

The weak approximation problem for manifold-valued Sobolev mappings

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Sobolev maps into manifolds

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Definition

$$W^{1,p}(\mathcal{M}; \mathcal{N}) = \{u \in W^{1,p}(\mathcal{M}; \mathbb{R}^{\nu}) : u(x) \in \mathcal{N} \text{ for a.e. } x \in \mathcal{M}\}$$

Motivation

- Natural framework to study harmonic maps between manifolds (starting notably with Eells, Hildebrandt, Jost, Kaul, Lemaire, Sampson, and Widman).

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- And many more (Ginzburg–Landau, Cosserat materials, gauge theories, etc.).

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Question

Do we have $W^{1,p}(\mathcal{M}; \mathcal{N}) = H_S^{1,p}(\mathcal{M}; \mathcal{N})$?

Schoen and Uhlenbeck's seminal counterexample

Proposition (Schoen and Uhlenbeck (1983))

The map $u_0: \mathbb{B}^3 \rightarrow \mathbb{S}^2$ defined by $u_0(x) = x/|x|$ satisfies

$$u_0 \in W^{1,p}(\mathbb{B}^3; \mathbb{S}^2) \setminus H_S^{1,p}(\mathbb{B}^3; \mathbb{S}^2) \quad \text{whenever } 2 \leq p < 3.$$

The strong density theorem

Theorem (Bethuel (1991))

Then, $W^{1,p}(\mathbb{B}^m; \mathcal{N}) = H_S^{1,p}(\mathbb{B}^m; \mathcal{N})$ if and only if $\pi_{\lfloor p \rfloor}(\mathcal{N}) = \{0\}$ or $p \geq m$.

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This has been extended to $W^{s,p}$ for other values of s by Brezis and Mironescu ($0 < s < 1$, 2015), Bousquet, Ponce, and Van Schaftingen ($s = 2, 3, \dots$, 2015), and D. ($s > 1$ noninteger, 2023).

The weak approximation problem

We say that $(u_n)_{n \in \mathbb{N}}$ *weakly converges* to u in $W^{1,p}$, and we write $u_n \rightharpoonup u$, whenever $u_n \rightarrow u$ almost everywhere and

$$\sup_{n \in \mathbb{N}} \mathcal{E}^{1,p}(u_n, \mathcal{M}) = \sup_{n \in \mathbb{N}} \int_{\mathcal{M}} |Du_n|^p < +\infty.$$

Define

$$H_W^{1,p}(\mathcal{M}; \mathcal{N}) = \{u \in W^{1,p}(\mathcal{M}; \mathcal{N}) : \text{there exists } (u_n)_{n \in \mathbb{N}} \text{ in } C^\infty(\mathcal{M}; \mathcal{N}) \text{ such that } u_n \rightharpoonup u\}.$$

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Question

Does it hold that $H_W^{1,p}(\mathcal{M}; \mathcal{N}) = W^{1,p}(\mathcal{M}; \mathcal{N})$?

A topological obstruction: here we go again?

If $2 < p < 3$, then $\frac{x}{|x|} \notin H_W^{1,p}(\mathbb{B}^3; \mathbb{S}^2)$.

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If $p \notin \mathbb{N}$, then $H_W^{1,p}(\mathcal{M}; \mathcal{N}) = H_S^{1,p}(\mathcal{M}; \mathcal{N})$.

A new phenomenon: the case $p \in \mathbb{N}$

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- $H_W^{1,2}(\mathcal{M}; \mathcal{N}) = W^{1,2}(\mathcal{M}; \mathcal{N})$ for more general \mathcal{N} (Pakzad and Rivière (2003));
- $H_W^{2,2}(\mathbb{B}^5; \mathbb{S}^3) = W^{2,2}(\mathbb{B}^5; \mathbb{S}^3)$ (Hardt and Rivière (2015)).

Obstructions strike back: the analytical obstruction

Theorem (Bethuel (2020))

$$H_W^{1,3}(\mathbb{B}^4; \mathbb{S}^2) \subsetneq W^{1,3}(\mathbb{B}^4; \mathbb{S}^2)$$

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Global topological obstructions were already known (Hang and Lin (2003)).
Here, the obstruction is local.

Two families of analytical obstructions to the weak approximation property

Theorem (D. and Van Schaftingen (2024))

For every $p \in \mathbb{N} \setminus \{0, 1\}$, there exists a compact Riemannian manifold \mathcal{N} such that, if $\dim \mathcal{M} > p$, then

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Theorem (D. and Van Schaftingen (2024))

For every $d \in \mathbb{N}_*$, if $\dim \mathcal{M} > 4d - 1$, then

$$H_W^{1,4d-1}(\mathcal{M}; \mathbb{S}^{2d}) \subsetneq W^{1,4d-1}(\mathcal{M}; \mathbb{S}^{2d}).$$

Weak approximation of mappings into manifold with a finite homotopy group

Theorem (D. and Van Schaftingen (in preparation))

For every $p \in \mathbb{N}_*$, if $\pi_p(\mathcal{N})$ is finite, then $H_W^{1,p}(\mathbb{B}^m; \mathcal{N}) = W^{1,p}(\mathbb{B}^m; \mathcal{N})$.

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Combining this with a result of Serre, we obtain a complete answer for sphere-valued maps.

Corollary

For every $p \in \mathbb{N}_*$ and $\ell \in \mathbb{N}_*$, we have $H_W^{1,p}(\mathbb{B}^m; \mathbb{S}^\ell) = W^{1,p}(\mathbb{B}^m; \mathbb{S}^\ell)$ if and only if either ℓ is odd or $p \neq 2\ell - 1$.

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- Is it possible to characterize those maps which are weakly approximable, in case the weak approximation property fails?
- And many more!

Thank you for your attention!