$, we have

$$\int_0^\infty \chi_A(P, \varepsilon)\, \mathrm{d}\varepsilon = 0$$

(by (6.73)). We find that

$$\int_0^\infty \int_{\mathbb{R}^{k+1}} \chi_A(P, \varepsilon)\, \mathrm{d}P\, \mathrm{d}\varepsilon = \int_{\mathbb{R}^{k+1}} \int_0^\infty \chi_A(P, \varepsilon)\, \mathrm{d}\varepsilon\, \mathrm{d}P = 0,$$

and thus, for a.e. $\varepsilon > 0$, we have $\liminf_{\ell\to\infty} G_\ell(P, \varepsilon) = G(P, \varepsilon)$ for a.e. $P \in \mathbb{R}^{k+1}$. Similarly, for a.e. $\varepsilon > 0$, we have $\limsup_{\ell\to\infty} G_\ell(P, \varepsilon) = G(P, \varepsilon)$ for a.e. $P \in \mathbb{R}^{k+1}$. This completes the proof of (6.72). □

6.5 Approximation with maps induced by skeletons

In this section, we consider: (a) $0 < s < 1$ and $1 < p < \infty$ such that $1 \le k \le sp < k+1$; (b) an integer $N > k$. In this setting, we will present several results related to the approximation of maps in $W^{s,p}_1(\mathbb{R}^N; \mathscr{N})$. Most of these results (or at least variants of them) were essentially established (but possibly not stated) by Brezis and Mironescu [21]. Here, we adapt the statements therein to our setting and provide only the missing arguments.

For $\varepsilon > 0$ and a point P in the cube $Q_\varepsilon := [-\varepsilon, \varepsilon)^N$, let $\mathscr{C}^N_{\varepsilon,P}$ be the N-dimensional mesh in $\mathbb{R}^N$ with diameter 2ε and P as one of its centers, i.e., $\mathscr{C}^N_{\varepsilon,P}$ is the collection of cubes $P + 2\varepsilon K + Q_\varepsilon$, with $K \in \mathbb{Z}^N$.

Let $\mathscr{C}^{N-1}_{\varepsilon,P}$ be the $(N-1)$-dimensional skeleton associated to $\mathscr{C}^N_{\varepsilon,P}$, i.e., the collection of the boundaries of the cubes in $\mathscr{C}^N_{\varepsilon,P}$. Similarly, we let $\mathscr{C}^{N-2}_{\varepsilon,P}$ be the collection of the boundaries of the cubes in $\mathscr{C}^{N-1}_{\varepsilon,P}$, etc. With a slight abuse of notation, we identify $\mathscr{C}^{j}_{\varepsilon,P}$ with $\cup_{C^j\in\mathscr{C}^j_{\varepsilon,P}} C^j$.

We next discuss the properties of the restrictions of $W^{s,p}$ maps $f\colon \mathbb{R}^N \to \mathbb{R}^n$ to "generic" skeletons $\mathscr{C}^{j}_{\varepsilon,P}$. For this purpose, it will be convenient to consider f not as an equivalence class, but as an everywhere defined Borel function. Since f is a function, the restriction of f to any subset of $\mathbb{R}^N$, in particular to $\mathscr{C}^{j}_{\varepsilon,P}$, is unambiguously defined. For such f, the following holds [20, Appendix E]: For *every* $\varepsilon > 0$, for *almost every* $P \in Q_\varepsilon$, and for *every* cube $C^{k+1} \in \mathscr{C}^{k+1}_{\varepsilon,P}$, we have

$$f_{|\partial C^{k+1}} \in W^{s,p}(\partial C^{k+1}) \subset \mathrm{VMO}(\partial C^{k+1}). \tag{6.74}$$

As a consequence of (6.74), if, in addition, $f\colon \mathbb{R}^N \to \mathscr{N}$, then $f_{|\partial C^{k+1}}$ has a well-defined homotopy class in $\mathrm{VMO}(\partial C^{k+1};\mathscr{N})$ (see (2.34)).

We are now in position to state the main result of this section.

Theorem 6.16. *Let $f \in W^{s,p}_1(\mathbb{R}^N;\mathscr{N})$. If there exist $c_0 > 0$ and a sequence $\varepsilon_\ell \to 0$ such that the set*

$$A_\ell := \{P \in Q_{\varepsilon_\ell} \colon f_{|\partial C^{k+1}} \text{ is in } W^{s,p}(\partial C^{k+1};\mathscr{N}) \text{ and nullhomotopic}, \forall\, C^{k+1} \in \mathscr{C}^{k+1}_{\varepsilon_\ell,P}\}$$

satisfies $|A_\ell|/|Q_{\varepsilon_\ell}| > c_0$ for every ℓ, then

$$f \in \overline{C^\infty_1(\mathbb{R}^N;\mathscr{N})}^{W^{s,p}_1}.$$

Before proceeding to the proof of Theorem 6.16, we introduce some definitions used in the proof.

If $g\colon \mathscr{C}^{k}_{\varepsilon,P} \to \mathbb{R}^n$, we define its homogeneous extension $H^{k+1}(g)\colon \mathscr{C}^{k+1}_{\varepsilon,P} \to \mathbb{R}^n$ to the cubes in $\mathscr{C}^{k+1}_{\varepsilon,P}$ as follows. Let $x \in C^{k+1} \in \mathscr{C}^{k+1}_{\varepsilon,P}$. If x is not the center $\mathscr{O}$ of C^{k+1}, we let

$$y := \mathscr{O} + \frac{\varepsilon(x-\mathscr{O})}{|x-\mathscr{O}|_\infty} \in \mathscr{C}^{k}_{\varepsilon,P}$$

and set

$$H^{k+1}(g)(x) := g(y).$$

The definition does not depend on the choice of the cube $C^{k+1} \in \mathscr{C}^{k+1}_{\varepsilon,P}$ such that $x \in C^{k+1}$, and $H^{k+1}(g)$ is "locally 0−homogeneous", in the sense that $H^{k+1}(g)$ is constant along the "ray" $(\mathscr{O}, y]$. We also note that $H^{k+1}(g)$ is well-defined except at the centers of the cubes in $\mathscr{C}^{k+1}_{\varepsilon,P}$.

Iterating the above construction, we obtain an a.e. defined map

$$H(g) \coloneqq H^N(H^{N-1}(\cdots(H^{k+1}(g))\cdots))\colon \mathbb{R}^N \to \mathbb{R}^n.$$

We next introduce a $W^{s,p}$-type seminorm adapted to skeletons. Given a map $g\colon \mathscr{C}^j_{\varepsilon,P} \to \mathbb{R}^n$, we let

$$|g|^p_{W^{s,p}(\mathscr{C}^j_{\varepsilon,P})} \coloneqq \iint\limits_{\mathscr{C}^j_{\varepsilon,P}\times\mathscr{C}^j_{\varepsilon,P}} \frac{|g(x)-g(y)|^p}{|x-y|^{j+sp}_\infty}\,\mathrm{d}\mathscr{H}^j(x)\,\mathrm{d}\mathscr{H}^j(y)$$

and define

$$W^{s,p}_1(\mathscr{C}^j_{\varepsilon,P}) \coloneqq \{g\colon \mathscr{C}^j_{\varepsilon,P} \to \mathbb{R}^n\colon |g|_{W^{s,p}(\mathscr{C}^j_{\varepsilon,P})} < \infty \text{ and } \exists\, c_g \in \mathbb{R}^n \text{ s.t. } \operatorname{supp}(g-c_g) \subset \mathbb{B}^N\}.$$

We have the following result.

Lemma 6.17. *If $f \in W^{s,p}_1(\mathbb{R}^N)$, then, for every $\varepsilon > 0$ and almost every $P \in Q_\varepsilon$, we have*

$$f_{|\mathscr{C}^j_{\varepsilon,P}} \in W^{s,p}_1(\mathscr{C}^j_{\varepsilon,P}),\ \forall\, 0 \le j \le N. \tag{6.75}$$

Proof. In what follows, $C_i = C_i(N, j, s, p, \varepsilon)$ denotes a finite constant (possibly depending on ε).

Using, when $|x-y|_\infty \ge 2\varepsilon$, the inequality

$$|f(x)-f(y)| \le |f(x)-c_f| + |f(y)-c_f|,$$

we find that

$$|f|^p_{W^{s,p}(\mathscr{C}^j_{\varepsilon,P})} \le \iint\limits_{\substack{\mathscr{C}^j_{\varepsilon,P}\times\mathscr{C}^j_{\varepsilon,P}\\ |x-y|_\infty<2\varepsilon}} \frac{|f(x)-f(y)|^p}{|x-y|^{j+sp}_\infty}\,\mathrm{d}\mathscr{H}^j(x)\,\mathrm{d}\mathscr{H}^j(y) + C_1\int_{\mathscr{C}^j_{\varepsilon,P}} |f(x)-c_f|^p\,\mathrm{d}\mathscr{H}^j(x),$$

and therefore

$$\int_{Q_\varepsilon} |f|^p_{W^{s,p}(\mathscr{C}^j_{\varepsilon,P})}\,\mathrm{d}P \le \int_{Q_\varepsilon} \iint\limits_{\substack{\mathscr{C}^j_{\varepsilon,P}\times\mathscr{C}^j_{\varepsilon,P}\\ |x-y|_\infty<2\varepsilon}} \frac{|f(x)-f(y)|^p}{|x-y|^{j+sp}_\infty}\,\mathrm{d}\mathscr{H}^j(x)\,\mathrm{d}\mathscr{H}^j(y)\,\mathrm{d}P + C_1\|f-c_f\|^p_p. \tag{6.76}$$

By [21, Lemma 6.1], the integral on the right-hand side of (6.76) is dominated by $C_2|f|^p_{W^{s,p}(\mathbb{R}^N)}$. Combining this with the fact that $f = c_f$ outside $\mathbb{B}^N$, we obtain

$$\int_{Q_\varepsilon} |f|^p_{W^{s,p}(\mathscr{C}^j_{\varepsilon,P})}\, \mathrm{d}P \le C_3|f|^p_{W^{s,p}(\mathbb{R}^N)} + C_1\|f - c_f\|^p_p \le C_4|f|^p_{W^{s,p}(\mathbb{R}^N)},$$

which implies (6.75). □

We will call a mesh $\mathscr{C}^N_{\varepsilon,P}$ such that (6.75) holds a "good mesh".

Proof of Theorem 6.16. With no loss of generality, we may assume that $c_f = 0$, and thus $f \in W^{s,p}(\mathbb{R}^N)$.

We divide the proof into 4 steps.

Step 1. If $sp \ge 1$ then, for any $f \in W^{s,p}_1(\mathbb{R}^N;\mathbb{R}^n)$ (not necessarily $\mathscr{N}$-valued), there exist sets $D_\varepsilon \subset Q_\varepsilon$ such that: (j) $|D_\varepsilon|/|Q_\varepsilon| \to 1$ as $\varepsilon \to 0$; (jj) for every $P_\varepsilon \in D_\varepsilon$, we have

$$H(f_{|\mathscr{C}^k_{\varepsilon,P_\varepsilon}}) \to f \text{ in } W^{s,p}(\mathbb{R}^N) \text{ as } \varepsilon \to 0.$$

Indeed, define

$$D_\varepsilon := \left\{P \in Q_\varepsilon\colon \|f - H(f_{|\mathscr{C}^k_{\varepsilon,P}})\|^p_{W^{s,p}(\mathbb{R}^N)} \le \left(\fint_{Q_\varepsilon} \left\|f - H(f_{|\mathscr{C}^k_{\varepsilon,P}})\right\|^p_{W^{s,p}(\mathbb{R}^N)} \mathrm{d}P\right)^{1/2}\right\}.$$

We then have

$$\int_{Q_\varepsilon} \left\|f - H(f_{|\mathscr{C}^k_{\varepsilon,P}})\right\|^p_{W^{s,p}(\mathbb{R}^N)} \mathrm{d}P \ge (|Q_\varepsilon| - |D_\varepsilon|)\left(\fint_{Q_\varepsilon} \left\|f - H(f_{|\mathscr{C}^k_{\varepsilon,P}})\right\|^p_{W^{s,p}(\mathbb{R}^N)} \mathrm{d}P\right)^{1/2},$$

which implies that

$$|D_\varepsilon|/|Q_\varepsilon| \ge 1 - \left(\fint_{Q_\varepsilon} \left\|f - H(f_{|\mathscr{C}^k_{\varepsilon,P}})\right\|^p_{W^{s,p}(\mathbb{R}^N)} \mathrm{d}P\right)^{1/2}. \tag{6.77}$$

On the other hand, we have the following result [21, (5.54)].

Lemma 6.18. *If $sp \ge 1$ then, for every $f \in W^{s,p}(\mathbb{R}^N;\mathbb{R}^n)$,*

$$\lim_{\varepsilon\to 0} \fint_{Q_\varepsilon} \left\|f - H(f_{|\mathscr{C}^k_{\varepsilon,P}})\right\|^p_{W^{s,p}(\mathbb{R}^N)} \mathrm{d}P = 0. \tag{6.78}$$

We complete Step 1 *via* (6.77) and (6.78).

Step 2. Under the assumptions of the theorem, there exists a sequence $(\mathscr{C}_{\varepsilon_\ell,P_\ell})$ of good meshes such that

$$H(f_{|\mathscr{C}^k_{\varepsilon_\ell,P_\ell}}) \to f \text{ in } W^{s,p}(\mathbb{R}^N;\mathscr{N}) \tag{6.79}$$

and

$$f_{|\partial C^{k+1}} \text{ is nullhomotopic}, \forall\, C^{k+1} \in \mathscr{C}^{k+1}_{\varepsilon_\ell,P_\ell}. \tag{6.80}$$

Indeed, since $|A_\ell|/|Q_{\varepsilon_\ell}| > c_0$ for every ℓ, then, for ℓ sufficiently large, we have $|A_\ell \cap D_{\varepsilon_\ell}| > 0$. We complete Step 2 by choosing, for large ℓ, $P_\ell \in A_\ell \cap D_{\varepsilon_\ell}$.

From now on, we fix ℓ sufficiently large such that

$$\operatorname{supp} f \subset B_{1-9\sqrt{N}\varepsilon_\ell}(0). \tag{6.81}$$

By a standard smoothing argument, it suffices to prove that, under the assumptions (6.79) and (6.80), $H(f_{|\mathscr{C}^k_{\varepsilon_\ell,P_\ell}})$ can be approximated with Lipschitz maps from $\mathbb{R}^N$ to $\mathscr{N}$ supported in $\mathbb{B}^N$. This will be proved for each fixed ℓ. In order to lighten the notation, we write ε, respectively P, instead of ε_ℓ, respectively P_ℓ.

Step 3. If (6.80) and (6.81) hold, then $H(f_{|\mathscr{C}^k_{\varepsilon,P}})$ can be approximated with $H(g)$ for some Lipschitz map $g\colon \mathscr{C}_{\varepsilon,P} \to \mathscr{N}$ satisfying

$$\operatorname{supp} g \subset B_{1-7\sqrt{N}\varepsilon}(0). \tag{6.82}$$

For this purpose, we first approximate $f_{|\mathscr{C}^k_{\varepsilon,P}}$ with Lipschitz maps on $\mathscr{C}^k_{\varepsilon,P}$ by the means of the following lemma.

Lemma 6.19. *There exists a sequence of Lipschitz maps $(g_i) \subset \operatorname{Lip}(\mathscr{C}^k_{\varepsilon,P};\mathscr{N})$ such that*

$$g_i \to f_{|\mathscr{C}^k_{\varepsilon,P}} \textit{ in } W^{s,p}_1(\mathscr{C}^k_{\varepsilon,P};\mathscr{N}) \textit{ as } i \to \infty$$

and, for any cube $C^k \in \mathscr{C}^k_{\varepsilon,P}$, if $f = 0$ in C^k, then $g_i = 0$ in C^k for all i.

Granted Lemma 6.19, we complete Step 3 by the following continuity property of H.

Lemma 6.20. *For maps g and (g_i) in $W^{s,p}_1(\mathscr{C}^k_{\varepsilon,P};\mathbb{R}^n)$ and supported in $B_{1-7\sqrt{N}\varepsilon}(0)$, it holds that*

$$[g_i \to g \textit{ in } W^{s,p}(\mathscr{C}^k_{\varepsilon,P};\mathbb{R}^n)] \;\Rightarrow\; [H(g_i) \to H(g) \textit{ in } W^{s,p}(\mathbb{R}^N;\mathbb{R}^n)].$$

In view of Steps 2 and 3, we complete the proof of Theorem 6.16 *via*

Step 4. Under the assumptions (6.79)–(6.81), for large i, the map $H(g_i)$ can be approximated in $W^{s,p}$ with Lipschitz maps with support in $\mathbb{B}^N$.

For this purpose, we first notice that for all cubes $C^{k+1} \in \mathscr{C}^{k+1}_{\varepsilon,P}$, we have $W^{s,p}(\partial C^{k+1}) \hookrightarrow (\mathrm{VMO} \cap \mathscr{L}^1)(\partial C^{k+1})$. Combining this with Lemma 2.35, and the fact that f takes nonzero values only on finitely many cubes, we get that for i sufficiently large and for all cubes $C^{k+1} \in \mathscr{C}^{k+1}_{\varepsilon,P}$, $g_{i|\partial C^{k+1}} \sim f_{|\partial C^{k+1}}$, and thus $g_{i|\partial C^{k+1}}$ is nullhomotopic. From now on, we consider such i's.

We next adapt to our setting an approximation result initially obtained by Bethuel [5, Section II]. This is the content of the following

Lemma 6.21. *Let $g \in \mathrm{Lip}(\mathscr{C}^k_{\varepsilon,P}; \mathscr{N})$ be such that $g_{|\partial C^{k+1}}$ is nullhomotopic for all cubes $C^{k+1} \in \mathscr{C}^{k+1}_{\varepsilon,P}$ and $\operatorname{supp} g \subset B_{1-7\sqrt{N}\varepsilon}(0)$. For $1 \le q < k+1$, the map $H(g)$ is a strong limit in $W^{1,q}$ of maps in $\mathrm{Lip}(\mathbb{R}^N; \mathscr{N})$ with value 0 outside of a compact subset of $\mathbb{B}^N$.*

We complete Step 4 (and the proof of Theorem 6.16) by combining Lemma 6.21 with the Gagliardo–Nirenberg embedding

$$W^{1,q} \cap \mathscr{L}^\infty \hookrightarrow W^{s,p}, \forall\, sp < q < k+1. \qquad \square$$

We now justify Lemmas 6.19 and 6.21 used in the proof of Theorem 6.16. They are variants of [21, Lemma 7.1] and [49, Proposition 2.8]. (However, in [49] the topological setting is different.) We adapt here the "local" arguments in [21, 49] to the case of maps defined in the full space and constant at infinity.

Proof of Lemma 6.19. It suffices to repeat the proof of Lemma 7.1 in [21]. There, the maps are defined only on a cube. However, applied to our situation, the construction in [21] yields a map g satisfying (6.82). □

Proof of Lemma 6.21. It suffices to repeat the proof of Proposition 2.8 in [49] with two changes: (i) in the first step of the proof, we obtain the existence of a Lipschitz extension $h : \mathscr{C}^{k+1}_{\varepsilon,P} \to \mathscr{N}$ of g using the assumption that $g_{|\partial C^{k+1}}$ is nullhomotopic (in [49, Proposition 2.8], the assumption is $\pi_k(\mathscr{N}) = \{0\}$); (ii) in the second step of the proof, we consider a different homotopy G, designed to preserve the property that we approximate with compactly supported Lipschitz maps. More specifically, instead of requiring, as in [49, proof of Proposition 2.8] that, when θ close to 1, $G(x,\theta) = a$ for some $a \in \mathscr{C}^{k+1}$, we require that $G(x,\theta)$ stays outside $\operatorname{supp} g$. For this purpose, we consider the map G defined in Lemma 6.22 below (with $j := k$ and, in (4), $r := 1 - 5\sqrt{N}\varepsilon$). For this G and each $P \in Q_\varepsilon$, we have $B_{1-7\sqrt{N}\varepsilon}(0) \subset B_{1-5\sqrt{N}\varepsilon}(P) \subset B_{1-3\sqrt{N}\varepsilon}(0)$. Using this fact, it is straightforward that the approximating sequence considered in the third step of the proof of [49, Proposition 2.8] is supported in $B_{1-\sqrt{N}\varepsilon}(0)$. □

The following lemma relies only on the topological structure of a bounded mesh, so that, for simplicity, we assume that $\varepsilon = 1$ and $P = 0$. Let $\mathscr{C}_M^N$ be the collection of cubes $2K + Q_1$, with $K \in \{-M, \dots, M\}^N$, and let $\mathscr{C}_M^j$ for $0 \le j \le N-1$ be the corresponding j-skeleton. We identify $\mathscr{C}_M^N$ with the union of its cubes, which is Q_{2M+1}.

Lemma 6.22. *Let $0 \le j \le N-1$. There exists a Lipschitz homotopy $G = G(x, \theta)\colon \mathscr{C}_M^N \times [0,1] \to \mathscr{C}_M^N$ such that:*

(1) $G(x, 0) = x, \forall\, x \in \mathscr{C}_M^N$;

(2) $G(x, \theta) \in \partial Q_{2M+1}, \forall\, x \in \mathscr{C}_M^N \setminus Q_{1/2}, \forall\, \theta \ge 1/2$;

(3) $G(x, \theta) \in \mathscr{C}_M^{j+1}, \forall\, x \in \mathscr{C}_M^j, \forall\, \theta$;

(4) *for each $r > 0$ and each cube $C \in \mathscr{C}_M^j$, if $C \cap B_r(0) = \varnothing$, then $G(x, \theta) \notin B_r(0)$, $\forall\, x \in C$, $\forall\, \theta$.*

Proof. Consider the Lipschitz map $g\colon [-(2M+1), 2M+1] \times [0,1] \to [-(2M+1), 2M+1]$ given by

$$g(a, \theta) \coloneqq \operatorname{sgn}(a) \min((4M\theta + \theta + 1)|a|, 2M+1).$$

For $1 \le i \le N$, set

$$G_i(x^1, \dots, x^{i-1}, x^i, x^{i+1}, \dots, x^N, \theta) \coloneqq (x^1, \dots, x^{i-1}, g(x^i, \theta), x^{i+1}, \dots, x^N).$$

Clearly, G_i satisfies

(j) $G_i(x, 0) = x, \forall\, x \in \mathscr{C}_M^N$;

(jj) $G_i(x, 1) \in \mathscr{C}_M^j \cap \{x^i = 2M+1\}, \forall\, x \in \mathscr{C}_M^j$ with $x^i \ge 1/2$;

(jjj) $G_i(x, 1) \in \mathscr{C}_M^j \cap \{x^i = -2M-1\}, \forall\, x \in \mathscr{C}_M^j$ with $x^i \le -1/2$;

(jjjj) $G_i(x, \theta) \in \mathscr{C}_M^{j+1}, \forall\, x \in \mathscr{C}_M^j, \forall\, \theta$.

Let

$$G(x, \theta) \coloneqq \begin{cases} G_1(x, 2N\theta), & \text{if } \theta \le 1/(2N) \\ G_2(G(x, 1/(2N)), 2N\theta - 1), & \text{if } 1/(2N) < \theta \le 1/N \\ \dots & \\ G_N(G(x, (N-1)/(2N)), 2N\theta - N + 1), & \text{if } (N-1)/(2N) < \theta \le 1/2 \\ G(x, 1/2), & \text{if } 1/2 < \theta \le 1 \end{cases}.$$

Using (j)–(jjjj), we easily find that $G(x, \theta)$ satisfies all the required properties. ◻

6.6 Approximation with smooth maps to $\mathscr{N}$

Recall that we consider: (a) $0 < s < 1$ and $1 < p < \infty$ such that $sp = k$; (b) an integer $N > k$.

Only in this section, we make the extra assumption that "the cohomology of $\mathscr{N}$ sees its homotopy". More specifically, we assume that $\mathscr{N}$ has the following property:

$$\begin{aligned}&\text{for each } f \in C^\infty(\mathbb{S}^k;\mathscr{N}), \text{ we have}\\ &\left[\int_{\mathbb{S}^k} f^*\omega = 0, \forall \text{ smooth closed } k\text{-form } \omega \text{ on } \mathscr{N}\right] \implies f \text{ is nullhomotopic.}\end{aligned} \tag{6.83}$$

Standard results in algebraic topology provide sufficient conditions for the validity of (6.83). We briefly discuss this in Appendix A. To give a flavor of that discussion, we note here that $\mathscr{N} := \mathbb{S}^k$ satisfies (6.83), as we already mentioned in Example 5.5.

In this section, we prove the following theorem.

Theorem 6.23. *Assume that* (6.83) *holds. Let $f \in W^{s,p}_1(\mathbb{R}^N;\mathscr{N})$ and let $T = T_\omega$ be defined as in Section* 6.2*. Then*

$$f \in \overline{C^\infty_1(\mathbb{R}^N;\mathscr{N})}^{W^{s,p}_1} \Leftrightarrow [\forall \textit{ smooth closed } k\textit{-form } \omega \textit{ on } \mathscr{N}, T_\omega f = 0].$$

Combining: (i) Remark 6.2; (ii) Corollary 6.6; (iii) the fact that $H^k_{\mathrm{dR}}(\mathbb{S}^k) = \mathbb{R}$ is generated by the standard volume form $\omega_{\mathbb{S}^k}$, we deduce the following corollary of Theorem 6.23.

Corollary 6.24. *Let $f \in W^{s,p}_1(\mathbb{R}^N;\mathbb{S}^k)$. Then*

$$f \in \overline{C^\infty_1(\mathbb{R}^N;\mathbb{S}^k)}^{W^{s,p}_1} \Leftrightarrow \operatorname{Jac} f = 0.$$

Thus, as we said in the introduction, in the special case of sphere-valued maps, our result contains as a particular case the fact that the distributional Jacobian detects the closure of smooth maps, as announced by Mucci [54].

Before proving Theorem 6.23, we present a (equivalent) form of (6.83) adapted to our context.

Lemma 6.25. *Assume that* (6.83) *holds. Let C be a cube in $\mathbb{R}^{k+1}$. Then*

$$\begin{aligned}&\textit{for each } f \in \mathrm{VMO}(\partial C;\mathscr{N}), \textit{ we have}\\ &\left[\mathscr{I}_{\partial C,\omega}(f) = 0, \forall \textit{ smooth closed } k\textit{-form } \omega \textit{ on } \mathscr{N}\right] \implies f \textit{ is nullhomotopic.}\end{aligned}$$

Proof. First, let us note that, if (6.83) holds, then it also holds for continuous maps. (This follows from a standard smoothing argument and Corollary 3.28.)

By Corollary 3.29, there exists ε_1 such that

$$\mathcal{I}_{\partial C,\omega}(f) = \mathcal{I}_{\partial C,\omega}(f^{\varepsilon}), \forall \text{ smooth closed } k\text{-form } \omega \text{ on } \mathcal{N}, \forall\, \varepsilon < \varepsilon_1. \tag{6.84}$$

Let $\Psi\colon \mathbb{S}^k \to \partial C$ be a bi-Lipschitz orientation preserving map. By Corollary 3.32, we have

$$\mathcal{I}_{\partial C,\omega}(g) = \mathcal{I}_{\mathbb{S}^k,\omega}(g \circ \Psi), \forall\, g \in C(\partial C; \mathcal{N}). \tag{6.85}$$

The conclusion of the lemma follows from (6.84), (6.85) (with $g := f^{\varepsilon}$), and the validity of (6.83) for continuous maps. □

Proof of Theorem 6.23. "⇒" (Here, we do not rely on (6.83).) This follows from Theorem 6.1 when $k \geq 2$, respectively Theorem 6.5 when $k = 1$, since for $f \in C_1^{\infty}(\mathbb{R}^N; \mathcal{N})$ and $\zeta \in \mathrm{Lip}(\mathbb{R}^N; \Lambda^{N-k-1})$, we have

$$\langle Tf, \zeta\rangle = \int_{\mathbb{R}^N} f^*\omega \wedge \mathrm{d}\zeta = (-1)^k \int_{\mathbb{R}^N} \mathrm{d}(f^*\omega \wedge \zeta) = 0.$$

"⇐" We divide the proof into three steps: dimensional reduction, proof in the special case where $N = k+1$, and a cohomology argument.

For simplicity, in the proof we denote points in $\mathbb{R}^N$ as (x, y), with $x \in \mathbb{R}^{k+1}$ and $y \in \mathbb{R}^{N-k-1}$. We note that, if $f \in W_1^{s,p}(\mathbb{R}^N; \mathcal{N})$ then, for a.e. $y_0 \in \mathbb{R}^{N-k-1}$, we have $f(\cdot, y_0) \in W_1^{s,p}(\mathbb{R}^{k+1}; \mathcal{N})$.

Step 1. Fix ω. By Proposition 6.11 (applied with $\ell = k+1$ and $\nu = N-k-1$), for a.e. $y_0 \in \mathbb{R}^{N-k-1}$, we have $T_\omega f(\cdot, y_0) = 0$.

Step 2. Let $N = k+1$ and fix ω. Let $f \in W_1^{s,p}(\mathbb{R}^{k+1}; \mathcal{N})$ satisfy $T_\omega f = 0$. Then, by Proposition 6.15, for a.e. $\varepsilon > 0$, we have

$$\mathcal{I}_{P+\partial Q_\varepsilon,\omega}(f_{|P+\partial Q_\varepsilon}) = 0 \text{ for a.e. } P \in \mathbb{R}^{k+1}. \tag{6.86}$$

Step 3. We complete the proof of Theorem 6.23 *via* Remark 6.12, (6.86), (6.83), Lemma 6.25, and Theorem 6.16. □

A Reading homotopy from integral invariants

In this appendix, we study some necessary conditions that ensure the validity of the assumption (6.83), which plays a crucial role in Section 6.6.

We first recall that, given a smooth Riemannian manifold $\mathcal{N}$, there exists a map $\mathfrak{hur}\colon \pi_k(\mathcal{N}) \to H_k(\mathcal{N};\mathbb{Z})$, called the *Hurewicz homomorphism*, that maps a homotopy class $[f] \in \pi_k(\mathcal{N})$ to the cycle $f_\sharp[\mathbb{S}^k]$.

The following proposition characterizes the validity of (6.83) (and even slightly more) in terms of the Hurewicz map.

Proposition A.1. *Assume that $H_k(\mathcal{N};\mathbb{Z})$ is torsionfree. Then,*

$$\begin{gathered}\text{for each } f, g \in C^\infty(\mathbb{S}^k;\mathcal{N}), \text{ we have}\\ \left[\int_{\mathbb{S}^k} f^*\omega = \int_{\mathbb{S}^k} g^*\omega, \forall \text{ smooth closed } k\text{-form } \omega \text{ on } \mathcal{N}\right] \implies f \sim g\end{gathered} \tag{A.1}$$

if and only if $\mathfrak{hur}$ is injective.

In particular, (A.1) implies (6.83), specializing to g being a constant map. Proposition A.1 is well-known to experts, but for the sake of completeness we present here an argument, using as little technology as possible.

Proof. By the de Rham theorem, there exists an identification $\mathcal{J}_\mathcal{N}\colon H^k_{\mathrm{dR}}(\mathcal{N}) \to H^k(\mathcal{N};\mathbb{R})$ between the de Rham cohomology and the singular cohomology. If a cycle σ is associated with a sufficiently smooth domain of $\mathcal{N}$, then

$$\langle \mathcal{J}_\mathcal{N}(\alpha), \sigma\rangle = \int_\sigma \alpha,$$

with the integral being defined as in Section 3.5.

"$\Leftarrow$" Let $f, g \in C^\infty(\mathbb{S}^k;\mathcal{N})$. Our proof is in two steps: we first prove that

$$\int_{\mathbb{S}^k} f^*\omega = \langle \mathcal{J}_\mathcal{N}(\omega), f_\sharp[\mathbb{S}^k]\rangle, \forall \text{ closed } k\text{-form } \omega \text{ on } \mathcal{N}, \tag{A.2}$$

then find, using (A.2) and the assumptions on $\mathcal{N}$, that $f \sim g$.

Step 1. To prove (A.2), we start from the fact that $\mathcal{J}_\mathcal{N}$ is a natural transformation between the de Rham cohomology functor and the singular cohomology functor, that is, $\mathcal{J}_{\mathbb{S}^k} \circ f^* = f^\sharp \circ \mathcal{J}_\mathcal{N}$ (see, e.g., Lee [46, Proposition 18.13]). Hence, we find that

$$\int_{\mathbb{S}^k} f^*\omega = \langle \mathcal{J}_{\mathbb{S}^k}(f^*\omega), [\mathbb{S}^k]\rangle = \langle f^\sharp \mathcal{J}_\mathcal{N}(\omega), [\mathbb{S}^k]\rangle, \forall \text{ smooth closed } k\text{-form } \omega \text{ on } \mathcal{N}.$$

Now, we recall that, thanks to the universal coefficients theorem for cohomology (see, e.g., Hatcher [43, Theorem 3.2]), we have

$$H^k(\mathscr{N};\mathbb{R}) \cong \mathrm{Hom}(H_k(\mathscr{N};\mathbb{Z});\mathbb{R}).$$

Moreover, this correspondence is natural, meaning that the map $f^\sharp$ induced in cohomology by f is dual to the map $f_\sharp$ induced in homology; see, e.g., [43, Page 196]. Therefore,

$$\langle f^\sharp \mathscr{I}_{\mathscr{N}}(\omega), [\mathbb{S}^k]\rangle = \langle \mathscr{I}_{\mathscr{N}}(\omega), f_\sharp[\mathbb{S}^k]\rangle, \forall \text{ smooth closed } k\text{-form } \omega \text{ on } \mathscr{N}.$$

This concludes the proof of (A.2).

Step 2. The de Rham homomorphism being an isomorphism, (A.2) and the fact that

$$\int_{\mathbb{S}^k} f^*\omega = \int_{\mathbb{S}^k} g^*\omega, \forall \text{ smooth closed } k\text{-form } \omega \text{ on } \mathscr{N} \tag{A.3}$$

imply that $f_\sharp[\mathbb{S}^k]$ and $g_\sharp[\mathbb{S}^k]$ coincide when evaluated against any homomorphism from $H_k(\mathscr{N};\mathbb{Z})$ to $\mathbb{R}$. But, since $H_k(\mathscr{N};\mathbb{Z})$ is torsionfree, it is isomorphic to $\mathbb{Z}^j$ for some $j \in \mathbb{N}$. Hence, $f_\sharp[\mathbb{S}^k] = g_\sharp[\mathbb{S}^k]$.

Therefore, if $\mathfrak{hur}$ is injective, then (A.3) implies that $[f] = [g]$ in $\pi_k(\mathscr{N})$, showing that (A.1) holds.

"⇒" We have to prove that, if $f \in C^\infty(\mathbb{S}^k;\mathscr{N})$ is such that $f_\sharp[\mathbb{S}^k] = 0$, then f is nullhomotopic. By (A.2), we find that

$$\int_{\mathbb{S}^k} f^*\omega = 0, \forall \text{ smooth closed } k\text{-form } \omega \text{ on } \mathscr{N},$$

and hence (A.1) applied with g a constant map implies that f is nullhomotopic. We observe that the proof of this implication does not rely on the fact that $H_k(\mathscr{N};\mathbb{Z})$ is torsionfree. □

Combining Proposition A.1 with the Hurewicz theorem, see, e.g., [43, Theorem 4.37], which asserts that $\mathfrak{hur}$ is an isomorphism whenever either $k \geq 2$ and $\mathscr{N}$ is $(k-1)$-connected, or $k = 1$ and $\pi_1(\mathscr{N})$ is abelian, we obtain the following, more readable, sufficient condition for (A.1) to hold.

Proposition A.2. *Assume that $\pi_k(\mathscr{N})$ is torsionfree, and that either $k \geq 2$ and $\mathscr{N}$ is $(k-1)$-connected, or $k = 1$ and $\pi_1(\mathscr{N})$ is abelian. Then* (A.1) *holds.*

Let us give examples of some typical situations that illustrate the various assumptions above.

Example A.3. Let $\mathcal{N} = \mathbb{T}^2$ be the 2-dimensional torus and $k = 2$. Since $\pi_2(\mathbb{T}^2) = \{0\}$, every map $f : \mathbb{S}^2 \to \mathbb{T}^2$ is nullhomotopic. Therefore, (A.1) trivially holds.

On the other hand, $\pi_1(\mathbb{T}^2) = \mathbb{Z}^2$ is nontrivial, whence Proposition A.2 does not apply. Actually, $H_2(\mathbb{T}^2; \mathbb{Z}) = \mathbb{Z}$, so that the Hurewicz homomorphism is not an isomorphism. It is nevertheless injective, as it is nothing else but the zero map $\{0\} = \pi_2(\mathbb{T}^2) \to H_2(\mathbb{T}^2; \mathbb{Z})$.

This highlights the fact that the assumptions in Proposition A.2 are more stringent than the ones of Proposition A.1, and that only the *injectivity* of $\mathfrak{hur}$ matters.

One can obtain a less trivial example, where *there* actually *is* some topology to be detected, by taking for instance $\mathbb{T}^2 \times \mathbb{S}^2$. □

Example A.4. Let $\mathcal{N} = \mathbb{RP}^2$ be the 2-dimensional projective plane and $k = 1$. Since $\pi_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$, there is a homotopically nontrivial smooth map $f : \mathbb{S}^1 \to \mathbb{RP}^2$. On the other hand, since $H^1_{\mathrm{dR}}(\mathbb{RP}^2) = \{0\}$, all smooth closed 1-forms ω are exact. Therefore, (A.3) trivially holds true for any pair of maps, and hence (A.1) fails.

The issue here is that $H_1(\mathbb{RP}^2; \mathbb{Z}) = \pi_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$ has torsion. This is actually a more general phenomenon, since the de Rham cohomology does not see torsion. This highlights why it is crucial to assume, in Proposition A.1, that the relevant homology group is torsionfree. □

Example A.5. Let $\mathcal{N} = \mathbb{S}^1 \vee \mathbb{S}^1$ be a bouquet of two circles and $k = 1$. Strictly speaking, this is not a manifold, but one can easily work instead with a manifold with the same relevant properties by considering for instance a torus with two holes.

In this case, we have $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1) = \mathbb{Z} * \mathbb{Z} \neq \mathbb{Z}^2 = H_1(\mathbb{S}^1 \vee \mathbb{S}^1; \mathbb{Z})$. But there is no injective group morphism $\mathbb{Z} * \mathbb{Z} \to \mathbb{Z}^2$. Indeed, if a and b are generators of $\mathbb{Z} * \mathbb{Z}$ and $g : \mathbb{Z} * \mathbb{Z} \to \mathbb{Z}^2$ is a morphism, then $g(aba^{-1}b^{-1}) = g(a) + g(b) - g(a) - g(b) = 0$ and thus g is not injective. In particular, the Hurewicz homomorphism is not injective. The issue here is that $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1)$ is not abelian, while homology groups are always abelian. We note that this may only arise when $k = 1$, as π_k is always abelian when $k \geq 2$.

On the other hand, if $f : \mathbb{S}^1 \to \mathbb{S}^1 \vee \mathbb{S}^1$ realizes the commutator $[a, b] = aba^{-1}b^{-1}$ in $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1)$, one has

$$\int_{\mathbb{S}^1} f^*\omega = 0, \forall \text{ smooth closed 1-form } \omega \text{ on } \mathbb{S}^1 \vee \mathbb{S}^1,$$

showing that (A.1) fails in this situation. This highlights the importance of the abelian assumption when $k = 1$. □

B Further results

This appendix is devoted to the proof of Lemma 3.37 and an improvement of Theorem 4.1, in the spirit of Bourgain, Brezis, and Nguyen [12].

Proof of Lemma 3.37. We use the same notation as in Sections 3.1 and 3.5. Let $\overline{f}_{i,\varepsilon}$ and $\overline{f}_{\varepsilon}$ be as in the proof of Lemma 3.35. Then

$$\overline{f}_{i,\varepsilon} \circ \varphi_i = (\xi_i \circ \varphi_i)(\overline{f}_i * \rho_\varepsilon) \longrightarrow (\xi_i \circ \varphi_i)\overline{f}_i = (\xi_i \circ \varphi_i)(f \circ \varphi_i) \text{ in } W^{s,p}(V_i)$$

as $\varepsilon \to 0$. Since φ_i is bi-Lipschitz, this implies that $\overline{f}_{i,\varepsilon} \to \xi_i f$ in $W^{s,p}(U_i)$. Convergence also holds in $W^{s,p}(\mathscr{M})$, since

$$\begin{aligned}
&\int_{\mathscr{M}}\int_{\mathscr{M}} \frac{|(\overline{f}_{i,\varepsilon}(x) - \xi_i(x)f(x)) - (\overline{f}_{i,\varepsilon}(y) - \xi_i(y)f(y))|^p}{\operatorname{dist}(x,y)^{k+sp}} \, d\mathscr{H}^k(x)\, d\mathscr{H}^k(y) \\
&\le 2\int_{\mathscr{M}\setminus U_i}\int_{U_i} \frac{|\overline{f}_{i,\varepsilon}(x) - \xi_i(x)f(x)|^p}{\operatorname{dist}(x,y)^{k+sp}} \, d\mathscr{H}^k(x)\, d\mathscr{H}^k(y) + |\overline{f}_{i,\varepsilon} - \xi_i f|^p_{W^{s,p}(U_i)} \\
&\le C(\varepsilon_1)\|\overline{f}_{i,\varepsilon} - \xi_i f\|_{\mathscr{L}^p(U_i)} + |\overline{f}_{i,\varepsilon} - \xi_i f|^p_{W^{s,p}(U_i)} \longrightarrow 0 \text{ when } \varepsilon \to 0.
\end{aligned}$$

(In the last inequality, we have used the fact that for ε small and any i, $\operatorname{supp} \overline{f}_{i,\varepsilon}$ is contained in a fixed compact subset of U_i'.)

Therefore, when $\varepsilon \to 0$, we have $\overline{f}_\varepsilon \to f$ in $W^{s,p}$. By (4.1), this implies that $\overline{f}_\varepsilon \to f$ in $\mathrm{BMO} \cap \mathscr{L}^1$. We are now in position to repeat the proof of Lemma 3.35 and find that, for small ε, one can define $\Pi \circ \overline{f}_\varepsilon$, which is Lipschitz and $\mathscr{N}$-valued. By Lemma 5.12, we have $\Pi \circ \overline{f}_\varepsilon \to f$ in $W^{s,p}$. □

We next present an improvement, inspired by [12], of estimate (4.2) in Theorem 4.1. The setting is the one of Section 4: (a) $\mathscr{M}$ is a compact k-dimensional Lipschitz manifold oriented by a finite chart structure $\{(U_i, V_i, \varphi_i)\}_{i\in I}$; (b) $\mathscr{N}$ is a closed manifold; (c) ω is a smooth *closed* k-form on $\mathscr{N}$; (d) $0 < s < 1$ and $1 < p < \infty$ are such that $sp = k$; (e) $\mathscr{I}(f)$ is the invariant whose existence is proved in Theorem 4.1; (f) δ is as in Definition 2.10.

Theorem B.1. *For $0 < \eta < \delta$, there exists a finite constant $C_\eta = C(\mathscr{M}, \mathscr{N}, \omega, s, p, \eta)$ such that*

$$|\mathscr{I}(f)| \le C_\eta \iint_{\{(x,y)\in\mathscr{M}\times\mathscr{M} : |f(y)-f(x)|>\eta\}} \frac{1}{[\operatorname{dist}(x,y)]^{2k}} \, d\mathscr{H}^k(x)\, d\mathscr{H}^k(y), \quad \forall f \in W^{s,p}(\mathscr{M};\mathscr{N}). \tag{B.1}$$

In order to see that (B.1) is indeed a refinement of (4.2), it suffices to note the obvious inequalities

$$\begin{aligned}&\iint_{\{(x,y)\in\mathscr{M}\times\mathscr{M}:|f(y)-f(x)|>\eta\}}\frac{1}{[\operatorname{dist}(x,y)]^{2k}}\,\mathrm{d}\mathscr{H}^k(x)\,\mathrm{d}\mathscr{H}^k(y)\\&\leq\frac{1}{\eta^p}\iint_{\{(x,y)\in\mathscr{M}\times\mathscr{M}:|f(y)-f(x)|>\eta\}}\frac{|f(x)-f(y)|^p}{[\operatorname{dist}(x,y)]^{2k}}\,\mathrm{d}\mathscr{H}^k(x)\,\mathrm{d}\mathscr{H}^k(y)\leq\frac{1}{\eta^p}|f|^p_{W^{s,p}}.\end{aligned}$$

When $k\geq 2$ and $\mathscr{M}=\mathbb{S}^k$, estimate (B.1) is due to Van Schaftingen [63, Theorem 6.2]. For more subtle questions as the range of the η's such that (B.1) holds and the optimal dependence of C_η on η, see Nguyen [55, 56] and [63].

Proof. Let $F=F(x,\varepsilon)$ be as in (4.4), with $f_\varepsilon=f_\varepsilon(x)$ as in (2.9).

Let $0<\beta<\delta-\eta$ and set

$$h_\beta(x):=\inf\{\varepsilon>0\colon \operatorname{dist}(F(x,\varepsilon),\mathscr{N})\geq\delta-\beta\}.$$

Let $\widetilde{\Pi}_\beta\in C^\infty_c(\mathbb{R}^n;\mathbb{R}^n)$ be such that $\widetilde{\Pi}_\beta(z)=\Pi(z)$, $\forall\, z\in\mathscr{N}_{\delta-\beta}$. By repeating the proof of Theorem 4.1 (see the proof of (4.24)), we have

$$|\mathscr{I}(f)|\leq C_1\int_{\mathscr{M}}\frac{1}{[h_\beta(x)]^k}\,\mathrm{d}\mathscr{H}^k(x).\tag{B.2}$$

(Here and in what follows, constants do not depend on f.)

By the proof of (4.30), for a.e. $x\in\mathscr{M}$ we have

$$\begin{aligned}\delta-\beta&\leq|F(x,h_\beta(x))-f(x)|\\&\leq\int_{\{y\in\mathscr{M}:|f(y)-f(x)|>\eta\}}\widetilde{\rho}(x,h_\beta(x),y)|f(y)-f(x)|\,\mathrm{d}\mathscr{H}^k(y)+\eta.\end{aligned}\tag{B.3}$$

Combining (B.3), (4.5), and (2.13), we have

$$(\delta-\beta-\eta)\mathscr{H}^k(B_{h_\beta(x)}(x))\leq C_2\mathscr{H}^k(\{y\in B_{h_\beta(x)}(x)\colon|f(y)-f(x)|>\eta\}).$$

This implies that

$$\begin{aligned}&\int_{\{y\in\mathscr{M}:|f(y)-f(x)|>\eta\}}\frac{1}{[\operatorname{dist}(x,y)]^{2k}}\,\mathrm{d}\mathscr{H}^k(y)\\&\geq\frac{C_3\mathscr{H}^k(B_{h_\beta(x)}(x))}{[\min(h_\beta(x),\operatorname{diam}(\mathscr{M}))]^{2k}}(\delta-\beta-\eta).\end{aligned}\tag{B.4}$$

On the other hand, (3.4) implies that

$$\frac{1}{[h_\beta(x)]^k} \le C_4 \frac{\mathscr{H}^k(B_{h_\beta(x)}(x))}{[\min(h_\beta(x), \operatorname{diam}(\mathscr{M}))]^{2k}}. \tag{B.5}$$

We obtain (B.1) from (B.2), (B.4) (integrated in x), and (B.5). □

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