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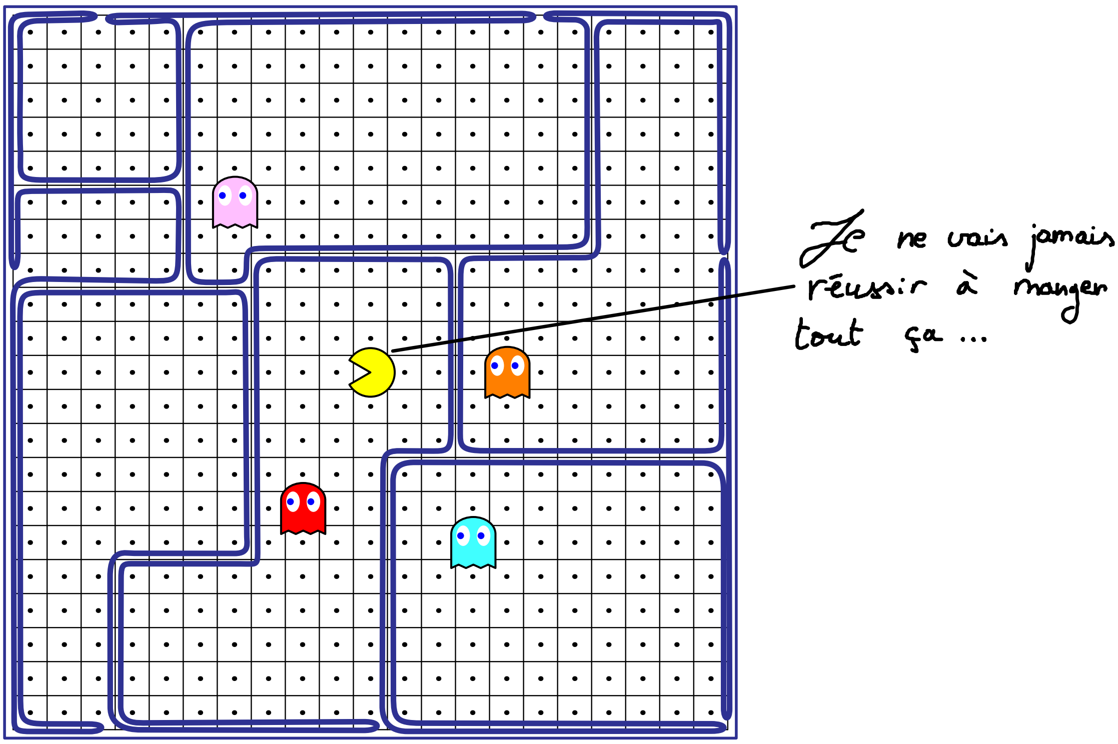
Espaces de Sobolev à valeurs variétés



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Thèse de doctorat

Université Claude Bernard Lyon 1

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Résumé substantiel en français

Dans cette thèse, on s'intéresse à certaines propriétés des espaces de Sobolev d'applications *entre variétés*. Plus particulièrement, un problème modèle sur lequel on se concentre est le suivant :

> étant donnée une application $u\colon \mathscr{M} \to \mathscr{N}$ de régularité $W^{s,p}$ entre deux variétés Riemanniennes compactes $\mathscr{M}$ et $\mathscr{N}$, u peut-elle être approximée par des applications lisses ?

Déjà au début des années 80, une petite communauté de mathématiciennes et mathématiciens avait saisi l'intérêt de considérer des applications de Sobolev *à valeurs dans une variété*, en raison notamment de leur forte connection avec des problèmes issus de la géométrie, de la physique, ou encore du calcul numérique. Bien que définis comme des sous-espaces métriques d'espaces de Sobolev classiques de fonctions à valeurs vectorielles, les espaces $W^{s,p}(\mathscr{M};\mathscr{N})$ exhibent des *différences qualitatives frappantes* avec ces derniers. Un exemple typique d'une telle différence est le suivant, observé par R. Schoen et K. Uhlenbeck en 1983 : il existe une application $u \in W^{1,2}(\mathbb{B}^3;\mathbb{S}^2)$ qui ne peut être obtenue comme une limite d'applications lisses à valeurs dans la sphère $\mathbb{S}^2$.

Partant de ce constat que, contrairement au cas classique de fonctions à valeurs réelles (ou vectorielles), la densité des applications lisses peut échouer en présence d'une contrainte variété, un pan entier de recherche s'ouvrit, se focalisant notamment sur les quatre questions suivantes :

(Q1) caractériser les s, p, $\mathscr{M}$, et $\mathscr{N}$ pour lesquels il y a densité des applications lisses ;

(Q2) trouver une classe convenable d'*applications presque lisses* qui est toujours dense dans $W^{s,p}(\mathscr{M};\mathscr{N})$;

(Q3) lorsque la densité forte échoue, caractériser la clôture des applications lisses ;

(Q4) étudier ce qu'il advient lorsqu'on considère une notion plus faible de convergence.

L'objectif de cette thèse est de présenter une contribution à l'étude de chacune de ces questions.

Dans le Chapitre 2, on s'intéresse aux deux premières questions, qui constituent ce qu'on nomme le *problème de la densité forte*. Une contribution fondamentale dans cette direction est celle de F. Bethuel en 1991, dans laquelle il apporta une réponse complète au problème lorsque $s = 1$ et le domaine est une boule. Plus précisément, F. Bethuel a montré que les applications lisses sont denses dans $W^{1,p}(\mathbb{B}^m; \mathscr{N})$ *si et seulement si* $\pi_{\lfloor p \rfloor}(\mathscr{N})$ est trivial, et que la classe des applications $W^{1,p}(\mathbb{B}^m; \mathscr{N})$ qui sont lisses en dehors d'un ensemble singulier constitué d'une union finie de sous-variétés de $\mathbb{B}^m$ de dimension $m - \lfloor p \rfloor - 1$ est toujours dense. L'argument de F. Bethuel repose notamment sur l'utilisation de la *méthode des bons et mauvais cubes*. Le cas d'un domaine général fut compris par Hang F. et Lin F. en 2003, où ils obtinrent la densité forte de la même classe d'applications presque lisses, ainsi qu'une condition nécessaire et suffisante pour la densité des fonctions lisses, formulée uniquement en termes de la topologie de $\mathscr{M}$ et $\mathscr{N}$. Cette condition inclut $\pi_{\lfloor p \rfloor}(\mathscr{N}) = \{0\}$, et lui est équivalente lorsque le domaine est une boule par exemple.

Après une série d'importantes contributions partielles, le résultat de F. Bethuel fut étendu en toute généralité, pour $0 < s < 1$ par H. Brezis et P. Mironescu, et pour $s \in \mathbb{N}_*$ par P. Bousquet, A. Ponce, et J. Van Schaftingen. Le cas $s \in \mathbb{N}_*$ fut traité à l'aide de la méthode des bons et mauvais cubes développée par F. Bethuel, complémentée par de nouveaux outils, permettant notamment de gérer la plus grande rigidité des espaces de Sobolev d'ordre supérieur. Le cas $0 < s < 1$ fut quant à lui résolu par une méthode complémentaire, conceptuellement plus simple, mais strictement limitée à la gamme $0 < s < 1$ pour des raisons techniques.

La contribution de la présente thèse est d'apporter la pierre manquante à l'édifice, en résolvant le cas $s > 1$ non entier. Partant de la méthode des bons et mauvais cubes ainsi que des outils supplémentaires introduits par P. Bousquet, A. Ponce, et J. Van Schaftingen, auxquels sont ajoutées de nouvelles idées permettant notamment de gérer le caractère non local des espaces d'ordre fractionnaire, on montre que les applications lisses sont denses dans $W^{s,p}(\mathbb{B}^m; \mathscr{N})$ si et seulement si $\pi_{\lfloor sp \rfloor}(\mathscr{N}) = \{0\}$, et on établit la densité des applications presque lisses. La méthode est suffisamment robuste pour couvrir de façon unifiée la gamme complète $0 < s < +\infty$, y compris le cas $0 < s < 1$, résolu par H. Brezis et P. Mironescu *via* une autre stratégie, et pour couvrir également le cas d'un domaine arbitraire, dans l'esprit des travaux de Hang F. et Lin F. pour $s = 1$.

Dans le Chapitre 3, on pousse plus loin l'étude de la densité de la classe des fonctions presque lisses. En particulier, la définition de cette classe permet à l'ensemble singulier des fonctions qui la composent d'exhiber des croisements. Par ailleurs, les deux stratégies de démonstrations mentionnées plus haut produisent *effectivement* une approximation par des fonctions dont l'ensemble singulier possède des croisements. Une question, posée par H. Brezis et P. Mironescu dans leur livre, demande si on peut obtenir la densité de la classe plus réduite des fonctions lisses en dehors d'un ensemble singulier constitué d'*une seule* sous-variété de dimension $m - \lfloor sp \rfloor - 1$, ce qui interdit les croisements. Cette question est supportée par des réponses positives connues dans certains cas particuliers, notamment à l'aide de la *méthode de la projection singulière*, qui s'applique lorsque la topologie de la cible est suffisamment simple.

Dans cette thèse, on présente une nouvelle technique topologique de décroisement, qui donne la densité des fonctions presque lisses sans croisement dans $W^{s,p}(\mathbb{B}^m;\mathscr{N})$ pour une cible $\mathscr{N}$ arbitraire, répondant ainsi à la question de H. Brezis et P. Mironescu. Le cas d'un domaine général reste quant à lui ouvert.

Le Chapitre 4 constitue un *intermezzo*, dans lequel on emploie certaines des techniques développées et utilisées dans les chapitres qui précèdent pour résoudre un problème ayant trait à l'unicité des applications harmoniques minimisantes. Ceci correspond à un travail en collaboration avec K. Mazowiecka. Il s'agit d'un exemple d'une situation où *la boucle est bouclée* : l'étude des applications harmoniques minimisantes, utilisant les espaces de Sobolev entre variétés comme un cadre de travail, fut une des motivations pour l'étude des propriétés théoriques de ces derniers, et en retour, les outils développés pour ce faire permettent des progrès dans l'étude des problèmes de départ.

Après cet interlude, dans le Chapitre 5, on s'intéresse au problème de l'*approximation faible*. Plus précisément, la question est de savoir ce qu'il advient si, au lieu d'exiger d'avoir une approximation forte — c'est-à-dire par rapport à la convergence en norme — par des applications lisses, on requiert simplement une approximation *faible*. Ici, on dit qu'une suite converge faiblement dans $W^{s,p}$ lorsqu'elle est uniformément bornée dans $W^{s,p}$ et qu'elle converge presque partout. Lorsque $sp \notin \mathbb{N}_*$, une application $u \in W^{s,p}(\mathscr{M};\mathscr{N})$ est une limite faible d'applications lisses si et seulement si elle est une limite forte d'applications lisses, comme montré par F. Bethuel en 1991. Ainsi, dans ce cas, la réponse complète au problème de l'approximation faible est connue comme conséquence directe de la réponse correspondante pour le problème de la densité forte.

Le cas où $sp \in \mathbb{N}_*$ est quant à lui bien plus délicat. Il fut notamment observé par F. Bethuel en 1990 que la propriété d'approximation faible est valide dans $W^{1,2}(\mathbb{B}^3;\mathbb{S}^2)$, alors que la densité forte échoue. Une série de résultats partiels positifs vinrent complémenter cette observation, dus notamment à F. Bethuel, P. Hajłasz, R. Pakzad et T. Rivière, R. Hardt et T. Rivière, et P. Bousquet, A. Ponce, et J. Van Schaftingen. Cette succession de progrès dans la direction positive conduisit les experts du domaine à conjecturer que la propriété d'approximation faible devait être toujours valide lorsque $sp \in \mathbb{N}_*$, avant qu'un contre-exemple inattendu de F. Bethuel en 2020 n'apporte une réponse négative à cette conjecture : la propriété d'approximation faible échoue dans $W^{1,3}(\mathbb{B}^4;\mathbb{S}^2)$.

Dans cette thèse, on présente deux familles d'obstructions, montrant que ce phénomène n'est pas spécifique à $sp = 3$, mais que des obstructions existent pour toute valeur de $sp \in \mathbb{N} \setminus \{0,1\}$. Plus précisément, sous l'hypothèse que $s \geq 1$, (i) on construit, pour chaque nombre naturel $n \geq 3$, une variété cible $\mathscr{N}$ pour laquelle la propriété d'approximation faible échoue dans $W^{s,p}(\mathbb{B}^n; \mathscr{N})$ lorsque $sp = n-1$; et (ii) on montre que la propriété d'approximation faible échoue dans $W^{s,p}(\mathbb{B}^{4n}; \mathbb{S}^{2n})$ lorsque $sp = 4n-1$. En particulier, la seconde famille contient le contre-exemple de F. Bethuel comme un cas particulier lorsque $n = 1$ et $s = 1$. L'ingrédient clé de la démonstration est une construction périodique sur une grille, ainsi qu'un résultat précis de *bubbling* couplé à des estimations d'énergie tenant soigneusement compte de la contribution combinée de plusieurs singularités en différents centres de la grille. Ceci correspond à un travail en collaboration avec J. Van Schaftingen.

Finalement, le Chapitre 6 est dédié à la question de caractériser la clôture des applications lisses en l'absence de la propriété de densité forte. Autrement dit, là où le problème de la densité forte demande quelles sont les situations où *toutes* les applications de Sobolev sont des limites de fonctions lisses, ce dernier problème cherche un critère permettant de déterminer, pour *une* application de Sobolev u *donnée*, si u est ou non une limite d'applications lisses.

Dans cette entreprise, essentiellement deux directions émergent. La première se préoccupe de caractérisations dites *topologiques*, où l'approximabilité de u est déterminée selon le type d'homotopie de ses restrictions à des sous-ensembles *génériques* bien choisis, par exemple des sphères. Une telle direction a notamment été explorée dans la toute récente contribution de P. Bousquet, A. Ponce, et J. Van Schaftingen, où ils établissent un cadre rigoureux pour la notion de généricité, et énoncent et démontrent la caractérisation correspondante dans la gamme $s \in \mathbb{N}_*$.

L'avantage de cette première approche est qu'elle s'applique à une cible $\mathscr{N}$ arbitraire. En contrepartie, le résultat est une caractérisation souvent délicate à énoncer et à manipuler, du fait de sa nature topologique et de l'usage de notions de généricité. La seconde direction explore donc des caractérisations de type *analytique*. Typiquement, on cherche à construire une distribution associée à u, qui encode l'ensemble de ses singularités topologiques de façon robuste — ici, on entend que l'objet obtenu est stable par rapport à la convergence $W^{s,p}$ — avec l'espoir que l'objet ainsi obtenu s'annule si et seulement si u est une limite d'applications lisses. L'archétype de tels objets est donné par le *Jacobien*, dont la définition distributionnelle est attribuée à J. Ball et dont le premier usage en lien avec le problème de la densité forte remonte aux travaux de H. Brezis, J.-M. Coron, et E. Lieb, qui furent suivis de nombreuses autres contributions importantes poursuivant dans cette direction. L'inconvénient de cette seconde approche réside dans le fait qu'elle est la plupart du temps restreinte à des cibles particulières où, en termes vagues, l'homotopie peut être détectée par la cohomologie, ce qui permet de décrire les quantités topologiques pertinentes dans l'étude de ce problème par des objets intégraux.

Dans cette thèse, on se propose d'aborder ce problème dans la gamme $0 < s < 1$. Dans la première direction, on établit une caractérisation topologique de la clôture des applications lisses en termes de restrictions génériques sur des bords de cubes. Par rapport au cas d'espaces d'ordre supérieur, une telle entreprise est facilitée par l'utilisation de la stratégie de densité forte développée par H. Brezis et P. Mironescu, dont la construction rend plus aisé tant d'énoncer que de démontrer la caractérisation correspondante. En ce qui concerne les caractérisations analytiques, on présente soigneusement un cadre de travail pour l'étude d'*invariants intégraux* pour les applications de faible régularité, permettant notamment de définir de façon robuste des quantités construites à partir de tirés en arrière de formes différentielles par des applications $W^{s,p}$. La première difficulté en ce sens surgit dès la *définition* des objets en question, la définition classique requérant l'usage d'une dérivée entière de la fonction u, et devant donc être adaptée dans la gamme $0 < s < 1$. Les objets ainsi construits sont ensuite utilisés pour obtenir une caractérisation de la clôture des applications lisses dès que l'homotopie de la cible est décrite par sa cohomologie. Une telle démarche se place dans la lignée des travaux notamment de F. Bethuel, J.-M. Coron, F. Demengel, et F. Hélein, J. Bourgain, H. Brezis, et P. Mironescu, P. Bousquet et P. Mironescu, et D. Mucci. Ceci correspond à un travail en collaboration avec P. Mironescu et Xiao K.

Substantial summary in English

This thesis is concerned with the study of several properties of Sobolev spaces of mappings *between manifolds*. More specifically, a model problem on which we focus is the following:

> given a mapping $u \colon \mathscr{M} \to \mathscr{N}$ of $W^{s,p}$ regularity between two compact Riemannian manifolds $\mathscr{M}$ and $\mathscr{N}$, can u be approximated with smooth maps?

Already in the early 80's, a small community of mathematicians had understood the interest of considering Sobolev mappings *with values into a manifold*, notably motivated by their strong connections with problems coming from geometry, physics, or numerical methods. Although being defined as metric subspaces of classical Sobolev spaces of vector-valued functions, the spaces $W^{s,p}(\mathscr{M};\mathscr{N})$ exhibit *striking qualitative differences* with the former ones. A typical instance of such a difference is the following one, first observed by R. Schoen and K. Uhlenbeck in 1983: there exists a mapping $u \in W^{1,2}(\mathbb{B}^3;\mathbb{S}^2)$ which is not a limit of smooth mappings with values into $\mathbb{S}^2$.

Starting from this observation that, unlike what happens for the classical case of real (or vector) valued functions, smooth maps *need not* be dense in presence of a manifold constraint, a whole program of research was initiated, focusing notably on the following four questions:

(Q1) characterize those s, p, $\mathscr{M}$, and $\mathscr{N}$ for which strong density of smooth maps *does* occur;

(Q2) find a suitable class of *almost smooth maps* which is always dense in $W^{s,p}(\mathscr{M};\mathscr{N})$;

(Q3) when strong density fails, characterize the closure of smooth maps;

(Q4) determine what happens if strong convergence is replaced by a weaker notion.

This thesis aims at presenting a contribution in the study of each of these questions.

In Chapter 2, we investigate the first two questions, which constitute what is called the *strong density problem*. A seminal contribution in this direction is the work by F. Bethuel in 1991, where he gave a complete answer to this problem when $s = 1$ and the domain is a ball. More specifically, F. Bethuel showed that smooth maps are dense in $W^{1,p}(\mathbb{B}^m;\mathscr{N})$ *if and only if* $\pi_{\lfloor p \rfloor}(\mathscr{N})$ is trivial, and that the class of those $W^{1,p}(\mathbb{B}^m;\mathscr{N})$ that are smooth outside of a finite union of $(m - \lfloor p \rfloor - 1)$-dimensional submanifolds of $\mathbb{B}^m$

is always dense. The argument by F. Bethuel relies notably on the so-called *method of good and bad cubes*. The case of a general domain was understood by Hang F. and Lin F. in 2003, where they obtained the strong density of the same class of almost smooth maps, as well as a necessary and sufficient condition for the density of smooth maps, formulated only in terms of the topology of $\mathscr{M}$ and $\mathscr{N}$. This condition encompasses $\pi_{\lfloor p \rfloor}(\mathscr{N}) = \{0\}$, and is equivalent to it for instance if the domain is a ball.

After many important partial contributions, Bethuel's result was extended in full generality, for $0 < s < 1$ by H. Brezis and P. Mironescu, and for $s \in \mathbb{N}_*$ by P. Bousquet, A. Ponce, and J. Van Schaftingen. The case where $s \in \mathbb{N}_*$ was handled thanks to Bethuel's good and bad cubes method, supplemented with new tools, allowing notably to deal with the additional rigidity of higher order Sobolev spaces. The case $0 < s < 1$ was tackled via a complementary approach, conceptually simpler, but limited to the range $0 < s < 1$ for technical reasons.

The contribution of this thesis is to lay the final stone to the study of this problem, by solving the missing case $s > 1$ noninteger. Starting with the method of good and bad cubes by F. Bethuel and the additional tools by P. Bousquet, A. Ponce, and J. Van Schaftingen, that we complement with new ideas to handle notably the nonlocal character of fractional order spaces, we show that smooth maps are dense in $W^{s,p}(\mathbb{B}^m; \mathscr{N})$ if and only if $\pi_{\lfloor sp \rfloor}(\mathscr{N}) = \{0\}$, and we obtain the density of almost smooth maps. Our method is sufficiently robust so that it covers the full range $0 < s < +\infty$ with a unified approach, including the case where $0 < s < 1$, that was already solved by H. Brezis and P. Mironescu via an alternative approach, and so that it extends to the case of an arbitrary domain, in the spirit of the work by Hang F. and Lin F. for $s = 1$.

In Chapter 3, we extend further our study of the density of almost smooth maps. In particular, the definition of this class allows the singular set of the mappings that belong to it to exhibit crossings. Moreover, the two strategies of proof mentioned above yield approximating maps with a singular set which *does* exhibit crossings. A question, raised by H. Brezis and P. Mironescu in their book, asks whether the smaller class of mappings that are smooth outside of a singular set made of only *one* $(m - \lfloor sp \rfloor - 1)$-dimensional submanifold is dense in $W^{s,p}(\mathscr{M}; \mathscr{N})$, which in particular forbids crossings. This question is supported by some partial positive answers in several special cases, notably using the *method of singular projection*, which applies as soon as the target has a sufficiently simple topology. In this thesis, we introduce a new topological uncrossing strategy, which gives the strong density of the class of almost smooth maps without crossing in $W^{s,p}(\mathbb{B}^m; \mathscr{N})$ for a general target $\mathscr{N}$, hence answering Brezis and Mironescu's question. The case of a general domain remains open.

Chapter 4 is an intermezzo, in which we apply some techniques introduced and used in the previous chapters to study a problem related to uniqueness questions for minimizing harmonic maps. This corresponds to a joint work with K. Mazowiecka. It is an instance of a situation where *the circle is complete*: the study of minimizing harmonic maps, using Sobolev spaces of mappings as a framework, motivated the study of their theoretical properties, and in turn, the tools developed to do so can be used to make progress on the study of the original problems.

After this interlude, in Chapter 5, we turn to the *weak approximation problem*. More precisely, the question is what happens if, instead of requesting strong approximation — that is, with respect to the norm convergence — by smooth mappings, one simply requires a weak approximation. Here, we say that a sequence weakly converges in $W^{s,p}$ whenever it is uniformly bounded in $W^{s,p}$ and converges almost everywhere. When $sp \notin \mathbb{N}_*$, a mapping $u \in W^{s,p}(\mathscr{M};\mathscr{N})$ is a weak limit of smooth maps if and only if it is a strong limit of smooth maps, as shown by F. Bethuel in 1991. The complete answer to the weak approximation problem therefore follows from the corresponding answer to the strong density problem.

The case where $s \in \mathbb{N}_*$ is much more delicate. It was notably observed by F. Bethuel in 1990 that the weak approximation property holds in $W^{1,2}(\mathbb{B}^3;\mathbb{S}^2)$, while the strong density property fails. Several partial positive results complemented this observation, notably by F. Bethuel, P. Hajłasz, R. Pakzad and T. Rivière, R. Hardt and T. Rivière, and P. Bousquet, A. Ponce, and J. Van Schaftingen. This progress in the positive direction led the experts in the field to conjecture that the weak approximation property should always hold when $sp \in \mathbb{N}_*$, before a groundbreaking counterexample by F. Bethuel in 2020 gave a negative answer to this conjecture: the weak approximation property fails in $W^{1,3}(\mathbb{B}^4;\mathbb{S}^2)$.

In this thesis, we present two families of obstructions, showing that this phenomenon is not specific to $sp = 3$, but that obstructions arise for any value of $sp \in \mathbb{N} \setminus \{0,1\}$. More precisely, assuming that $s \geq 1$, (i) we construct, for every $n \geq 3$, a target manifold $\mathscr{N}$ such that the weak approximation property fails in $W^{s,p}(\mathbb{B}^n;\mathscr{N})$ when $sp = n-1$; and (ii) we show that the weak approximation property fails in $W^{s,p}(\mathbb{B}^{4n};\mathbb{S}^{2n})$ when $sp = 4n-1$. In particular, the second family contains Bethuel's counterexample as a special case when $n = 1$ and $s = 1$. The key ingredient in the proof is a periodic construction on a grid, along with a precise *bubbling* statement combined with careful joint energy estimates taking into account the contribution arising from different singularities in various centers of the grid. This corresponds to a joint work with J. Van Schaftingen.

Finally, Chapter 6 is devoted to the question of characterizing the closure of smooth maps when strong density fails. More specifically, while the strong density problem is concerned with the situations where *every* Sobolev map is a limit of smooth maps, this last problem is about devising criteria to determine, for a *given* Sobolev map u, whether or not u is a strong limit of smooth maps.

In this endeavor, essentially two directions emerge. The first one is concerned with *topological* characterizations, where the approximability of u is determined by its homotopy type when restricted to suitable *generic* subset, for instance spheres. Such a direction has notably been explored in the very recent contribution by P. Bousquet, A. Ponce, and J. Van Schaftingen, where they establish a rigorous framework for the notion of genericity, and state and prove the corresponding characterization in the range $s \in \mathbb{N}_*$.

The advantage of this first approach lies in the fact that it applies to an arbitrary target $\mathscr{N}$. On the other hand, the resulting characterization is often cumbersome to state and to manipulate, because of its topological nature and the use of notions of genericity. The second direction therefore explores characterizations of *analytical* nature. Typically, one tries to construct a distribution associated to u, which encodes the set of its topological singularities in a robust way — here, we mean that the object that is obtained is stable with respect to the $W^{s,p}$ convergence — with the hope that the resulting object vanishes if and only if u is a limit of smooth maps. A typical such object is the *Jacobian*, whose distributional definition is attributed to J. Ball, and whose first use in connection to strong density questions goes back to the work of H. Brezis, J.-M. Coron, and E. Lieb, which was followed by numerous other important contributions pursuing this direction of research. The drawback of this second approach is that it is most of the time restricted to special targets where, loosely speaking, the homotopy can be detected by the cohomology, allowing one to describe the relevant topological quantities by integral invariants.

In this thesis, we study this problem in the range $0 < s < 1$. In the first direction, we establish a topological characterization of the closure of smooth maps in terms of generic restrictions on boundaries of cubes. When compared to the case of higher order spaces, such an endeavor is made easier by the use of the strong density strategy devised by H. Brezis and P. Mironescu, whose construction facilitates both stating and proving the corresponding characterization. Concerning analytical characterizations, we carefully present a framework for the study of *integral invariants* for low regularity mappings, allowing notably to define in a robust way quantities constructed from pullbacks of differential forms by $W^{s,p}$ mappings. The first difficulty here arises already when *defining* those objects, as the usual definition requires one full derivative of u and therefore needs to be adapted in the range $0 < s < 1$. The objects constructed that way are then used to obtain a characterization of the closure of smooth mappings as soon as the homotopy of the target can be described by its cohomology. Such a program is in line with the work notably by F. Bethuel, J.-M. Coron, F. Demengel, and F. Hélein,

J. Bourgain, H. Brezis, and P. Mironescu, P. Bousquet and P. Mironescu, and D. Mucci. This corresponds to a joint work with P. Mironescu and Xiao K.

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Chapter 1

Introduction

Résumé

Dans cette introduction, on présente les objets et problèmes principaux qui sont étudiés dans ce travail, ainsi que les contributions principales de ce texte. Après une brève présentation des espaces de Sobolev classiques, avec un point de vue adapté aux problèmes présentés par la suite, on introduit et motive le concept d'application à valeurs dans une variété. Enfin, on présente les problèmes qui seront étudiés dans ce texte, avec pour point de départ l'observation de R. Schoen et K. Uhlenbeck selon laquelle les applications lisses ne sont pas toujours fortement denses dans les espaces de Sobolev à valeurs variété. Les problèmes sur lesquels on se concentre sont alors : (i) caractériser les situations où les applications lisses sont denses ; (ii) exhiber une classe convenable d'applications presques lisses qui est toujours dense ; (iii) lorsque la densité forte échoue, caractériser la clôture des applications lisses ; (iv) étudier le problème de la densité pour une notion de convergence plus faible que la convergence en norme.

Abstract

In this introduction, we present the main objects and problems that are studied in this work, as well as the main contributions of this text. After a short introduction to the concept of classical Sobolev spaces, with a viewpoint suited to the problems that are presented next, we introduce and motivate the concept of mapping with values into a manifold. Finally, we present the problems that are studied in this text, starting from the observation by R. Schoen and K. Uhlenbeck that smooth maps need not be strongly dense in Sobolev spaces of mappings into a manifold. The problems on which we focus are then: (i) characterize those instances where smooth maps are dense; (ii) find a suitable class of almost smooth maps that is always dense; (iii) when strong density fails, characterize the closure of smooth maps; (iv) study the density problem for a weaker notion of convergence than the norm convergence.

1.1 Overview

At the heart of this thesis lies the following model problem:

> given a mapping $u\colon \mathbb{B}^m \to \mathscr{N}$ of $W^{s,p}$ regularity with values into a compact Riemannian manifold $\mathscr{N}$, can u be approximated with smooth $\mathscr{N}$-valued maps?

Here, $0 < s < +\infty$ and $1 \leq p < +\infty$ are the Sobolev regularity and integrability parameters, $\mathbb{B}^m$ is the unit ball in $\mathbb{R}^m$, and we assume that the manifold $\mathscr{N}$ is isometrically embedded into $\mathbb{R}^\nu$, respectively, which is possible by virtue of the Nash embedding theorem [Nas54, Nas56]. It would also be possible to take the *domain* to be a compact Riemannian manifold $\mathscr{M}$ of dimension m, isometrically embedded into $\mathbb{R}^N$. For the sake of simplicity, and to emphasize the main contributions of this work, in this introduction, we will mostly restrict to *local* problems. At the end of this introduction, in Section 1.5, we will briefly explain the additional difficulties that arise when the domain is an arbitrary compact manifold, possibly with nontrivial topology.

This model problem stated above characterized by two key features: (i) the use of *Sobolev regularity*; and (ii) the use of mappings *with values into a manifold*.

Studying mappings between manifolds is naturally suggested by applications to problems coming, e.g., from physics or numerics, to take into account the geometry of the problem. A model example is the use of mappings with values into the sphere, which are natural objects to encode an information about orientation, for instance when modeling the arrangement of liquid crystals in physics or the attitudes of cubes in computer graphics. More details about the motivation for studying mappings into manifolds will be given in Section 1.3 of this introduction.

As far as Sobolev functions are concerned, their ubiquity in modern analysis is widely justified as they are a natural framework to investigate a broad range of problems, e.g., from calculus of variations or partial differential equations. This comes notably from the fact that, as a functions space, $W^{s,p}$ enjoys many desirable features, such as compactness properties or reflexivity (at least when $p > 1$), which are not shared by the spaces C^k of classically differentiable functions. This will be the topic of Section 1.2 in this introduction, where we recall and motivate one possible way of defining Sobolev spaces, with a point of view suited to our context.

As a trade-off, Sobolev functions can be harder to manipulate than smooth ones; for instance, they can be *everywhere discontinuous*. The approximation problem that we started with therefore asks if we can enjoy the best of both worlds: look for solutions to calculus of variations or partial differential equations problems in Sobolev spaces, by taking profit of their crucial properties for doing so, while working most of the time with smooth functions, much easier to manipulate, knowing that we can extend many of their properties to Sobolev functions by approximation arguments.

While the answer to this approximation problem is known to be positive for *real-valued* functions, the question is much more delicate for maps with values into a compact manifold. As first observed by R. Schoen and K. Uhlenbeck [SU83], obstructions to the approximation property may arise from the *topology* of the target manifold. Later on, F. Bethuel showed, in his seminal contribution [Bet91], that whether or not any map $u \in W^{1,p}(\mathbb{B}^m; \mathscr{N})$ can be strongly approximated with smooth $\mathscr{N}$-valued maps is *completely determined* by a topological property of $\mathscr{N}$, more precisely, by the triviality of the homotopy group $\pi_{\lfloor p \rfloor}(\mathscr{N})$.

This important result, far from closing the problem, paved the way for an important amount of subsequent research, opening the field of Sobolev mappings into manifolds as a research area on its own. This thesis is devoted to the study of some of the questions that originated from there, and that will be given a detailed presentation in Section 1.4 of this introduction.

As hinted just above, a pervasive feature in this field of research is the interplay between *topology* and *analysis*. This is apparent not only in the statements of most theorems in the area, which give topological conditions for analytical properties such as density to hold, but also in the proofs of those theorems. Typically, a challenging issue is to combine topological constructions, which are by nature *qualitative*, with analytical ones, which are on the contrary *quantitative*. For instance, whether or not a topological construction can be performed to satisfy a given estimate is often critical to decide whether a specific problem has a positive or negative answer. To illustrate this fact by some material from this thesis, let us compare Chapter 3 with Chapter 5. In Chapter 3, the fact that the uncrossing topological construction that we devise can be coupled with a Sobolev estimate is instrumental to yield the positive result contained in Theorem 3.1.2. On the contrary, in Chapter 5, the fact that the homotopical equivalence of the target manifold that we construct with a bouquet of spheres can *never* be achieved in a quantitative way is at the heart of the counterexample featured in Theorem 5.1.1.

A third discipline which is involved in the area of Sobolev mappings into manifolds and which interacts with analysis and topology is *Riemannian geometry*. This will be mainly featured in Chapter 6, where the use of notions such as *pullbacks of differential forms* and their action on *de Rham cohomology classes* will be instrumental in answering the problem of *detecting which Sobolev maps can be approximated with smooth maps*.

In the rest of this introduction, we try to present, in an as elementary as possible way, the main problems studied in this work, along with their context and motivation. This will also be the opportunity to present the notation that will be used in the whole document, and to recall any relevant concept. To do so, we start with a brief presentation of *classical* Sobolev spaces of real-valued functions, with a point of view suited to our context. We then pursue with some motivation for introducing maps *with values into a manifold*, and a typical problem that triggered their intensive study. We finally present the main problems that will be studied in this thesis, their context and motivation, the state of the art, and the contributions of the present work.

This introduction is intended to be accessible without any prerequisite on the theory of manifold-valued Sobolev mappings. In this spirit, we avoid presenting any technical argument. However, we present a few proofs of important results in model cases, whenever they make it possible to illustrate a key idea that will be used later on in the technical part of this text.

1.2 From a problem of calculus of variations to Sobolev spaces

This section aims at briefly introducing and motivating classical Sobolev spaces, for the readers not well-acquainted with this notion. The way we introduce them also eases the transition towards the main problems that we will study in this work. We do not aim at giving a detailed presentation: the reader may refer, e.g., to [Ada75, Bre11, Maz11, Wil22] for a thorough exposition of Sobolev spaces and their properties.

There would be many ways to start with in order to introduce Sobolev spaces. In this text, having in mind the developments we aim at, we choose to start from the following problem:

$$\text{find a function } u\colon \mathbb{B}^m \to \mathbb{R} \text{ minimizing } \int_{\mathbb{B}^m} |\mathrm{D}u|^p, \text{ and satisfying } u = g \text{ on } \partial\mathbb{B}^m, \tag{1.2.1}$$

for some prescribed boundary datum g.

Despite its apparent simplicity, this problem is of great importance, both as a typical question in *calculus of variations*, and by virtue of its ubiquity, e.g., in physics. Indeed, this problem is related to the *least action principle* in physics, according to which nature always tries to choose, among all possible configurations of a system, the one which minimizes the energy. For instance, if $u(x)$ represents the quantity of heat at the point x, and if the temperature on the boundary is kept constant in time and represented by g, then the observed configuration of heat will be a solution of the above minimization problem. Similarly, such a model can be used to describe the electrostatic potential in a region without charge, or the gravitational field in a region without mass.

A detail that we left aside above is the set among which the minimization is performed.

Given the formulation of our problem, we need to be able to define the derivative of the functions under consideration and to integrate them. A natural candidate for a suitable set of competitors would then be the space $C^1(\overline{\mathbb{B}^m})$ of functions on $\mathbb{B}^m$ that are continuously differentiable up to the boundary. If one is willing to find a solution to the minimization problem (1.2.1), or even to prove the existence of a solution, the most classical way to proceed is via the *direct method of the calculus of variations*. Namely, one lets

$$c = \inf \int_{\mathbb{B}^m} |\mathrm{D}u|^p,$$

where the infimum runs over the set of all admissible competitors, one lets $(u_n)_{n\in\mathbb{N}}$ be a sequence of $C^1(\overline{\mathbb{B}^m})$ functions with $u_n = g$ on $\partial\mathbb{B}^m$ such that

$$\int_{\mathbb{B}^m} |\mathrm{D}u_n|^p \to c,$$

which is called a *minimizing sequence*, and one tries to prove that the sequence $(u_n)_{n\in\mathbb{N}}$ converges to a function u which is a minimizer.

To implement this strategy, one first needs to select the appropriate notion of convergence to work with. For this purpose, we endow $C^1(\overline{\mathbb{B}^m})$ with the norm

$$\|u\|_{W^{1,p}(\mathbb{B}^m)} = \left(\int_{\mathbb{B}^m} |u|^p + \int_{\mathbb{B}^m} |\mathrm{D}u|^p\right)^{\frac{1}{p}}.$$

This is a natural candidate to work with, as the assumption that the sequence $(u_n)_{n\in\mathbb{N}}$ is minimizing immediately provides a bound on the second integral along the sequence. The first term, involving the L^p norm of the function itself, ensures that the quantity we have defined is indeed a norm — otherwise it would vanish on constant functions — and is controlled along a minimizing sequence thanks to the boundary condition, by virtue of a Poincaré inequality; see, e.g., [Bre11, Corollary 9.19] or [Wil22, Theorem 6.4.7]. Therefore, any minimizing sequence for the problem (1.2.1) is bounded in the $W^{1,p}$ norm. One would only need to obtain some compactness to reach the desired conclusion.

However, the main issue here is that $C^1(\overline{\mathbb{B}^m})$ is not even *complete* with respect to the $W^{1,p}$ norm. For instance, it is easy to show that the sequence $(v_n)_{n\in\mathbb{N}}$ in $C^1(\overline{\mathbb{B}^m})$ defined by

$$v_n(x) = \sqrt{|x|^2 + n^{-1}}$$

converges in L^p to the function v given by $v(x) = |x|$, which is not differentiable at 0.

Therefore, $(v_n)_{n\in\mathbb{N}}$ cannot converge to a limit in the $W^{1,p}$ norm in the space $C^1(\overline{\mathbb{B}^m})$.

This observation is the motivation to define Sobolev spaces. Namely, we let $W^{1,p}(\mathbb{B}^m)$ be the completion of $C^1(\overline{\mathbb{B}^m})$ under the $W^{1,p}$ norm. With this definition, elements in $W^{1,p}(\mathbb{B}^m)$ are equivalence classes of Cauchy sequences in $C^1(\overline{\mathbb{B}^m})$. However, by virtue of the fact that L^p is complete, one shows that any element in $W^{1,p}(\mathbb{B}^m)$ is uniquely represented by a function $u \in L^p(\mathbb{B}^m)$, which is the common limit of all the Cauchy sequences in the associated equivalence class. Moreover, by the same argument, one shows that the corresponding Cauchy sequences of the derivatives also converge to a common limit in L^p, that we denote by $\mathrm{D}u$. By passing to the limit in the integration by parts formula, one finds the fundamental relation

$$\int_{\mathbb{B}^m} u\mathrm{D}\varphi = -\int_{\mathbb{B}^m} \varphi\mathrm{D}u \quad \text{for every } \varphi \in C^\infty_{\mathrm{c}}(\mathbb{B}^m). \tag{1.2.2}$$

Actually, this integration by parts formula is usually used to *define* $W^{1,p}$: one says that an L^p function u belongs to $W^{1,p}$ whenever there exists a function $\mathrm{D}u$ in L^p such that the relation (1.2.2) holds, in which case this function is unique, and called the *weak derivative* of u. The fact that $W^{1,p}$ coincides with the closure of C^1 functions up to the boundary is then a theorem, called the *density theorem*; see e.g. [Bre11, Corollary 9.8] or [Wil22, Theorem 6.3.2]. In this text, we chose to *define* $W^{1,p}$ as the completion of C^1 functions, for the sake of pedagogy, but most importantly to ease the connection with the problems that will be presented later on.

To come back to the minimization problem (1.2.1), it turns out that the space $W^{1,p}$ that we just defined is the suitable framework to work with in order to implement the direct method of the calculus of variations. Indeed, by moving from C^1 to $W^{1,p}$, we now have at our disposal a compactness theorem, which ensures that from any bounded sequence $(v_n)_{n\in\mathbb{N}}$ in $W^{1,p}$, one can extract a subsequence $(v_{n_k})_{k\in\mathbb{N}}$ that converges in L^p to a limiting function v which belongs to $W^{1,p}$. Moreover, if $1 < p < +\infty$, it holds that

$$\int_{\mathbb{B}^m} |\mathrm{D}v|^p \leq \liminf_{k\to+\infty} \int_{\mathbb{B}^m} |\mathrm{D}v_{n_k}|^p.$$

We refer the reader, e.g., to [Wil22, Theorem 6.4.6 and Theorem 6.1.7] for a proof of both these properties. Applied to our minimizing sequence, one deduces that the limiting function $u \in W^{1,p}(\mathbb{B}^m)$ is a minimizer of the problem (1.2.1). Then, there is a whole area called *regularity theory*, whose endeavor is to prove that the minimizer u is actually more regular than merely $W^{1,p}$, in order to deduce the existence of a minimizer in a nicer class of functions. In the present situation, one may show that the minimizer is actually smooth. However, we shall not expand further on these matters in this introduction.

Although the above motivational problem is focused on functions with one derivative defined on the unit ball $\mathbb{B}^m$, one may define Sobolev spaces of higher regularity and on more general domains. More precisely, given $k \in \mathbb{N}_*$, we define the norm

$$\|u\|_{W^{k,p}(\mathscr{M})} = \left(\sum_{j=0}^{k} \int_{\mathscr{M}} |\mathrm{D}^j u|^p\right)^{\frac{1}{p}},$$

and the corresponding Sobolev space $W^{k,p}(\mathscr{M})$ can be defined as the completion of $C^\infty(\mathscr{M})$ with respect to this norm.

It is also of interest to define spaces of *intermediate* regularity, described by any positive *real* parameter. Given $\sigma \in (0,1)$, we define the *Gagliardo seminorm* as

$$|u|_{W^{\sigma,p}(\mathscr{M})} = \left(\int_{\mathscr{M}} \int_{\mathscr{M}} \frac{|u(x) - u(y)|^p}{\operatorname{dist}(x,y)^{m+\sigma p}} \,\mathrm{d}x \,\mathrm{d}y\right)^{\frac{1}{p}}.$$

Then, if $0 < s < +\infty$ is not an integer, we decompose $s = k + \sigma$ with $k \in \mathbb{N}$ and $\sigma \in (0,1)$, and we define the Sobolev norm

$$\|u\|_{W^{s,p}(\mathscr{M})} = \left(\|u\|^p_{W^{k,p}(\mathscr{M})} + |u|^p_{W^{\sigma,p}(\mathscr{M})}\right)^{\frac{1}{p}}.$$

Once again, the corresponding Sobolev space $W^{s,p}(\mathscr{M})$ can be defined as the completion of $C^\infty(\mathscr{M})$ with respect to this norm.

Apart from the fact that fractional Sobolev spaces provide a continuous scale of regularity between integer order spaces, a notion which is formalized in *interpolation theory*, the historical interest for studying them also comes from *trace theory*.

To start with, let us come back to our model problem (1.2.1). One last point that we did not make precise is the regularity which is allowed for the boundary datum g. In order for the minimization problem to be well-posed, it is necessary that the set of competitors is nonempty. Otherwise stated, there should exist at least one $W^{1,p}(\mathbb{B}^m)$ function u such that $u = g$ on $\partial\mathbb{B}^m$. A first issue comes from the meaning to give to the condition $u = g$ on the boundary. Indeed, $\partial\mathscr{M}$ is a null set, while Sobolev functions, being essentially L^p functions with some extra regularity, are defined only up to a null set. Therefore, it is not clear what should be the appropriate definition of u on $\partial\mathscr{M}$.

This is where *trace theory* comes into play. Indeed, the trace theorem asserts that there is a continuous linear operator $W^{1,p}(\mathscr{M}) \to L^p(\partial\mathscr{M})$, called the *trace operator* and denoted tr, which coincides with the restriction on the boundary for $C^1(\mathscr{M})$ functions; see, e.g., [Bre11, Lemma 9.9] or [Wil22, Theorem 6.3.4]. Therefore, the condition $u = g$ on the boundary should be interpreted as $\operatorname{tr} u = g$.

However, this does not settle the issue of the nonemptyness of the set of admissible competitors. In other words, a valid boundary datum should be a map g which belongs to the range of the trace operator. As a matter of fact, the trace operator is not surjective when the codomain is taken to be L^p. An important theorem due to E. Gagliardo [Gag57] asserts that the right target to use is the fractional Sobolev space $W^{1-1/p,p}(\mathscr{M})$: the trace operator maps $W^{1,p}(\mathscr{M})$ to $W^{1-1/p,p}(\partial\mathscr{M})$, and there exists a continuous linear extension operator $\mathrm{Ext}\colon W^{1-1/p,p}(\partial\mathscr{M}) \to W^{1,p}(\mathscr{M})$ which is a right inverse for the trace operator. We mention that, when $p = 1$, then $1 - 1/p = 0$, and therefore the above statement has to be slightly modified. In this case, the trace operator is surjective into $L^1(\partial\mathscr{M})$ — the result is also due to E. Gagliardo [Gag57], and a very elegant alternative proof can be found in [Mir15] — but it is not possible to find a continuous *linear* right inverse [Pee79].

The conclusion of the above discussion is that fractional Sobolev spaces do not only provide one with a continuous scale of regularity between integer order Sobolev spaces, but they also are the appropriate framework to use when dealing with problems involving boundary data.

1.3 Problems with a geometric flavor: mappings to manifolds

In the previous section, we gave some motivation to illustrate how Sobolev spaces are a natural framework to study for example problems in calculus of variations. In this section, we introduce and motivate the main object of study in this text, namely Sobolev spaces of mappings *with values into manifolds*.

Let us come back to our motivation problem (1.2.1). As we explained, it is connected to the least action principle, with applications in various areas from physics for instance. However, such problems may exhibit some geometric flavor, which needs to be taken into account in the model. As a typical example, let us consider the modeling of a field of *liquid crystals*.

Namely, one is willing to describe the configuration adopted by a field of liquid crystals located in the unit ball $\mathbb{B}^3$, and which are forced to adopt a given configuration on the boundary $\partial\mathbb{B}^3$, described by a map g. The feature that we wish to describe is the direction of the crystals at each point, which suggests to consider *vector-valued* mappings $u\colon \mathbb{B}^3 \to \mathbb{R}^3$, so that $u(x)$ is a vector indicating the direction taken by the field at the point x. However, this choice is not totally accurate. Indeed, a vector in $\mathbb{R}^3$ carries at the same time a *direction* and a *length* information, the latter being irrelevant to our situation. To obtain only a direction information, one is therefore led to work with *sphere-valued* mappings $u\colon \mathbb{B}^3 \to \mathbb{S}^2$.

The corresponding least action principle then dictates that the effective configuration should be a solution of the problem

$$\min\left\{\int_{\mathbb{B}^3} |\mathrm{D}u|^p : u \in W^{1,p}(\mathbb{B}^3;\mathbb{S}^2),\ \operatorname{tr} u = g\right\}, \tag{1.3.1}$$

where g is some fixed $W^{1-1/p,p}(\partial\mathbb{B}^3;\mathbb{S}^2)$ map. As a matter of fact, this model corresponds exactly to a particular case of the so-called *Oseen–Frank* model, which is in use to describe the physics of liquid crystals. We refer the reader to [HK88] for a nice presentation of this topic.

This problem belongs more generally to the field of *ordered media physics*, which provides various other natural targets that arise in applications. Again when modeling liquid crystals, one might be willing to make the description even more accurate by taking into account the fact that the crystals are not oriented, that is, $-u(x)$ describes the same configuration as $u(x)$. For this purpose, it is natural to work instead with the target being the 2-dimensional projective plane $\mathbb{RP}^2$ which is precisely constructed from $\mathbb{S}^2$ by identifying all pairs of antipodal points together. Other physical models featuring more involved target manifolds comprise *biaxial liquid crystals*, having two preferred directions, and which may be described by mappings with values into $\mathbb{S}^3/H$, with H being the finite group of quaternions; and also several phases of superfluid helium, involving targets built upon groups of rotations. We refer the reader to [Mer79] for a complete survey of this topic, and to [BC07] for a short presentation.

Another similar direction of research that has attracted a lot of interest recently is concerned with the study of *heterogeneous materials*, namely, composite bodies formed by several different materials, typically two with one of them arranged in the other one according to a periodic structure. One may then study the physical properties of the material.

For instance, if the materials have ferromagnetic properties, one may then try to describe the state of the composite when subjected to an external magnetic field. In this situation, physical models assert that the appropriate quantity to study is the so-called *magnetization field*, which is a vector field M encoding the relevant magnetic properties of the body. But, a principle from micromagnetics, called the *fundamental constraint of micromagnetics theory*, asserts that the magnetization field M is saturated, which means that $|M| = M_0$, for some positive constant M_0 depending on the temperature of the body. To summarize, up to normalization, the relevant quantity to study such a model is again a mapping with values into the sphere $\mathbb{S}^2$.

Once more, by virtue of a least action principle postulated by the physical models, the actual configuration that will be observed may be obtained as a minimizer of a calculus of variations problem similar to (1.3.1). More precisely, the quantity in (1.3.1) would account for some part of the energy to be minimized, called the *internal energy*, and one would need to complete the functional with additional terms to account for the contribution of the other physically relevant quantities.

We refer the reader to [ADF15] for one typical instance of the type of research which is performed in this direction, as well as to [GHP24, DH24] for very recent developments that build a bridge between the theory of Sobolev mappings into manifolds and the study of heterogeneous materials, relying on several theoretical tools that will show up later on in this work, such as the *method of singular projection*.

We conclude this short exposition with an example from a totally different area, namely, numerical methods. Also in this discipline, mappings to manifolds have proved their usefulness, notably for their ability to encode an orientation when the target is chosen to be a sphere, a projective plane, or a group of rotations. We refer the reader to [HTWB11] for such an application to the description of the attitudes of a cube for computer graphics, or to the *Hextreme* project [1] for applications to the generation of meshes for finite elements methods. In the latter project, nice meshes were notably generated through solving a suitable Ginzburg–Landau-type problem, which is once again a calculus of variations problem with a geometric constraint in the spirit of (1.3.1).

We do not claim here to give an exhaustive survey of possible applications of mappings to manifolds, but only a few, hopefully representative, examples. Other instances would include the study of deformations in *Cosserat materials* [ET58], involving mappings into $\mathbb{R}^3 \times SO(3)$, or the study of the Ginzburg–Landau functional, which constitutes a whole area of research on its own, and involves notably the study of $\mathbb{S}^1$-valued maps.

The common feature in all these examples is that they involve mappings whose target is a Riemannian manifold, most of the time compact — with the notable exception of Cosserat materials, where the target nevertheless boils down to a product of a compact manifold with a Euclidean space. This suggests to investigate the more general framework of the space $W^{s,p}(\mathscr{M};\mathscr{N})$ of $W^{s,p}$ mappings from $\mathscr{M}$ to $\mathscr{N}$, where $0 < s < +\infty$ and $1 \leq p < +\infty$ are real numbers, and $\mathscr{M}$ and $\mathscr{N}$ are compact Riemannian manifolds, isometrically embedded into $\mathbb{R}^N$ and $\mathbb{R}^\nu$, respectively. As we work with an embedded target, it is straightforward to define the space of mappings into $\mathscr{N}$, letting

$$W^{s,p}(\mathscr{M};\mathscr{N}) = \{u \in W^{s,p}(\mathscr{M};\mathbb{R}^\nu)\colon u(x) \in \mathscr{N} \text{ for a.e. } x \in \mathscr{M}\},$$

as already explained at the beginning of this introduction.

For the record, we mention that this definition is called *extrinsic*, as it relies on the target being embedded. Some research has been performed to define and study *intrinsic* Sobolev spaces of mappings. We shall not expand on these matters in this work, and

1. https://www.hextreme.eu/

we refer the reader to the work of A. Convent and J. Van Schaftingen, see, e.g., [CVS16]. We simply mention that, if either $s = 1$ or the target is compact, then both definitions coincide. In this thesis, we shall work with the extrinsic definition without any further comment.

1.4 The density problem

1.4.1 The genesis: harmonic maps

Despite being defined as subsets of classical Sobolev spaces of vector-valued functions, Sobolev spaces of mappings into manifolds may exhibit *striking qualitative differences* in comparison with the former ones. The reason behind this fact is the nonlinearity induced by the constraint that the maps should take their values into a compact manifold. Because of this, most of the constructions that are usual in the theory of linear Sobolev spaces cannot be implemented as such in presence of the geometric constraint. For instance, convolutions or convex combinations of mappings into a given manifold *need not* take their values into the manifold.

For this reason, and motivated by the numerous applications, some of which having been presented in Section 1.3, an important amount of research was devoted to the understanding of problems revolving around the spaces $W^{s,p}(\mathscr{M};\mathscr{N})$ in the last decades. At first, this research was mainly focused on the study of (minimizing) harmonic maps, in the line of the model problem (1.3.1) that we presented previously. There, Sobolev spaces of mappings were the framework in which the research took place, rather than the main object of study. Pioneering contributions in this direction inlcude notably, but not only, the work of J. Eells, S. Hildebrandt, J. Jost, H. Kaul, L. Lemaire, J. Sampson, and K.-O. Widman [ES64, HKW77, EL78, Jos84].

To illustrate the issues at stake, let us come back once again to our initial, linear, model problem (1.2.1). For simplicity, let us further assume that $p = 2$. It is well-established that the minimizers of such a problem enjoy three fundamental properties: existence, uniqueness, and regularity. More precisely, for any g in the suitable trace space, there exists one and only one $W^{1,2}$ minimizer to the problem, which is C^{∞} in the interior of the domain. The existence follows from the direct method of the calculus of variations, as we explained in Section 1.2, the uniqueness follows from the strict convexity of the functional being minimized, while the regularity follows from the fact that the associated *Euler–Lagrange equation* satisfied by the minimizer is $\Delta u = 0$, which is the prototypical example of an elliptic equation for which there is a standard and powerful regularity theory.

Things are much more complicated in the presence of a geometric constraint: existence, uniqueness, and regularity fail in general. A first issue comes from the very existence of an admissible competitor. Indeed, it needs no longer be true that the trace operator is surjective into the space of $W^{1-1/p,p}$ maps on the boundary in the context of mappings into a manifold; the answer depends on the target $\mathscr{N}$. This was notably studied in the seminal contribution of R. Hardt and Lin F. [HL87], and gave rise to the so-called *extension problem*, see notably [BD95, Bet14, MVS21b, VS24]. In general, the uniqueness is lost as well due to the fact that the geometric constraint destroys the convex character of the problem. Starting in the late 80's, several more or less dramatic failures of the uniqueness of minimizers were exhibited. This shall be the topic of Chapter 4, where we will give a more precise list of references. Concerning regularity, again due to the nonlinearity induced by the geometric constraint, the Euler–Lagrange equation verified by the minimizer is modified. For $W^{1,2}$ mappings into the sphere, for instance, it now writes $-\Delta u = |Du|^2 u$. This is a highly nonlinear equation, which compromises the regularity of the solutions. It was already observed that this equation *may indeed* admit discontinuous solutions by S. Hildebrandt, H. Kaul, and K.-O. Widman [HKW77]: the map $\mathbb{B}^3 \to \mathbb{S}^2$, $x \mapsto \frac{x}{|x|}$ is a solution with an essential discontinuity at 0. It was later proved by H. Brezis, J.-M. Coron, and E. Lieb [BCL86] that it is even a minimizer of the $W^{1,2}$ energy with boundary datum $g = \mathrm{id}$. For the record, we mention that the study of the minimizing properties of this map — which is sometimes nicknamed as *the hedgehog* — for more general dimensions of the domain and target and different values of p received a special attention, even recently; see e.g. [JK83, Lin87, AL88, CG89, HLW98, Hon01, MM23, Nak24].

Considering these observations, an important effort was accomplished towards the study of the regularity of minimizing harmonic maps. After some partial results, in two highly influential contributions [SU82, SU83], R. Schoen and K. Uhlenbeck presented a very general regularity theory for harmonic maps in $W^{1,p}(\mathscr{M};\mathscr{N})$. We shall not try to give a precise description of their results here, and we limit ourselves to the following loose principle: minimizing harmonic maps are smooth in their domain, except on some singular set, depending on the map, and whose Hausdorff dimension can be upper bounded depending on m and p.

In their second paper, R. Schoen and K. Uhlenbeck made what was at that moment a mere remark, but which was going to give rise to a new direction of research, where Sobolev spaces of mappings into manifolds would be the main object of study, and not only a suitable framework to investigate other problems. The next section explores their remark in detail.

1.4.2 Schoen and Uhlenbeck's observation

To state the remark by R. Schoen and K. Uhlenbeck, let us come back to the process that we used to define Sobolev mappings to manifolds. The definition of the space $W^{s,p}(\mathscr{M};\mathscr{N})$ was in two steps: we defined it as a subset of the linear space $W^{s,p}(\mathscr{M};\mathbb{R}^{\nu})$, which was itself defined as the closure of $C^{\infty}(\mathscr{M};\mathscr{N})$ under the $W^{s,p}$ norm. However, we could have used an alternative path, in one step, by considering the closure of $C^{\infty}(\mathscr{M};\mathscr{N})$ under the $W^{s,p}$ norm. Let us denote by $H^{s,p}_{S}(\mathscr{M};\mathscr{N})$ the space obtained in this fashion. (The letter H refers to the so-called $H = W$ theorem by N. Meyers and J. Serrin [MS64].) It is clear that

$$H^{s,p}_{S}(\mathscr{M};\mathscr{N}) \subset W^{s,p}(\mathscr{M};\mathscr{N}). \tag{1.4.1}$$

When the target is $\mathbb{R}$ — or more generally $\mathbb{R}^{\nu}$ — both spaces coincide: $H^{s,p}_{S}(\mathscr{M};\mathbb{R}^{\nu}) = W^{s,p}(\mathscr{M};\mathbb{R}^{\nu})$. (In this text, it is merely the definition of $W^{s,p}(\mathscr{M};\mathbb{R}^{\nu})$. With the usual definition, this is the content of the *density theorem*, as we already explained.) A natural question is whether both constructions always lead to the same space or not, that is, if it does always hold that $H^{s,p}_{S}(\mathscr{M};\mathscr{N}) = W^{s,p}(\mathscr{M};\mathscr{N})$, irrespective of $\mathscr{N}$.

The seminal observation by R. Schoen and K. Uhlenbeck is that the inclusion in (1.4.1) *may indeed* fail to be an equality. Once again, this is one of the striking examples that, although being defined as a subset of a linear Sobolev space, the space $W^{s,p}(\mathscr{M};\mathscr{N})$ may have qualitatively different properties than the former, and that the reader well-acquainted with the theory of linear Sobolev spaces should definitely expect surprises!

More precisely, consider again the hedgehog map $u_0\colon \mathbb{B}^3 \to \mathbb{S}^2$ defined by $u_0(x) = \frac{x}{|x|}$. Then, R. Schoen and K. Uhlenbeck obtained the following result.

Proposition 1.4.1. *If $2 \leq p < 3$, then $u_0 \in W^{1,p}(\mathbb{B}^3;\mathbb{S}^2) \setminus H^{1,p}_{S}(\mathbb{B}^3;\mathbb{S}^2)$.*

Proof. The fact that $u_0 \in W^{1,p}(\mathbb{B}^3;\mathbb{S}^2)$ is standard, relying on the estimate

$$|\mathrm{D}u_0(x)| \lesssim \frac{1}{|x|}.$$

Assume by contradiction that $u_0 \in H^{1,p}_{S}(\mathbb{B}^3;\mathbb{S}^2)$. Therefore, there exist maps $u_\eta \in C^{\infty}(\overline{\mathbb{B}^3};\mathbb{S}^2)$ such that $u_\eta \to u_0$ as $\eta \to 0$. By a genericity argument, up to extraction of a subsequence, it holds that

$$u_{\eta|\partial B^3_r} \to u_{0|\partial B^3_r} \quad \text{in } W^{1,p}\text{, for almost every } r \in (0,1).$$

The proof of this claim essentially boils down to the Fubini–Tonelli theorem, or more precisely, the polar integration formula.

The gain of proceeding to this reduction of dimension is that, now, we are working with mappings defined on a domain with dimension less that the regularity parameter p. A crucial feature in this range is that Sobolev mappings are more regular that expected a priori: they are essentially continuous. More precisely, assuming that actually $2 < p < 3$, the Morrey–Sobolev embedding theorem, see, e.g., [Bre11, Theorem 9.12] or [Wil22, Lemma 6.4.3], ensures that $u_{\eta|\partial B^3_r} \to u_{0|\partial B^3_r}$ uniformly. We conclude by observing that, by construction, the maps $u_{\eta|\partial B^3_r}$ are the restriction to ∂B^3_r of continuous maps defined on the whole $\overline{B^3_r}$, and are henceforth nullhomotopic. On the other hand, the map $u_{0|\partial B^3_r}$ is essentially the identity on $\mathbb{S}^2$, which is not nullhomotopic. We obtain the desired contradiction, as homotopy classes are stable under uniform convergence.

This concludes the proof when $2 < p < 3$. In the limiting case $p = 2$, the argument follows the same scheme, with the exception that the Morrey–Sobolev embedding fails, so that we cannot obtain uniform convergence anymore. It is nevertheless possible to show that the $W^{1,2}$ convergence in dimension 2 still preserves homotopy classes, and to conclude the proof as above. Roughly speaking, this relies on the fact that, when p coincides with the dimension of the domain, $W^{1,p}$ is not embedded in C^0, but in a space very close to it. Such considerations, revolving around the study of robust homotopical invariants for Sobolev mappings, will be the main topic of Chapter 6, and we omit the details for the moment. □

For the reader unfamiliar with the notions of topology used in the proof that precedes, and to fix the terminology in use, let us briefly recall the notions of topology that were used above. Given two topological spaces X and Y, we say that two continuous maps $f, g \colon X \to Y$ are *homotopic*, and we write $f \sim g$, whenever there exists a continuous map $H \colon X \times [0,1] \to Y$ such that $H_0 = f$ and $H_1 = g$, where we denote $H_t = H(\cdot, t)$. Such a map is called a *homotopy* from f to g. A map is said to be *nullhomotopic* when it is homotopic to a constant map. Being homotopic is an equivalence relation on the set of continuous maps between two given spaces, and the homotopy class of a map f is denoted by $[f]$.

For a continuous map $f \colon \mathbb{S}^l \to Y$, it is equivalent to be nullhomotopic and to admit a continuous extension $h \colon \overline{\mathbb{B}^{l+1}} \to Y$. This can be seen by associating to such an extension the homotopy H defined by $H_t(x) = h((1-t)x)$, and vice-versa.

When Y is a compact embedded manifold, a very important property of homotopy classes is that they are stable under uniform convergence. More precisely, there is a positive number $\delta_{\mathscr{N}} > 0$, depending only on $\mathscr{N}$, such that whenever $\|f - g\|_{L^\infty} \leq \delta_{\mathscr{N}}$, then $f \sim g$. This is a straightforward consequence of the following statement.

Proposition 1.4.2. *There exists a positive number $\iota > 0$ depending only on $\mathscr{N}$ such that, if $\mathscr{N}_\iota = \mathscr{N} + B_\iota$, then there exists a well-defined smooth map $\Pi_{\mathscr{N}} \colon \mathscr{N}_\iota \to \mathscr{N}$ such that $\Pi_{\mathscr{N}} = \mathrm{id}_{\mathscr{N}}$ on*

$\mathscr{N}$. The set $\mathscr{N}_\iota$ is called a tubular neighborhood *of $\mathscr{N}$, and $\Pi_{\mathscr{N}}$ is called a* smooth retraction *onto $\mathscr{N}$.*

Despite being very classical in differential geometry, it seems difficult to find a proof in the literature. We refer the reader to [Foo84].

Actually, the existence of such a retraction Π defined on a uniform neighborhood of the target is the only assumption needed to obtain the stability of homotopy classes under uniform convergence. Indeed, for sufficiently close continuous maps f and g, a homotopy is then constructed by letting $H_t = \Pi((1-t)f + tg)$.

There is something remarkable about the obstruction highlighted by Proposition 1.4.1: even though the question comes from analysis, namely the density of one space into another, the obstruction comes from the *topology* of the target. More precisely, at the heart of the counterexample lies the fact that the $W^{1,p}$ regularity in this situation is at the same time sufficiently mild to allow for singularities around which the map realizes nontrivial homotopy classes, while being at the same time sufficiently strong to preserve homotopy classes around a point through strong convergence, hence ruling out the density of smooth maps, the latter not being allowed to display topological singularities.

When being faced with the failure of equality in (1.4.1) in some special situations, the following questions arise naturally.

(Q1) Characterize those s, p, $\mathscr{M}$, and $\mathscr{N}$ for which strong density of smooth maps *does* occur.

(Q2) Find a suitable class of *almost smooth maps* which is always dense in $W^{s,p}(\mathscr{M};\mathscr{N})$.

(Q3) When strong density fails, characterize the space $H^{s,p}_S(\mathscr{M};\mathscr{N})$.

(Q4) What happens if strong convergence is replaced by a weaker notion?

These questions are at the heart of this work, which aims at presenting contributions in each of these four directions of research. In the next sections, we explain these problems into more details, presenting the state of the art in each of these directions, before we conclude this introduction with a plan of this text as well as the main contributions it contains.

1.4.3 The strong density problem

In the two next sections, we explore into more detail (Q1) and (Q2). As we will see, these questions are strongly tied together, which makes studying them simultaneously natural.

To understand what is the general phenomenon at work that should be expected, let us take a more careful look at the counterexample by R. Schoen and K. Uhlenbeck,

$u_0(x) = \frac{x}{|x|}$. Here, two comments are in order: (i) the map only has an essential discontinuity at one point, and is smooth everywhere else; and (ii) the obstruction for approximating u_0 strongly by smooth maps comes from the fact that it realizes a topologically nontrivial map $\mathbb{S}^2 \to \mathbb{S}^2$ around its singularity.

The second observation is readily turned into a general obstruction to strong density. For this purpose, let us introduce one more classical notion from topology. Given $j \in \mathbb{N}_*$, the *j-th homotopy group* of a topological space Y is the set of all homotopy classes of continuous maps $\mathbb{S}^j \to Y$, and it is denoted by $\pi_j(Y)$. As the name suggests, it can be endowed with a group structure, but we shall not make use of this in most of this text, so we omit the definition. Actually, for most of this work, it suffices to know what it means for a homotopy group to be trivial: $\pi_j(Y) = \{0\}$ whenever every continuous map $f\colon \mathbb{S}^j \to Y$ is homotopic to a constant map, or equivalently, whenever any such map admits a continuous extension $g\colon \overline{\mathbb{B}^{j+1}} \to Y$.

The key feature in Proposition 1.4.1 is the fact that $\pi_2(\mathbb{S}^2)$ is nontrivial, and more precisely, that $\mathrm{id}_{\mathbb{S}^2}$ is a nontrivial element of this group. For a more general target $\mathscr{N}$, if $\pi_d(\mathscr{N}) \neq \{0\}$, then we may take a smooth map $f\colon \mathbb{S}^d \to \mathscr{N}$ that is not homotopic to a constant, and define $u_0(x) = f\bigl(\frac{x}{|x|}\bigr)$ on $\mathbb{B}^{d+1}$. Then, for $d \leq sp < d+1$, a variation of the argument used to prove Proposition 1.4.1 shows that $u_0 \notin H^{s,p}(\mathbb{B}^{d+1};\mathscr{N})$. This construction can be transported to a higher-dimensional domain simply by addition of dummy variables. An important observation there is that, for a domain of dimension m, the map constructed by this procedure will be smooth outside of an affine space of dimension $m-d-1$.

This general argument was first coined out by F. Bethuel and Zheng X. [BZ88] for $s = 1$, and then generalized by M. Escobedo [Esc88] for arbitrary s, leading to the following theorem.

Theorem 1.4.3. *Assume that $sp < m$. If $H^{s,p}_S(\mathbb{B}^m;\mathscr{N}) = W^{s,p}(\mathbb{B}^m;\mathscr{N})$, then $\pi_{\lfloor sp \rfloor}(\mathscr{N}) = \{0\}$.*

This remarkably provides us with a *necessary condition* for the strong density of smooth maps given purely in terms of the topology of the target. Before we turn to the much more difficult question of its sufficiency, let us briefly comment on the assumption $sp < m$ in the above theorem.

This assumption ensures that the map u_0 that is constructed in the proof of Theorem 1.4.3 indeed belongs to the appropriate Sobolev space. It is not merely an artifact in the proof, as when $sp \geq m$, strong density *always holds*. This is an instance of a more general principle: *when $sp \geq m$, Sobolev spaces of mappings essentially behave like classical linear spaces*. This comes from the embedding $W^{s,p} \hookrightarrow L^\infty$ when $sp > m$, or its counterpart in the limiting case $sp = m$. Let us illustrate the phenomenon at work here by proving the following theorem.

Theorem 1.4.4. *Assume that $sp \geq m$. Then, $H^{s,p}_S(\mathscr{M};\mathscr{N}) = W^{s,p}(\mathscr{M};\mathscr{N})$.*

Proof when $sp > m$. Let $u \in W^{s,p}(\mathscr{M};\mathscr{N})$, and let $(u_\eta)_{\eta>0}$ in $C^\infty(\mathscr{M};\mathbb{R}^\nu)$ be such that $u_\eta \to u$ as $\eta \to 0$, obtained from the linear strong density theorem. Due to the Morrey–Sobolev embedding $W^{s,p} \hookrightarrow L^\infty$ when $sp > m$, for η sufficiently small, we have $u_\eta \in \mathscr{N}_\iota$, where $\iota > 0$ is the radius of a tubular neighborhood of $\mathscr{N}$ as in Proposition 1.4.2. Therefore, the map $\Pi \circ u_\eta$ is well-defined and belongs to $C^\infty(\mathscr{M};\mathscr{N})$. To conclude, we invoke the continuity of the composition operator on $W^{s,p} \cap L^\infty$, see [BM01], to deduce that $\Pi \circ u_\eta \to \Pi \circ u = u$ as $\eta \to 0$, which completes the proof. □

As we see, the key ingredient in the proof is to show that a sequence of $\mathbb{R}^\nu$-valued smooth maps approaching u ends up getting uniformly close to $\mathscr{N}$, so that we can project it back onto $\mathscr{N}$ via a smooth retraction Π onto $\mathscr{N}$. In the case $sp > m$, this is obtained by relying on the fact that the u_η uniformly converge to u, which is $\mathscr{N}$-valued.

When $sp = m$, the Morrey–Sobolev embedding fails, so that $W^{s,p}$ convergence does not imply uniform convergence anymore. It is nevertheless possible to show that the approximating sequence gets uniformly close to the target manifold, although not uniformly close to u, provided that we choose a suitable approximation process. We provide a proof in the special case where $s = 1$ and $\mathscr{M} = \overline{\mathbb{B}^m}$, to give a flavor of some arguments that will be of crucial importance in several parts of this thesis.

Sketch of the proof when $s = 1$ and $p = m$. Let $u \in W^{s,p}(\mathbb{B}^m;\mathscr{N})$, and let $\varphi \in C^\infty_c(\mathbb{B}^m)$ be a standard mollifier, that is, $\varphi \geq 0$, and $\int_{\mathbb{B}^m} \varphi = 1$. We define $u_\eta = \varphi_\eta * u$, where $\varphi_\eta(x) = \frac{1}{\eta^m}\varphi(\frac{x}{\eta})$. We leave aside the slight technical issue that this map is not well-defined near the boundary of $\mathbb{B}^m$, that can easily be fixed by an extension by dilation argument.

We now estimate the distance from u_η to $\mathscr{N}$. As $u \in \mathscr{N}$ a.e., we have

$$\operatorname{dist}(u_\eta(x), \mathscr{N}) \leq |u_\eta(x) - u(y)| \quad \text{for a.e. } y \in \mathbb{B}^m.$$

We take the average over the ball $B^m_\eta(x)$, and we use the definition of the convolution to find

$$\operatorname{dist}(u_\eta(x), \mathscr{N}) \leq \fint_{B^m_\eta(x)} |u_\eta(x) - u(y)|\,\mathrm{d}y \lesssim \fint_{B^m_\eta(x)} \fint_{B^m_\eta(x)} |u(z) - u(y)|\,\mathrm{d}y\,\mathrm{d}z.$$

Now, the Poincaré–Wirtinger inequality ensures that

$$\fint_{B^m_\eta(x)} \fint_{B^m_\eta(x)} |u(z) - u(y)|\,\mathrm{d}y\,\mathrm{d}z \lesssim \int_{B^m_\eta(x)} |\mathrm{D}u|^m. \tag{1.4.2}$$

As $u \in W^{1,m}$ by virtue of the assumption $p = m$, Lebesgue's lemma implies that the right-hand side converges to 0 as $\eta \to 0$, uniformly in x. Therefore, for η sufficiently small, we conclude that $\sup_{x \in \mathscr{M}} \operatorname{dist}(u_\eta(x), \mathscr{N}) < \iota$, and we conclude by projection as in the previous case. □

The ingredients to handle the case $sp = m$ were essentially already contained in the work by R. Schoen and K. Uhlenbeck [SU83], who proved Theorem 1.4.4 when $s = 1$ and $p = 2 = m$. Inequality (1.4.2) combined with Lebesgue's lemma actually shows that $W^{1,p} \hookrightarrow \mathrm{VMO}$, which is the limiting case of the Morrey–Sobolev embedding. The crucial observation that this can be connected with the distance from a convoluted map to the target manifold is due to H. Brezis and L. Nirenberg [BN95], and shall be of great use all along this work.

In view of Theorem 1.4.4, from now on, we shall focus on the more difficult case $sp < m$. The content of Theorem 1.4.3 is to provide a necessary condition for the strong density of smooth maps. What is extremely remarkable is that the topological obstruction encoded by this condition is the *only* local obstruction to the strong density of smooth maps. Namely, we have the following theorem.

Theorem 1.4.5. *Assume that $sp < m$. Then, $H^{s,p}_S(\mathbb{B}^m; \mathscr{N}) = W^{s,p}(\mathbb{B}^m; \mathscr{N})$ if and only if $\pi_{\lfloor sp \rfloor}(\mathscr{N}) = \{0\}$.*

This theorem is the result of quite a number of important contributions. The model case $s = 1$ was proved by F. Bethuel in his seminal paper [Bet91]. Prior to this, some partial results for sphere-valued maps had been obtained by F. Bethuel and Zheng X. [BZ88]. Bethuel's proof is technically involved, and relies on the so-called *method of good and bad cubes*, that he introduced in this context. Shortly after, P. Hajłasz gave an alternative, simpler, argument, called the *method of almost projection*, to prove Theorem 1.4.5 when $s = 1$ in the special case where $\mathscr{N}$ is $(\lfloor p \rfloor - 1)$-connected, that is, when $\pi_1(\mathscr{N}) = \cdots = \pi_{\lfloor p \rfloor - 1}(\mathscr{N}) = \{0\}$. Hajłasz's approach was subsequently extended to $s \geq 1$ by P. Bousquet, A. Ponce, and J. Van Schaftingen [BPVS13].

The general case of an arbitrary domain was obtained by Hang F. and Lin F. [HL03a] for $s = 1$. We shall comment later on, in Section 1.5, on the additional difficulties that might happen when considering a general domain, which are of *global* nature and arise from the interplay between the topology of the domain and the target.

After that, the case $0 < s < 1$ received special attention. Partial results were obtained, either relying on the fact that $W^{s,p}$ is a trace space, or using the so-called *method of singular projection*, notably by F. Bethuel [Bet95], T. Rivière [Riv00], P. Bousquet [Bou07], D. Mucci [Muc09], and P. Bousquet, A. Ponce, and J. Van Schaftingen [BPVS13]. The general case was finally obtained by H. Brezis and P. Mironescu [BM15], using an approach based on *homogeneous extension*, complementary to the orginal strategy of F. Bethuel. In parallel, Theorem 1.4.5 was obtained for higher order integer Sobolev spaces, i.e. for $s \in \mathbb{N}_*$, by P. Bousquet, A. Ponce, and J. Van Schaftingen [BPVS15]. Their approach relies on the method of good and bad cubes introduced by F. Bethuel, supplemented with several new tools to deal with the higher rigidity of higher order Sobolev spaces. A typical issue occurs for instance when trying to glue constructions performed on neighboring cubes: if $s \geq 1 + \frac{1}{p}$, then the coincidence of the traces on the common boundary is no longer sufficient to ensure the appropriate regularity of the resulting map, one also needs to make sure that the derivatives match.

The contribution of this work is to provide a proof of the missing case, when $s > 1$ is not an integer, allowing to finally write Theorem 1.4.5 in its full generality. Our approach follows the strategy of good and bad cubes introduced by F. Bethuel, incorporates the tools suggested by P. Bousquet, A. Ponce, and J. Van Schaftingen, and adds some new ideas to obtain the required estimates on fractional quantities that are needed to make the argument work. Here, the main challenge is the *nonlocal* character of the Gagliardo seminorm. Moreover, we will show that our argument carries on to the case $0 < s < 1$, providing a *unified* proof of Theorem 1.4.5, also covering the case already obtained by H. Brezis and P. Mironescu via a different approach.

1.4.4 A strongly dense class of "almost smooth maps"

We now turn to the problem (Q2). For this purpose, we come back to the first observation at the beginning of Section 1.4.3: in the seminal counterexample by R. Schoen and K. Uhlenbeck, the map that is constructed is smooth outside of a point singularity. As we observed later on, in the general case, obstructions are built as maps which are singular on an affine space of dimension $m - \lfloor sp \rfloor - 1$ in the flat case — in the case where the domain is a manifold, the singular set will then be a submanifold. A remarkable fact is that, in some sense, this is the worst that may happen.

More precisely, we introduce the following class of mappings. We define the class $\mathscr{R}_i(\mathscr{M};\mathscr{N})$ as the set of maps $u\colon \mathscr{M} \to \mathscr{N}$ which are smooth on $\mathscr{M} \setminus \mathscr{S}$, where $\mathscr{S} = \mathscr{S}_u$ is a finite union of i-dimensional submanifolds of $\mathscr{M}$, depending on u, and such that for every $j \in \mathbb{N}_*$ and every $x \in \mathscr{M} \setminus \mathscr{S}$,

$$|\mathrm{D}^j u(x)| \leq C_j \frac{1}{\operatorname{dist}(x,\mathscr{S})^j} \quad \text{for some constant } C_j > 0 \text{ depending on } j. \tag{1.4.3}$$

When we want to talk about this class without making the dimension explicit, we shall simply refer to *the class* $\mathscr{R}$.

A prototypical version of the class $\mathscr{R}_i$, with $i = 0$, was introduced by F. Bethuel and Zheng X. [BZ88], before the importance of the general version of this class was highlighted in Bethuel's seminal work [Bet91], where he proved the following theorem with $s = 1$.

Theorem 1.4.6. *If $sp < m$, then $\mathscr{R}_{m-\lfloor sp \rfloor-1}(\mathbb{B}^m;\mathscr{N})$ is dense in $W^{s,p}(\mathbb{B}^m;\mathscr{N})$.*

As we said at the beginning of Section 1.4.3, (Q1) and (Q2) are strongly tied together. In fact, the density of the class $\mathscr{R}$ does not only provide us with a convenient substitute for smooth maps when $H^{s,p}(\mathbb{B}^m;\mathscr{N}) \subsetneq W^{s,p}(\mathbb{B}^m;\mathscr{N})$, but it is also a crucial tool in the *proof* of the density of smooth maps. Indeed, a common strategy to prove Theorem 1.4.5 is to first show the density of the class $\mathscr{R}$, a step which does not rely on the topological assumption, and then to deduce from it the density of smooth maps using the topological assumption. This was for instance the strategy followed by F. Bethuel [Bet91], H. Brezis and P. Mironescu [BM15] ($0 < s < 1$), and P. Bousquet, A. Ponce, and J. Van Schaftingen [BPVS15] ($s \in \mathbb{N}_*$). This is also the strategy we follow in Chapter 2: we first prove the general version of Theorem 1.4.6, showing its validity in the full range $0 < s < +\infty$, and then we deduce Theorem 1.4.5 under the necessary topological assumption on $\mathscr{M}$ and $\mathscr{N}$. For this reason, we refer the reader to the previous section for an history of problem (Q2), which essentially coincides with the history of the problem of strong density of smooth maps. A notable exception to this is the contribution of P. Hajłasz [Haj94] and its higher order counterpart [BPVS13], where the method proves directly the density of smooth maps without obtaining the density of the class $\mathscr{R}$ on the way.

Having obtained a suitable strongly dense class of almost smooth maps, the natural question that arises is whether or not this class may be improved. A first natural guess would be to look for a singular set of lower dimension. However, it turns out that the value $i = m - \lfloor sp \rfloor - 1$ is the only possible value that makes $\mathscr{R}_i$ dense in $W^{s,p}$. Indeed, for a smaller i, the same topological obstruction as for the density of smooth maps would also prevent the strong density of the class $\mathscr{R}_i$, while for larger i, the class $\mathscr{R}_i$ would not even be a subset of $W^{s,p}$. We refer the reader to the discussion in [BPVS15, Section 6] for more details.

There is however another direction of improvement. Indeed, in the definition of the class $\mathscr{R}$, the singular set is required to be a *union* of submanifolds of $\mathscr{M}$. In particular, it may exhibit crossings, where several of these submanifolds intersect. Moreover, the singular set of the maps in the class $\mathscr{R}$ that are constructed in the usual proofs of Theorem 1.4.6, such as the one we will present in Chapter 2, *does* exhibit crossings. A natural question is therefore whether one may improve this result by obtaining the strong density of maps that are smooth outside of a singular set being made of only *one* submanifold, hence ruling out the possibility of crossings. More specifically, we introduce the class $\mathscr{R}_i^{\mathrm{uncr}}(\mathscr{M};\mathscr{N})$ as the set of all $u \in \mathscr{R}_i(\mathscr{M};\mathscr{N})$ such that the singular set $\mathscr{S}_u$ of u is made of only *one* i-dimensional submanifold of $\mathscr{M}$, and we ask whether $\mathscr{R}_{m-\lfloor sp\rfloor-1}^{\mathrm{uncr}}(\mathscr{M};\mathscr{N})$ is dense in $W^{s,p}(\mathscr{M};\mathscr{N})$.

This question was notably raised by H. Brezis and P. Mironescu [BM21, Chapter 10], and is supported by the fact that, in some special cases, it was known that the answer is positive, via an alternative proof of the density of the class $\mathscr{R}$ that provides a singular set without crossings. Let us exemplify this when the target is the sphere. Let $m \geq d+1$ and $u \in W^{1,d}(\mathbb{B}^m;\mathbb{S}^d)$ — here, the value $p = d$ is chosen so that density of smooth maps fails, as $\pi_d(\mathbb{S}^d) = \mathbb{Z}$. The idea is to start with the classical, linear approximation process, by defining $u_\eta = \varphi_\eta * u$, with φ a standard mollifier. The map u_η is smooth, but need not be $\mathbb{S}^d$-valued. To correct this, let us reproject it onto $\mathbb{S}^d$, by defining

$$v_{\eta,a} = \frac{u_\eta - a}{|u_\eta - a|}, \quad \text{with } a \in \mathbb{B}^{d+1}.$$

This corresponds to performing a radial projection centered at a. The issue is that, for a fixed a, the map u_η might take the value a on a large set, so that $v_{\eta,a}$ would not even be well-defined. The idea in order to overcome this issue is to let the value of a vary. The first observation is that, by Sard's lemma, for almost every $a \in \mathbb{B}^{d+1}$, $u_\eta^{-1}(\{a\})$ is an $(m-d-1)$-dimensional submanifold, so that $v_{\eta,a}$ is defined almost everywhere, and belongs to the class $\mathscr{R}_{m-d-1}^{\mathrm{uncr}}$, provided that we show the corresponding estimate for the blow up of derivatives near the singular set, which we omit in this introduction. We now show a model computation to prove that $v_{\eta,a}$ satisfies a suitable Sobolev estimate, relying on an ingenious averaging argument. We estimate

$$\int_{\mathbb{B}^{d+1}} \left(\int_{\mathbb{B}^m} |\mathrm{D}v_{\eta,a}|^d \right) \mathrm{d}a \lesssim \int_{\mathbb{B}^{d+1}} \left(\int_{\mathbb{B}^m} \frac{|\mathrm{D}u_\eta|^d}{|u_\eta - a|^d} \right) \mathrm{d}a$$

We now invoke Fubini–Tonelli's theorem, and find

$$\int_{\mathbb{B}^{d+1}} \left(\int_{\mathbb{B}^m} \frac{|\mathrm{D}u_\eta|^d}{|u_\eta - a|^d} \right) \mathrm{d}a = \int_{\mathbb{B}^m} \left(\int_{\mathbb{B}^{d+1}} \frac{|\mathrm{D}u_\eta|^d}{|u_\eta - a|^d} \,\mathrm{d}a \right) \lesssim \int_{\mathbb{B}^m} |\mathrm{D}u_\eta|^d. \tag{1.4.4}$$

Therefore, there exists a subset A of $\mathbb{B}^{d+1}$ of positive measure such that, for any $a \in A$,

$$\int_{\mathbb{B}^m} |\mathrm{D}v_{\eta,a}|^d \lesssim \int_{\mathbb{B}^m} |\mathrm{D}u_\eta|^d.$$

Using the same idea, with additional work, it is possible to prove that there exists points a_η, with $a_\eta \to 0$ as $\eta \to 0$, such that $v_{\eta,a_\eta} \to u$ in $W^{1,d}$. This proves the strong density of the class $\mathscr{R}^{\mathrm{uncr}}_{m-d-1}(\mathbb{B}^m;\mathbb{S}^d)$ in $W^{1,d}(\mathbb{B}^m;\mathbb{S}^d)$.

This technique is called the *method of singular projection*. It was introduced in the context of Sobolev mappings to manifolds by R. Hardt and Lin F. [HL87], and relies on the *averaging argument* introduced by H. Federer and W. Fleming [FF60] in their seminal contribution on normal and integral currents. This method was subsequently used to prove strong density results notably by F. Bethuel and Zheng X. [BZ88] for $W^{1,p}(\mathbb{B}^m;\mathbb{S}^{m-1})$ when $m-1 \leq p < m$, T. Rivière [Riv00] for $W^{\frac{1}{2},2}(\mathbb{S}^2;\mathbb{S}^1)$, J. Bourgain, H. Brezis, and P. Mironescu [BBM05] for $W^{s,p}(\mathbb{S}^m;\mathbb{S}^{m-1})$ when $0 < s < 1$ and $sp < m$ (see also [BBM04] for the case $s = \frac{1}{2}$), P. Bousquet [Bou07] for $W^{s,p}(\mathbb{S}^m;\mathbb{S}^1)$ when $1 \leq sp < 2$, and P. Bousquet, A. Ponce, and J. Van Schaftingen [BPVS14] for an $(\lfloor sp \rfloor - 1)$-connected target $\mathscr{N}$ when $0 < s < 1$.

However, regarding the question of the density of class $\mathscr{R}^{\mathrm{uncr}}$, there is no hope that this method can give a complete answer. Indeed, it can only be applied to $(\lfloor sp \rfloor - 1)$-connected targets, and even in this situation, its validity has not yet been established in full generality in the existing literature. Moreover, even if the target is $(\lfloor sp \rfloor - 1)$-connected, it might happen that the method of projection produces a singular set with crossings.

We tackle this question in Chapter 3. First, we establish the validity of the method of singular projection in full generality for an $(\lfloor sp \rfloor - 1)$-connected target. Second, we devise a new geometric uncrossing construction, that we use to prove the strong density of the class $\mathscr{R}^{\mathrm{uncr}}$ for a general target $\mathscr{N}$, not necessarily $(\lfloor sp \rfloor - 1)$-connected target, in the local situation. The case of a domain with nontrivial topology remains open in full generality.

Gathering the contributions announced in this section and the previous one, this work lays the last stone in a program of research aiming at a complete answer to (Q1) and (Q2), with a improved answer with respect to the expected one for (Q2) when the domain has a simple topology. In the two next sections, we will present in more detail the picture of (Q3) and (Q4), which remains widely open.

1.4.5 Topological and analytical characterizations of the strong closure of smooth maps

The characterization provided by Theorem 1.4.5 of situations where strong density of smooth maps occurs, although being very remarkable from a theoretical point of view, turns out to be somehow disappointing in practice. Already when the target is a sphere $\mathbb{S}^d$, in the range $sp \geq d+1$ which cannot be handled via the method of singular projection, there are actually many values of $\lfloor sp \rfloor$ and d such that $\pi_{\lfloor sp \rfloor}(\mathbb{S}^d) \neq \{0\}$; see for example the table of homotopy groups in [Hat02, Page 339]. This supports the idea that it would be of great use to be able to decide, for a *given* map $u \in W^{s,p}(\mathscr{M};\mathscr{N})$, whether or not $u \in H^{s,p}_S(\mathscr{M};\mathscr{N})$. Otherwise stated, this raises the question of characterizing $H^{s,p}_S(\mathscr{M};\mathscr{N})$.

Essentially two directions have been followed to address this question. The first one deals with topological characterizations. It starts from the observation that, following the proof of Proposition 1.4.1, the obstruction to the strong approximability of the hedgehog map arises from the fact that it realizes a homotopically nontrivial map on a *generic* sphere centered around its singularity at the origin. This might suggest that, to decide whether or not a map u belongs to $H^{s,p}_S(\mathscr{M};\mathscr{N})$, it suffices to look at the restrictions of u on a generic $\lfloor sp \rfloor$-dimensional sphere. (In the sequel, it will be more convenient to work rather on boundaries of $(\lfloor sp \rfloor+1)$-dimensional cubes.) This is made even more apparent after a careful examination of the usual proofs of the strong density theorem, such as the one we present in Chapter 2.

However, going from this intuitive idea to a rigorous statement along with a proof is a highly nontrivial task. Already giving a precise meaning to the term *generic* requires some work. This is especially difficult in the range $s \geq 1$, when working with the method of good and bad cubes by F. Bethuel and its extensions by P. Bousquet, A. Ponce, and J. Van Schaftingen [BPVS15] and in Chapter 2, as the construction *significantly modifies* the values of the map u under consideration on boundaries of $(\lfloor sp \rfloor + 1)$-dimensional spheres. The task of characterizing the space $H^{s,p}_S(\mathscr{M};\mathscr{N})$ in this context was accomplished for $s \in \mathbb{N}_*$ by P. Bousquet, A. Ponce, and J. Van Schaftingen in a very recent work [BPVS25], relying on the concept of *Fuglede map* and *detector* that they developed especially for this purpose. In this work, we address the case $0 < s < 1$. In this setting, the task is considerably simplified by the use of the alternative approach for strong density by H. Brezis and P. Mironescu [BM15]. Indeed, this approach essentially leaves unchanged the map under consideration on the boundary of the $(\lfloor sp \rfloor+1)$-dimensional cubes used in the proof. This makes easier both to define a suitable notion of genericity to work with, and to prove the corresponding characterization of $H^{s,p}_S(\mathscr{M};\mathscr{N})$. This is one of the main advantages of this alternative, conceptually more simple approach, which is however restricted to the range $0 < s < 1$, rather than working with the methodology based on good and bad cubes, which covers the full range $0 < s < +\infty$. More specifically, in Chapter 6, we prove essentially the following theorem.

Theorem 1.4.7. *Let $u \in W^{s,p}(\mathbb{B}^m;\mathscr{N})$, with $0 < s < 1$ and $1 \leq d \leq sp < d+1$. If there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ converging to 0 such that, for every n, for almost every $P \in \mathbb{B}^m$, u is nullhomotopic when restricted to the boundary of a $(d+1)$-dimensional cube of sidelength ε_n centered at P and aligned with the coordinate axis, then $u \in H^{s,p}_S(\mathbb{B}^m;\mathscr{N})$.*

Although it provides a complete description of $H^{s,p}_S(\mathbb{B}^m;\mathscr{N})$ for a general target $\mathscr{N}$, the topological characterization may appear as somehow unsatisfactory, given how involved it is to formulate the notion of generic restrictions, and as it might be cumbersome to manipulate. For this reason, an important effort has been invested in obtaining different, *analytic*, descriptions of $H^{s,p}_S(\mathscr{M};\mathscr{N})$.

A promising candidate for doing so is the *Jacobian*. Let us give an illustration in a model situation. Given $u \in W^{1,n-1}(\mathbb{B}^{d+1};\mathbb{S}^d)$, we define a distribution Ju by

$$\langle Ju,\alpha\rangle = -\int_{\mathbb{B}^{d+1}} \mathrm{d}\alpha \wedge u^\sharp\omega \quad \text{for every } \alpha \in C^\infty_{\mathrm{c}}(\mathbb{B}^{d+1}),$$

where ω is the standard volume form of $\mathbb{S}^d$, given by

$$\omega = \sum_{j=1}^{d+1}(-1)^{j-1}\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{j-1} \wedge \mathrm{d}x_{j+1} \wedge \cdots \wedge \mathrm{d}x_{d+1}.$$

Here, the sharp denotes the pullback, that is,

$$u^\sharp\omega = \sum_{j=1}^{d+1}(-1)^{j-1}u_j\mathrm{d}u_1 \wedge \cdots \wedge \mathrm{d}u_{j-1} \wedge \mathrm{d}u_{j+1} \wedge \cdots \wedge \mathrm{d}u_{d+1}. \tag{1.4.5}$$

In particular, by virtue of Hölder's inequality, $u^\sharp\omega$ is an L^1 function, so that Ju is well-defined, and corresponds actually to $\mathrm{d}(u^\sharp\omega)$ in the sense of distributions.

We shall not attempt to draw the (very rich) history of the Jacobian. We simply mention that it originates from the pioneering work of C. G. Jacobi, and that its distributional definition is due to J. Ball [Bal76]. We refer the reader to the survey of H. Brezis, J. Mawhin, and P. Mironescu [BMM24] and the numerous references therein for a detailed history of this object.

Let us prove two important properties of the Jacobian that suggest that it is indeed a promising candidate to characterize $H^{1,d}_S(\mathbb{B}^{d+1};\mathbb{S}^d)$. We note that $H^{1,d}(\mathbb{B}^{d+1};\mathbb{S}^d) \subsetneq W^{1,d}(\mathbb{B}^{d+1};\mathbb{S}^d)$, as $\pi_d(\mathbb{S}^d) = \mathbb{Z}$.

Lemma 1.4.8. *If $u \in H^{1,d}(\mathbb{B}^{d+1};\mathbb{S}^d)$, then $Ju = 0$.*

Proof. It is a direct consequence of Hölder's inequality, the dominated convergence theorem, and its partial converse that J is continuous with respect to the $W^{1,d}$ convergence. We therefore only need to prove that it vanishes on smooth maps. For this purpose, let $u \in C^\infty(\mathbb{B}^{d+1};\mathbb{S}^d)$, and let $\alpha \in C^\infty_{\mathrm{c}}(\mathbb{B}^{d+1})$. By standard rules of exterior calculus,

$$\mathrm{d}(\alpha \wedge u^\sharp\omega) = \mathrm{d}\alpha \wedge u^\sharp\omega + \alpha \wedge \mathrm{d}u^\sharp\omega = \mathrm{d}\alpha \wedge u^\sharp\omega,$$

where we have used the fact that $\mathrm{d}\omega = 0$ and that the pullback commutes with the exterior differential. We conclude by invoking Stokes' formula:

$$0 = \int_{\mathbb{B}^{d+1}} \mathrm{d}(\alpha \wedge u^\sharp \omega) = -\langle Ju, \alpha \rangle. \qquad \square$$

Lemma 1.4.9. *If $u_0(x) = \frac{x}{|x|}$ denotes the hedgehog map, then $Ju_0 = |\mathbb{S}^d|\delta_0$.*

Here, δ_0 is the Dirac mass centered at 0.

Proof. We rely once again on Stokes' formula. However, as u is not smooth at 0, it is necessary to remove a small ball around the origin to be in position to apply it. Hence, we compute

$$\langle Ju_0, \alpha \rangle = -\lim_{r\to 0} \int_{\mathbb{B}^{d+1}\setminus B_r^{d+1}} \mathrm{d}\alpha \wedge u^\sharp \omega = \lim_{r\to 0} \int_{\partial B_r^{d+1}} \alpha \wedge u^\sharp \omega.$$

By continuity of α at 0, we get

$$\lim_{r\to 0} \int_{\partial B_r^{d+1}} \alpha \wedge u^\sharp \omega = \alpha(0) \lim_{r\to 0} \int_{\partial B_r^{d+1}} u^\sharp \omega.$$

Since u_0 is a homogeneous map, a simple change of variable yields

$$\int_{\partial B_r^{d+1}} u^\sharp \omega = |\mathbb{S}^d|. \tag{1.4.6}$$

We deduce that

$$\langle Ju_0, \alpha \rangle = |\mathbb{S}^d| \alpha(0),$$

which concludes the proof. $\square$

These two facts combined together suggest that $Ju = 0$ if and only if $u \in H_S^{1,d}(\mathbb{B}^{d+1}; \mathbb{S}^d)$, and that actually Ju detects the singular set of u. This is indeed the case, as proved by F. Bethuel in [Bet90] for $d = 2$, and by F. Bethuel, J.-M. Coron, F. Demengel, and F. Hélein [BCDH91] for general d. This therefore provides us with a characterisation of $H^{1,d}(\mathbb{B}^{d+1}; \mathbb{S}^d)$ by the means of an *analytical* object, namely a distribution.

This idea can be generalized to more general targets $\mathscr{N}$, upon appropriately modifying the analytical object that is used. The important feature about the sphere in the above example is that

$$\int_{\mathbb{S}^d} f^\sharp \omega = |\mathbb{S}^d| \deg f \quad \text{for every smooth } f\colon \mathbb{S}^d \to \mathbb{S}^d,$$

of which we have used a special case to establish (1.4.6). We will say that *the cohomology of $\mathscr{N}$ sees its homotopy*, by which we roughly mean that being nullhomotopic can be detected only by evaluating suitable integrals of pullbacks of closed forms. The importance of this criterion was first understood by F. Bethuel, J.-M. Coron, F. Demengel, and F. Hélein [BCDH91], who used this idea to establish an analytical characterization of $H^{1,p}_S(\mathscr{M};\mathscr{N})$ in the case of a $(\lfloor p \rfloor - 1)$-connected target whose $\pi_{\lfloor p \rfloor}(\mathscr{N})$ has no torsion. We also refer the reader to the work of R. Hardt and T. Rivière [HR03, HR08], which addresses the more general case of a target whose $\pi_{\lfloor p \rfloor}$ has no torsion (without simple connectedness assumption), relying on rational homotopy and the Novikov integral formula, and also to the work of M. Giaquinta, G. Modica, J. Souček, and collaborators, culminating at the monograph [GMS98a, GMS98b], giving a geometric measure theoretical viewpoint on this question, thanks to the notion of *Cartesian currents*.

We now turn to extensions of this approach to the whole range $0 < s < +\infty$. Here, there are two ranges to distinguish. The range $s \geq 1$ reduces to the case $s = 1$ by the combination of the following two facts: (i) $W^{s,p} \cap L^\infty \subset W^{1,sp}$ by the Gagliardo–Nirenberg inequality; and (ii) a mapping $u \in W^{s,p}(\mathscr{M};\mathscr{N})$ belongs to $H^{s,p}_S(\mathscr{M};\mathscr{N})$ if and only if it belongs to $H^{1,sp}_S(\mathscr{M};\mathscr{N})$. Indeed, this follows from the fact that the same objects that characterize $H^{1,sp}_S(\mathscr{M};\mathscr{N})$ can be used to characterize $H^{s,p}_S(\mathscr{M};\mathscr{N})$. Fact (ii) above is again a property which is intuitively apparent from the proof of the strong density theorem, such as the one we will present in Chapter 2, but whose complete proof requires a nontrivial argument. We refer the reader to [BPVS25, Corollary 1.15] for the case $s \in \mathbb{N}_*$. The case $s > 1$ noninteger is actually still a conjecture, although it seems quite obvious that a proof of it should only be a matter of combining the tools from [BPVS25] with the fractional estimates that we will present in Chapter 2.

The range $0 < s < 1$ turns out to be much more delicate. A first difficulty is how to even *define* Ju. Indeed, the definition of Ju over $W^{1,p}$ relies on the integration of $u^\sharp\omega$, and as (1.4.5) shows, this quantity requires one full derivative of u to be well-defined. Hence, the definition needs at least to be modified in the range $0 < s < 1$.

Once again, the solution shall come from the Stokes formula. Let us exemplify the idea by one formal computation, for a map $u \in W^{s,p}(\mathbb{B}^{d+1};\mathbb{S}^d)$. Assume that U is a smooth (non necessarily $\mathbb{S}^d$-valued) extension of u to $\mathbb{B}^{d+1} \times (0,+\infty)$, that is, $\operatorname{tr}_{|\mathbb{B}^{d+1}} U = u$. Such an extension can essentially be constructed by defining $U(x,t) = \varphi_t * u(x)$, with φ a standard mollifier. We also observe that the standard volume form ω on $\mathbb{S}^d$ extends naturally as a d-form on $\mathbb{R}^{d+1}$, given by the same formula. (However, it is no longer a closed form when considered as a form on $\mathbb{R}^{d+1}$.) Finally, given $\alpha \in C^\infty_c(\mathbb{B}^{d+1})$, we let $\tilde{\alpha}$ be a smooth extension to $\mathbb{B}^{d+1} \times (0, +\infty)$ that vanishes on $\partial\mathbb{B}^{d+1} \times (0, +\infty)$; for instance, $\tilde{\alpha}(x, t) = \alpha(x)$.

Using Stokes' formula, we compute formally

$$\langle Ju, \alpha\rangle = -\int_{\mathbb{B}^{d+1}} \mathrm{d}\alpha \wedge u^\sharp\omega = \int_{\mathbb{B}^{d+1}\times(0,+\infty)} \mathrm{d}(\mathrm{d}\tilde{\alpha} \wedge U^\sharp\omega)$$
$$= -\int_{\mathbb{B}^{d+1}\times(0,+\infty)} \mathrm{d}\tilde{\alpha} \wedge U^\sharp(\mathrm{d}\omega). \quad (1.4.7)$$

Now, the twist is that the right-hand side in (1.4.7) is well-defined, since U is a smooth map. A tentative strategy to define the Jacobian in the range $0 < s < 1$ is now clear: (i) prove rigorously the validity of (1.4.7) when $u \in W^{1,d}$; and (ii) prove that the right–hand side of (1.4.7) makes sense and is continuous in $W^{s,p}$ when $0 < s < 1$ and $sp = d$. As $W^{1,d}(\mathbb{B}^{d+1}; \mathbb{S}^d)$ is dense in $W^{s,p}(\mathbb{B}^{d+1}; \mathbb{S}^d)$ — since it contains the dense class $\mathscr{R}_0(\mathbb{B}^{d+1}; \mathbb{S}^d)$ — this program would show that the right-hand side of (1.4.7) is the unique continuous extension of the Jacobian to $W^{s,p}$. This idea was introduced by J. Bourgain, H. Brezis, and P. Mironescu [BBM05] precisely for mappings from balls to spheres. Further properties of the Jacobian in fractional spaces where studied by P. Bousquet and P. Mironescu [BM14]. In this text, we carefully explain the rigorous construction of Jacobian-like objects in $W^{s,p}(\mathscr{M}; \mathscr{N})$ for a general target manifold. We then show how to use these objects to give an analytical characterization of $H^{s,p}_S(\mathbb{B}^m; \mathscr{N})$ in the general case where the cohomology of $\mathscr{N}$ sees its homotopy, hence extending partial results by D. Mucci [Muc24] for sphere-valued maps.

1.4.6 The weak density problem

This last section is devoted to the study of problem (Q4). The motivation is the following: when faced with the issue that smooth maps may fail to be strongly dense in the corresponding Sobolev space of mappings, one natural approach is to try to weaken the notion of convergence in use, with the hope of obtaining an approximation property with respect to this weaker notion. In this text, we will focus on the notion that is commonly used for this purpose, namely the *weak convergence*.

Let us first give a definition of the Sobolev energy. Given $u\colon \mathscr{M} \to \mathscr{N}$, we define

$$\mathcal{E}^{s,p}(u,\mathscr{M}) = \begin{cases} \displaystyle\int_{\mathscr{M}} |\mathrm{D}^s u|^p & \text{if } s \in \mathbb{N}_*, \\ \displaystyle\int_{\mathscr{M}}\int_{\mathscr{M}} \frac{|\mathrm{D}^k u(x) - \mathrm{D}^k u(y)|^p}{\operatorname{dist}(x,y)^{m+\sigma p}}\,\mathrm{d}x\,\mathrm{d}y & \text{if } s \notin \mathbb{N}_*, \end{cases}$$

with the convention that $\mathcal{E}^{s,p}(u,\mathscr{M}) = +\infty$ if $u \notin W^{s,p}$. With this definition, we say that a sequence $(u_n)_{n\in\mathbb{N}}$ *weakly converges* to u, and we write $u_n \rightharpoonup u$, whenever $u_n \to u$ almost everywhere and

$$\sup_{n\in\mathbb{N}} \mathcal{E}^{s,p}(u_n,\mathscr{M}) < +\infty.$$

Otherwise stated, weak convergence replaces the convergence in norm by merely the boundedness of the energy, along with a very weak notion of convergence at the level of the functions themselves. Since with are working with maps with values into a compact target, we could as well have required convergence in measure or in L^1_{loc}, and this would have resulted in the same notion of convergence, possibly up to extracting subsequences.

An important remark concerns the relation between this notion and the usual notion of linear weak convergence, inherited from the dual space of L^p. When $1 < p < +\infty$, both these notions coincide thanks to the reflexivity of L^p, via the Banach–Alaoglu theorem. When $p = 1$ however, weak convergence is somewhat ill-behaved due to the lack of reflexivity of L^1, and both notions cease to coincide. Actually, the linear notion is stronger than the one stated above, and is equivalent to it under an equi-integrability requirement. From now on, we shall work with weak convergence as defined above without any further comments. It is more intrinsic in spirit, since it does not rely on the fact that $\mathscr{N}$ is embedded to make sense of the linear notion of weak convergence induced by the dual space.

It is straightforward from the definition that strong convergence implies weak convergence. Additionally, weak convergence is a natural notion to work with in the context of problems of partial differential equations or calculus of variations, since it is in general much easier to obtain weak convergence for sequences of approximate solutions or almost minimizers than strong convergence. It can even happen that weak convergence holds while strong convergence fails. Moreover, weak convergence may sometimes be sufficient to extend desirable properties from smooth to Sobolev mappings; see [MVS21a, Theorem 3] for one instance of such a phenomenon. All these reasons suggest that weak convergence, although being a milder notion than strong convergence, is still a meaningful notion to study.

With this definition arises the question of the weak density of smooth maps in Sobolev spaces of mappings. However, now there is a subtlety about the precise meaning to give to the word *density*. Indeed, we could either ask whether the closure of $C^\infty(\mathscr{M};\mathscr{N})$ in the topology induced by the weak convergence coincides with $W^{s,p}(\mathscr{M};\mathscr{N})$, or whether any map $u \in W^{s,p}(\mathscr{M};\mathscr{N})$ is the weak limit of a sequence of smooth mappings. Unlike with strong convergence, as weak convergence is not metrizable, both notions do not coincide. The answer to the first question is completely known: it has been shown by F. Bethuel [Bet91] that the weak closure of $C^\infty(\mathbb{B}^m;\mathscr{N})$ always coincides with $W^{1,p}(\mathbb{B}^m;\mathscr{N})$, without any assumption on $\mathscr{N}$; the argument can be adapted to deal with the case of a general domain $\mathscr{M}$ as well as the complete range $0 < s < +\infty$. The reason for this is that, on the ball $\mathbb{B}^m$, the mappings in the class $\mathscr{R}$ are weak limits of smooth maps. We will see later on in this section the key ingredients required to prove this claim.

Much more subtle is the second question, about weak approximability by sequences of smooth maps, and we shall therefore focus on that from now on. Let us define $H^{s,p}_W(\mathscr{M};\mathscr{N})$ as the set of all $W^{s,p}(\mathscr{M};\mathscr{N})$ mappings that are weak limits of maps in $C^\infty(\mathscr{M};\mathscr{N})$. The question is therefore to know whether or not $H^{s,p}_W(\mathscr{M};\mathscr{N}) = W^{s,p}(\mathscr{M};\mathscr{N})$, and if not, to characterize the space $H^{s,p}_W(\mathscr{M};\mathscr{N})$. In case equality holds, we say that *the weak approximation property holds*.

It turns out that, when $sp \notin \mathbb{N}$, then the weak and strong approximation problems coincide. Let us exemplify this situation by considering again the hedgehog map.

Example 1.4.10. For $2 < p < 3$, the map $u_0(x) = \frac{x}{|x|} \in W^{1,p}(\mathbb{B}^3;\mathbb{S}^2)$ does not belong to $H^{1,p}_W(\mathbb{B}^3;\mathbb{S}^2)$.

Indeed, assume by contradiction that there exists a family $(u_\eta)_{\eta>0}$ in $C^\infty(\mathbb{B}^3;\mathbb{S}^2)$ such that $u_\eta \rightharpoonup u$. By the polar integration formula and Fatou's lemma, we find

$$\int_0^1 \left(\liminf_{\eta\to 0} \int_{\partial B^3_r} |\mathrm{D}u_\eta|^p\right) \mathrm{d}r \leq \liminf_{\eta\to 0} \int_0^1 \left(\int_{\partial B^3_r} |\mathrm{D}u_\eta|^p\right) \mathrm{d}r = \liminf_{\eta\to 0} \int_{\mathbb{B}^3} |\mathrm{D}u_\eta|^p < +\infty.$$

Therefore, for every $r \in (0,1)$, up to extraction of a subsequence (here possibly depending on r), we have

$$\sup_{\eta>0} \int_{\partial B^3_r} |\mathrm{D}u_\eta|^p < +\infty.$$

Another easy Fubini–Tonelli argument provides convergence a.e., whence we deduce weak convergence on ∂B^3_r. By the Rellich–Kondrashov compactness theorem, this implies the uniform convergence of $u_{\eta|\partial B^3_r}$ to $u_{|\partial B^3_r}$, which allows to derive the required contradiction by a homotopy argument as for strong density. □

Actually, it was proved by F. Bethuel [Bet91] that, when $sp \notin \mathbb{N}$, then $H^{s,p}_W(\mathscr{M};\mathscr{N}) = H^{s,p}_S(\mathscr{M};\mathscr{N})$, whence the weak approximation problem is completely solved as well for $sp \notin \mathbb{N}_*$.

In the case $sp \in \mathbb{N}_*$ however, a new phenomenon occurs, as illustrated by the following example.

Example 1.4.11. The hedgehog map u_0 belongs to $H^{1,2}_W(\mathbb{B}^3;\mathbb{S}^2)$, while is does not belong to $H^{1,2}_S(\mathbb{B}^3;\mathbb{S}^2)$.

For this purpose, given any $\eta > 0$, we define a map $\varphi_\eta \colon \mathbb{S}^2 \to \mathbb{S}^2$ such that (i) $\varphi_\eta = \mathrm{id}$ outside of a ball of radius η around the North Pole; (ii) φ_η has degree zero; and (iii) $\sup_{\eta>0} \mathcal{E}^{1,2}(\varphi_\eta, \mathbb{S}^2) < +\infty$. For instance, we can rely on the map $\psi \colon \mathbb{B}^2 \to \mathbb{S}^2$ defined by

$$\psi(x) = \Big(\sin(\pi|x|) \cdot \frac{x}{|x|}, \cos(\pi(1 - |x|))\Big),$$

which has degree -1 and maps $\overline{\mathbb{B}^2}$ onto the North Pole. It then suffices to scale ψ to a ball of radius η, and then to proceed to a suitable truncation and rescaling to match it with the identity.

We then define $u_\eta \in W^{1,2}(\mathbb{B}^3;\mathbb{S}^2)$ as the homogeneous extension of φ_η:

$$u_\eta(x) = \varphi_\eta\Big(\frac{x}{|x|}\Big).$$

By the properties of the homogeneous extension, these maps enjoy the following features: (i) $\sup_{\eta>0} \mathcal{E}^{1,2}(u_\eta, \mathbb{S}^2) < +\infty$; (ii) $u_\eta \to u_0$ a.e.; and (iii) $u_{\eta|\partial B_r}$ has degree 0 for every η and every $r \in (0,1)$. By this third property, each u_η may be approximated *strongly* in $W^{1,2}$ by smooth mappings into $\mathbb{S}^2$. Intuitively, this follows from the fact that the central singularity is now topologically trivial, so that the obstruction to the strong approximation does not arise anymore. A rigorous argument follows by the shrinking procedure, that we will explain in Section 2.7. A diagonal argument allows us to conclude. □

The key difference between the case $sp \in \mathbb{N}_*$ and the case $sp \notin \mathbb{N}_*$ is that, in the former, the Sobolev energy in invariant by scaling in dimension $\lfloor sp \rfloor$. For instance, if $u \colon \mathbb{B}^2 \to \mathbb{R}$ and $u_r(x) = u(x/r)$, then

$$\mathcal{E}^{1,2}(u_r, B^2_r) = \mathcal{E}^{1,2}(u, \mathbb{B}^2),$$

while $\mathcal{E}^{1,p}(u_r, B^2_r)$ would blow up to $+\infty$ as $r \to 0$ for $p > 2$. This was crucial to obtain property (iii) of the map φ_η: if instead $p > 2$, then the $W^{1,p}$ energy of the map φ_η

construct with such a process would blow up as $\eta \to 0$.

The fact that, when moving from $2 < p < 3$ to $p = 2$, the hedgehog map u_0 is now a weak limit of smooth maps is not an isolated phenomenon: actually, it holds that $H^{1,2}_W(\mathbb{B}^3;\mathbb{S}^2) = W^{1,2}(\mathbb{B}^3;\mathbb{S}^2)$. This was proved by F. Bethuel [Bet90]. The strategy for proving the weak approximability of u_0 can be summarized in two steps: (i) connect the topological singularity of u_0 to the boundary via a straight line; and (ii) eliminate the singularity along this connection by inserting the inverse of the topological charge of the singularity in a small neighborhood of the line.

Below, we briefly sketch the key ingredients to apply this strategy to a general sphere-valued mapping. The first step essentially boils down to the following proposition, which ensures the existence of *connections with controlled length*.

Proposition 1.4.12. *Let $u \in \mathscr{R}_0(\mathbb{B}^3;\mathbb{S}^2)$. There exists $y \in \mathbb{S}^2$ a regular image for u such that the set $u^{-1}(\{y\})$ is a smooth 1-dimensional manifold whose length satisfies*

$$\mathscr{H}^1(u^{-1}(\{y\})) \lesssim \int_{\mathbb{B}^3} |\mathrm{D}u|^2.$$

Proof. Since u is smooth in $\mathbb{B}^3 \setminus \mathscr{S}_u$, it follows from Sard's lemma that almost every point $y \in \mathbb{S}^2$ is a regular image for u. For such points, $u^{-1}(\{y\})$ is indeed a smooth 1-dimensional manifold by virtue of the submersion theorem.

We now invoke Federer's co-area formula:

$$\int_{\mathbb{S}^2} \mathscr{H}^1(u^{-1}(\{y\}))\,\mathrm{d}y = \int_{\mathbb{B}^3} \mathcal{J}u,$$

where the Jacobian is computed as the determinant of the differential from $\mathbb{R}^3$ to the tangent bundle to $\mathbb{S}^2$. Since the Jacobian is the product of the singular values of $\mathrm{D}u$, while the norm of the differential is the square root of the sum of those singular values, the arithmetico-geometric inequality implies that

$$\mathcal{J}u \leq \frac{1}{2}|\mathrm{D}u|^2.$$

Hence,

$$\int_{\mathbb{S}^2} \mathscr{H}^1(u^{-1}(\{y\}))\,\mathrm{d}y \lesssim \int_{\mathbb{B}^3} |\mathrm{D}u|^2.$$

This ensures the existence of a point y, regular image of u, satisfying

$$\mathscr{H}^1(u^{-1}(\{y\}))\,\mathrm{d}y \lesssim \int_{\mathbb{B}^3} |\mathrm{D}u|^2. \qquad \square$$

The usefulness of this proposition in the study of the weak approximation problem comes from the fact that, for a map $u \in \mathscr{R}_0(\mathbb{B}^3;\mathbb{S}^2)$, the set $u^{-1}(\{y\})$ provides a connection between the topological singularities of u. In the language of geometric measure theory, this could be written as $\partial u^{-1}(\{y\}) = \mathscr{S}_u$, where the boundary is taken in the sense of currents. But in this situation, this can be phrased in an elementary way: when suitably endowed with an orientation, the set $u^{-1}(\{y\})$ is made of a finite union of smooth curves terminating either on the boundary or on singularities of u, and such that the number of curves ending at a given singularity, taking into account the orientation, coincides with the degree of u on a small sphere around the singularity. That is, on a singularity of degree 1, there could be two curves coming and one living, so that the sum of the contributions is equal to 1.

The second step in Bethuel's strategy for proving approximation, the elimination of topological singularities along a connection, relies on the so-called *dipole construction*, that we sketch next. The reader may bear in mind the construction performed in Example 1.4.11, which is a special case of this procedure.

The prototypical idea is to construct, for instance, mappings $u_\eta\colon \mathbb{B}^3 \to \mathbb{S}^2$ with the following properties:

(i) $u_\eta \in C^\infty(\mathbb{B}^3 \setminus \{A,B\};\mathbb{S}^2)$;

(ii) $u_\eta = a$ outside of a neighborhood of $[A,B]$ of width η;

(iii) $\deg_B(u) = j$ and $\deg_A(u) = -j$;

(iv) $\mathcal{E}^{1,2}(u_\eta,\mathbb{B}^3) \leq Cj|B-A|$.

Here, A, $B \in \overline{\mathbb{B}^3}$, $j \in \mathbb{N}$, and $a \in \mathbb{S}^2$ are given and u_η depends on those parameters, but the constant $C > 0$ is independent of them and η. The notation $\deg_P(v)$ stands for the degree of v on a small sphere around P, which does not depend on the choice of such a sufficiently small sphere. Such a map u_η is called a *dipole*, it is of great use to transfer topological singularities from one point to another, for instance to cancel together two opposite singularities, or to eliminate singularities by transferring them to the boundary. The energy of such a construction is controlled by the *weight* of the charge being moved, and the length of the *connection* between the two points. Such a map can be constructed by inserting a scaled copy of a degree j map $\mathbb{B}^2 \to \mathbb{S}^2$ with boundary value a on each sphere orthogonal to $[A,B]$, in a neighborhood of $[A,B]$ with the shape of a two-sided sharpened pencil of width η, with the leads located at A and B. We omit giving an explicit construction of such a map. We nonetheless mention that this construction can be adapted to fit in a pencil-shaped neighborhood of curves connecting A to B more general than lines, with $|B-A|$ being replaced by the length of the curve in the energy bound (iv).

The dipole construction has its roots in the work by H. Brezis, J.-M. Coron, and E. Lieb [BCL86]. We also refer the reader to [ABL88] and [ABO03] for a very nice depiction of the construction as well as a higher codimensional counterpart and related directions of research.

Combining the construction of connections with controlled length with the dipole construction along the connection, for a given $u \in \mathscr{R}_0(\mathbb{B}^3;\mathbb{S}^2)$, one may construct maps $u_\eta \in H^{1,2}_S(\mathbb{B}^3;\mathbb{S}^2)$ such that $u_\eta \rightharpoonup u$ as $\eta \to 0$, and with energy bound

$$\mathcal{E}^{1,2}(u_\eta,\mathbb{B}^3) \lesssim \mathscr{H}^1(u^{-1}(\{y\})) \lesssim \mathcal{E}^{1,2}(u,\mathbb{B}^3). \tag{1.4.8}$$

Since the class $\mathscr{R}_0$ is dense in $W^{1,2}(\mathbb{B}^3;\mathbb{S}^2)$, a diagonal argument allows to conclude that $H^{1,2}_W(\mathbb{B}^3;\mathbb{S}^2) = W^{1,2}(\mathbb{B}^3;\mathbb{S}^2)$. We importantly note that the linear bound (1.4.8) is crucial to conclude this argument. Indeed, without the control on the length of connections, one could still attempt to prove the weak approximation property as follows. First, prove that any map in $\mathscr{R}_0$ is a weak limit of smooth maps, by removing the singularities by a dipole construction along connections linking singularities to the boundary without control, which always exists. Then, conclude with a diagonal argument using the density of the class $\mathscr{R}_0$. However, the second step is not justified, as nothing in the above reasoning prevents that, for some map $u \in W^{1,2}(\mathbb{B}^3;\mathbb{S}^2)$, the minimum energy to perform the weak approximation blows up along any sequence of maps in $\mathscr{R}_0$ strongly converging to u. Although this does not happen in this situation, such possibility will be of crucial importance later on in this work.

The weak approximation result in $W^{1,2}(\mathbb{B}^3;\mathbb{S}^2)$ was later on extended, first by F. Bethuel to $W^{1,n-1}(\mathbb{B}^n;\mathbb{S}^{n-1})$ [Bet91], and then by P. Hajłasz [Haj94], who proved that $H^{1,p}_W(\mathscr{M};\mathscr{N}) = W^{1,p}(\mathscr{M};\mathscr{N})$ whenever $\mathscr{N}$ is $(p-1)$-connected. We note that the proof of P. Hajłasz relies on a different argument, based on the method of *almost projection*. This was later on extended to $W^{s,p}$ with $sp \in \mathbb{N}_*$ and $s \geq 1$ by P. Bousquet, A. Ponce, and J. Van Schaftingen [BPVS13]. In particular, the weak approximation property always holds when $sp = 1$ and $s \geq 1$.

In the same direction, R. Pakzad and T. Rivière [PR03] proved that $H^{1,2}_W(\mathscr{M};\mathscr{N}) = W^{1,2}(\mathscr{M};\mathscr{N})$ for more general targets $\mathscr{N}$ than those covered by the case $p = 2$ of Hajłasz's theorem, relying on the dipole construction and controlled connections approach in the spirit of Bethuel's strategy, extended to more general targets than spheres. More precisely, their approach recovers Hajłasz's result for simply connected targets, but also applies to some non-simply connected $\mathscr{N}$. Also, R. Hardt and T. Rivière [HR15] proved that $H^{2,2}_W(\mathbb{B}^5;\mathbb{S}^3) = W^{2,2}(\mathbb{B}^5;\mathbb{S}^3)$, again by elimination of singularities using the dipole construction along connections. Although their result is limited to $s = 2$, $p = 2$, and $\mathscr{N} = \mathbb{S}^3$, the key feature in their reasoning is that $\pi_4(\mathbb{S}^3) = \mathbb{Z}/2\mathbb{Z}$ is finite.

All these partial results could have led to think that it should always hold that $H_W^{s,p}(\mathbb{B}^m;\mathscr{N}) = W^{s,p}(\mathbb{B}^m;\mathscr{N})$ whenever $sp \in \mathbb{N}_*$. This was even stated as a conjecture by several experts in the area, including F. Bethuel [Bet91], or Hang F. and Lin F. [HL03a, Conjecture 7.1]. However, in a groundbreaking contribution, F. Bethuel [Bet20] showed the existence of *analytical-type* obstructions to the weak approximation property, proving that

$$H_W^{1,3}(\mathbb{B}^m;\mathbb{S}^2) \subsetneq W^{1,3}(\mathbb{B}^m;\mathbb{S}^2) \quad \text{whenever } m \geq 4.$$

A typical difference between analytical and topological-type obstructions can be observed by looking at the following sequence of inclusions:

$$H_W^{s,p}(\mathscr{M};\mathscr{N}) \subset \overline{H_W^{s,p}(\mathscr{M};\mathscr{N})} \subset W^{s,p}(\mathscr{M};\mathscr{N}).$$

In presence of only topological obstructions, the first inclusion is an equality, while the second one is strict. In particular, $H_W^{s,p}(\mathscr{M};\mathscr{N})$ is a *closed* subset of $W^{s,p}(\mathscr{M};\mathscr{N})$. On the contrary, analytical obstructions prevent the *first* above inclusion to be an equality, and actually, they even result in $H_W^{s,p}(\mathscr{M};\mathscr{N})$ being *meager* in $W^{s,p}(\mathscr{M};\mathscr{N})$.

To understand Bethuel's approach, let us come back to the strategy for proving the weak approximation property that we explained above in the context of $W^{1,2}(\mathbb{B}^3;\mathbb{S}^2)$. One of the key features of the argument was that, if $u \in \mathscr{R}_0(\mathbb{B}^3;\mathbb{S}^2)$, then to any connection of the topological singularities of u to the boundary, one can associate a family $(u_\eta)_{\eta>0}$ of smooth maps converging a.e. to u, and whose $W^{1,2}$ energy is controlled by the mass of the connection, computed by taking into account both the length of the curves and the charge they carry. There is a general principle that, conversely, to any family $(u_\eta)_{\eta>0}$ of smooth maps weakly converging to u, there is an associated connection of the singularities to the boundary, whose mass is controlled by the lim inf of the Sobolev energy of the u_η. In the context of $W^{1,2}(\mathbb{B}^3;\mathbb{S}^2)$, such a principle goes back to the work of H. Brezis, J.-M. Coron, and E. Lieb [BCL86]. We refer the reader to the work of R. Pakzad [Pak02] and U. Tarp-Ficenc [TF05] for generalizations to higher dimensional domains, and to M. Giaquinta and D. Mucci [GM05b] for a result for $W^{1,2}$ maps into more general targets than $\mathbb{S}^2$ — but sharing nonetheless important topological features with $\mathbb{S}^2$. One may also consult [Lin22] for a sketch of proof in the setting of $W^{1,d}$ mappings into $\mathbb{S}^d$. In the context of $W^{1,3}$ mappings into $\mathbb{S}^2$, the corresponding estimate has been proved by R. Hardt and T. Rivière [HR03], using the concept of *scans* that they developed for this purpose, and was instrumental in Bethuel's approach.

The key difference between the setting of $W^{1,2}(\mathbb{B}^3;\mathbb{S}^2)$ and $W^{1,3}(\mathbb{B}^4;\mathbb{S}^2)$ arises from the rate of growth of the relevant topological quantities. In the first case, the topological singularities that need to be dealt with stem from the degree of smooth mappings $f\colon \mathbb{S}^2 \to \mathbb{S}^2$, associated with $\pi_2(\mathbb{S}^2) = \mathbb{Z}$. It turns out that the energy grows *linearly* with respect to the degree. More precisely, it holds that

$$|\deg f| \lesssim \int_{\mathbb{S}^2} |\mathrm{D}f|^2,$$

and this estimate is optimal, in the sense that for any $j \in \mathbb{Z}$, there exists a smooth map $f_j\colon \mathbb{S}^2 \to \mathbb{S}^2$ with degree j and

$$\int_{\mathbb{S}^2} |\mathrm{D}f_j|^2 \simeq |j|.$$

This means that the problem of transporting the singularities to the boundary is a linear optimal transport problem: the cost of transporting two charges separately or together is the same.

On the contrary, in the case of $W^{1,3}(\mathbb{B}^4;\mathbb{S}^2)$, topological singularities come from the *Hopf degree* of mappings $f\colon \mathbb{S}^3 \to \mathbb{S}^2$, with again $\pi_3(\mathbb{S}^2) = \mathbb{Z}$. But now, as proved by T. Rivière [Riv98], the Sobolev energy grows *sublinearly* with respect to the Hopf degree: it holds that

$$|\deg_H(f)|^{3/4} \lesssim \int_{\mathbb{S}^3} |\mathrm{D}f|^3,$$

and this estimate is again optimal. This implies that the transportation problem is now a *branched transportation problem*, as described in [BCM09], where one saves some cost by transporting charges together rather than separately. More precisely, the cost of transporting j charges together grows like $j^{3/4}$.

It turns out that, in dimension $l = 4$, the exponent $\alpha = 3/4 = 1 - 1/l$ is the *critical exponent* for branched optimal transportation, starting from which the estimates for the cost of transportation start deteriorating. Combining this fact with estimate coming from Hardt and Rivière's theory of scans, F. Bethuel was able to construct a sequence $(u_n)_{n\in\mathbb{N}}$ of mappings in $W^{1,3}(\mathbb{B}^4;\mathbb{S}^2)$ — they even belong to the class $\mathscr{R}_0$ — such that $\mathcal{E}^{1,3}(u_n, \mathbb{B}^4) \lesssim n^3$, but for any $(u_\eta)_{\eta>0}$ in $C^\infty(\mathbb{B}^4;\mathbb{S}^2)$ with $u_\eta \to u_n$ a.e., it holds that

$$\lim_{\eta\to 0} \mathcal{E}^{1,3}(u_\eta, \mathbb{B}^4) \gtrsim n^3 \log n.$$

Hence, the energy for performing the weak approximation grows *superlinearly* with respect to the Sobolev energy, in contrast with the linear bound in (1.4.8). The *nonlinear uniform boundedness principle* finally allows one to conclude that there exists a map $u \in W^{1,3}(\mathbb{B}^4;\mathbb{S}^2)$ which is not a weak limit of smooth maps, and that such maps are actually dense in $W^{1,3}(\mathbb{B}^4;\mathbb{S}^2)$. This principle may be viewed as a nonlinear counterpart of the classical Banach–Steinhaus theorem, saying that one may glue together suitably scaled copies of the maps u_ℓ to construct a map $u \in W^{1,3}(\mathbb{B}^4;\mathbb{S}^2)$ with finite Sobolev energy, but such that the Sobolev energy along any sequence of smooth maps converging a.e. to u blows up. In the context of weak approximation, such a principle was established by Hang F. and Lin F. [HL03b, Theorem 9.6]. We refer the reader to the work of A. Monteil and J. Van Schaftingen [MVS19] for a general version, and to [Bet20] for an ad hoc variant.

Up to now, Bethuel's counterexample was the only known analytical obstruction to the weak approximation property. In Chapter 5, we present two families of such obstructions. We first prove that, for any integer $p \geq 2$, there exists a target manifold for which the weak approximation property fails.

Theorem 1.4.13. *For every $p \in \mathbb{N} \setminus \{0,1\}$, there exists a compact manifold $\mathscr{N}$ such that if $m > p$, then*

$$H^{1,p}_W(\mathbb{B}^m;\mathscr{N}) \subsetneq W^{1,p}(\mathbb{B}^m;\mathscr{N}).$$

In particular, Theorem 1.4.13 is the *first* instance of the (local) failure of the weak approximation property for $p \neq 3$.

The manifold $\mathscr{N}$ is defined explicitly, depending on p. It is essentially obtained as the p-skeleton of the $(p+1)$-dimensional torus. For instance, for $p = 2$, its construction amounts to take three faces of a cube of dimension 3 that share a common vertex, and gluing together all parallel edges. The result consists of three tori, one for each face, such that every two of them have a common circle, and the three of them have a common point. The construction is illustrated on Figure 1.1, where all edges labeled with the same arrow type are identified together.

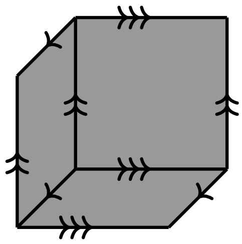

Figure 1.1 – A target such that the weak approximation property in $W^{1,2}$ fails

As we will see, the strategy behind the proof of Theorem 1.4.13 is similar in spirit to Bethuel's approach, in that it uses a topological quantity whose energy has a rate of growth corresponding to the critical exponent in a branched optimal transportation problem, in order to conclude via the nonlinear uniform boundedness principle. However, here, a careful localization argument allows us to rely on an elementary approach, without invoking the theory of branched transportation.

Even though Theorem 1.4.13 covers all the integer exponents $p \geq 2$, the resulting manifold $\mathscr{N}$ is not as simple as the sphere $\mathbb{S}^2$ in Bethuel's result [Bet20]. Using a variant of our construction, we also recover Bethuel's counterexample, which corresponds to our next result with $n = 1$.

Theorem 1.4.14. *For every $n \in \mathbb{N} \setminus \{0\}$, if $m > 4d - 1$, then*

$$H^{1,4d-1}_W(\mathbb{B}^m; \mathbb{S}^{2d}) \subsetneq W^{1,4d-1}(\mathbb{B}^m; \mathbb{S}^{2d}).$$

The relevant topological quantity involved in our construction is the Hopf invariant for mappings $f\colon \mathbb{S}^{4d-1} \to \mathbb{S}^{2d}$, so that the case $d = 1$ corresponds exactly to Bethuel's setting. Moreover, using a periodic construction inspired from our strategy for Theorem 1.4.13, we are able to significantly simplify Bethuel's construction, and eliminate the need of relying on the theory of scans by R. Hardt and T. Rivière.

1.5 What about general $\mathscr{M}$?

In this introduction, for the sake of simplicity, we have mainly focused on the study of *local* problems, that is, related to the Sobolev space $W^{s,p}(\mathbb{B}^m; \mathscr{N})$. When the domain is an arbitrary compact manifold $\mathscr{M}$ of dimension m, extra difficulties may arise.

Among the first observations of this fact is the seminal contribution by Hang F. and Lin F. [HL03a], where they observe that, for a general domain $\mathscr{M}$ and when $s = 1$, the condition $\pi_{\lfloor p \rfloor}(\mathscr{N})$ might *no longer* be sufficient to ensure that $H^{1,p}_S(\mathscr{M}; \mathscr{N}) = W^{1,p}(\mathscr{M}; \mathscr{N})$. Indeed, *global* obstructions may arise, coming from the interplay between the topology of the domain and the target. The model example involves Sobolev mappings between projective planes: they showed notably that

$$H^{1,2}_S(\mathbb{RP}^4; \mathbb{RP}^3) \subsetneq W^{1,2}(\mathbb{RP}^4; \mathbb{RP}^3),$$

while

$$H^{1,2}_S(\mathbb{B}^4; \mathbb{RP}^3) = W^{1,2}(\mathbb{B}^4; \mathbb{RP}^3).$$

Indeed, $\pi_2(\mathbb{RP}^3) = \{0\}$, which justifies the density of smooth maps in the local situation. The obstruction when the domain is $\mathbb{RP}^4$ is of global nature: it arises from a map u which is smooth outside of a point singularity, and that can be strongly approximated with smooth maps in the neighborhood of every point of $\mathbb{RP}^4$ — including the singular point — but this approximation process cannot be made global to approximate with smooth maps on the whole $\mathbb{RP}^4$. In contrast, the hedgehog map $u_0 = \frac{x}{|x|} \in W^{1,2}(\mathbb{B}^3; \mathbb{S}^2)$ cannot be approximated with smooth $\mathbb{S}^2$-valued maps in *any* small neighborhood of 0, highlighting the local nature of the obstruction.

In [HL03a], Hang F. and Lin F. proved a counterpart of Theorem 1.4.5 suited to the case of a general domain when $s = 1$. Their result gives a necessary and sufficient condition for the density of smooth maps to hold, in terms of the topology of both $\mathscr{M}$ and $\mathscr{N}$. This condition encompasses $\pi_{\lfloor p \rfloor}(\mathscr{N}) = \{0\}$, and is equivalent to it if $\mathscr{M}$ has a sufficiently simple topology: for instance, if $\mathscr{M} = \overline{\mathbb{B}^m}$, or $\mathscr{M} = \mathbb{S}^m$, or more generally if $\mathscr{M}$ is $\lfloor p-1 \rfloor$-connected. However, in general, it is more restrictive than $\pi_{\lfloor p \rfloor}(\mathscr{N}) = \{0\}$ alone. This is for instance the case for $\mathscr{M} = \mathbb{RP}^4$ and $\mathscr{N} = \mathbb{RP}^3$. We omit a precise statement of Hang and Lin's condition in this introduction, and refer the reader to Chapter 2 for more details and an extension to the whole range $0 < s < +\infty$.

In Chapter 2, we will see that the strategy on which we rely to prove our strong density theorems, based on the method of good and bad cubes devised by F. Bethuel as well as on the additional tool incorporated by P. Bousquet, A. Ponce, and J. Van Schaftingen to deal with higher order spaces, is so robust that it allows us to handle at once also the case where $\mathscr{M}$ is an arbitrary compact manifold. By contrast, our result in Chapter 3 concerning the density of an improved class of almost smooth maps relies on a new technique which is by nature local; we do not know whether or not this result extends to a general domain.

Concerning the weak approximation problem, it turns out that global obstructions also arise, even in the situation where $sp \in \mathbb{N}_*$, in which the assumptions for the weak and strong approximation properties to hold may differ. Also in [HL03a], it was notably proved that

$$H^{1,2}_W(\mathbb{RP}^3; \mathbb{RP}^2) \subsetneq W^{1,2}(\mathbb{RP}^3; \mathbb{RP}^2),$$

while

$$H^{1,2}_W(\mathbb{B}^3; \mathbb{RP}^2) = W^{1,2}(\mathbb{B}^3; \mathbb{RP}^2).$$

The equality in the second line can be proved by returning to the known case where the target is $\mathbb{S}^2$, using a lifting through the universal covering $\mathbb{S}^2 \to \mathbb{RP}^2$.

More generally, for $s = 1$ and $p \in \mathbb{N}_*$, a *necessary* condition for the weak approximation property to hold is the seemingly exotic condition

$$\operatorname{tr}_{\mathscr{M}^{p-1}}(C(\mathscr{M};\mathscr{N})) = \operatorname{tr}_{\mathscr{M}^{p-1}}(C(\mathscr{M}^{p};\mathscr{N})), \tag{1.5.1}$$

where $\mathscr{M}^{l}$ is an l-skeleton of $\mathscr{M}$. In more down-to-earth words, (1.5.1) asks that, for any continuous map $f\colon \mathscr{M}^{p} \to \mathscr{N}$, the restriction of f to the $(p-1)$-skeleton $\mathscr{M}^{p-1}$ can be extended as a continuous map $g\colon \mathscr{M} \to \mathscr{N}$ such that $g_{|\mathscr{M}^{p-1}} = f_{|\mathscr{M}^{p-1}}$. This condition is in particular satisfied, irrespective of $\mathscr{N}$, provided that $\mathscr{M}$ is $(p-1)$-connected. This includes the case of a cube or a ball of dimension at least p.

These obstructions are again of *global* nature, as they rely on the interplay between the topology of the domain and the target. On the contrary, the groundbreaking obstruction exhibited by F. Bethuel, as well as the families presented in Chapter 5, are of *local* nature: they already arise in a ball, and can be inserted in *any* domain.

1.6 Organization of this thesis

The main body is divided into five chapters, each of them being focused on one of the problems that have been presented in this introduction. Incidentally, each chapter also corresponds to one of the submitted contributions that arose from this thesis. The presentation of these chapters has been reworked with respect to the submitted version, in order to obtain a coherent presentation in this text, but we took care not to modify their mathematical content, for the sake of coherence with the already existing material. We conclude this introduction with a brief description of the content of each chapter.

In **Chapter 2**, we prove the strong density theorem, namely Theorems 1.4.5 and 1.4.6 (for a general $\mathscr{M}$) about the density of the class $\mathscr{R}$ and the necessary and sufficient condition for the density of smooth maps. For this purpose, we adapt the strategy of good and bad cubes by F. Bethuel, as well as the new tools introduced by P. Bousquet, A. Ponce, and J. Van Schaftingen, and we incorporate the additional necessary tools to enable them to work also in the fractional setting, the main challenge to overcome being the nonlocal character of the Gagliardo seminorm. This corresponds to the submitted work [Det23].

In **Chapter 3**, we tackle the problem of the density of the improved version of the class $\mathscr{R}$, namely the class $\mathscr{R}^{\mathrm{uncr}}$. In a first step, we extend the method of singular projection to its full range of expected applicability. In a second step, we devise a new geometric uncrossing procedure to deduce the density of the class $\mathscr{R}^{\mathrm{uncr}}$ in $W^{s,p}(\mathbb{B}^{m};\mathscr{N})$ for a general target $\mathscr{N}$, answering an open problem raised by H. Brezis and P. Mironescu [BM21, Chapter 10] in the local situation. The case of a general domain remains an open problem. This corresponds to the submitted work [Det25].

In **Chapter 4**, we apply several of the ideas that we used in the context of density questions to address a problem of calculus of variations with a geometric constraint. More precisely, we prove a *generic non-uniqueness* phenomenon for minimizing harmonic maps into the sphere. This improves an existing result by K. Mazowiecka and P. Strzelecki [MS17], the main new ingredient being a homotopy construction in the spirit of those used to handle strong density problems. This corresponds to a joint work with K. Mazowiecka [DM24].

In **Chapter 5**, we turn to the weak approximation problem. In this chapter, we prove Theorems 1.4.13 and 1.4.14 for general $\mathscr{M}$, along with their higher order counterparts, notably showing the existence, for any integer $p \in \mathbb{N} \setminus \{0, 1\}$, of a target manifold for which analytical obstructions to the weak approximation property arise. This corresponds to a joint work with J. Van Schaftingen [DVS24].

In **Chapter 6**, we focus on the problem of characterizing the strong closure of smooth maps $H^{s,p}(\mathscr{M}; \mathscr{N})$ in the range $0 < s < 1$. We develop the notion of pullback of closed differential forms by low regularity mappings for a general target manifold $\mathscr{N}$, carefully stating all the required notions to obtain a self-contained and rigorous presentation. Then, we obtain both a topological characterization of $H^{s,p}(\mathscr{M}; \mathscr{N})$ in terms of restrictions of maps on generic grids, valid for any target $\mathscr{N}$, and an analytical characterization formulated in terms of the vanishing of Jacobian-like quantities, valid whenever the cohomology of $\mathscr{N}$ sees its homotopy. This corresponds to a joint work with P. Mironescu and Xiao K. [DMX25].

Chapter 2

The strong density theorem

Résumé

Dans ce chapitre, on considère le problème de la densité forte, qui correspond aux questions (Q1) et (Q2). On donne une réponse complète à ces deux questions dans le cas manquant $s > 1$ non entier. Notre approche repose sur la méthode des bons et mauvais cubes développée par F. Bethuel ($s = 1$), à laquelle on incorpore les nouveaux outils suggérés par P. Bousquet, A. Ponce, et J. Van Schaftingen ($s = 2, 3, \ldots$) pour gérer la plus grande rigidité des espaces d'ordre supérieur, et que nous complémentons par de nouvelles idées pour les rendre compatibles avec le caractère non local de la semi-norme de Gagliardo. La stratégie de démonstration est suffisamment robuste pour couvrir d'une traite la gamme complète $0 < s < +\infty$ avec un argument unifié, incluant également le cas $0 < s < +\infty$ qui avait été traité par H. Brezis et P. Mironescu *via* une approche différente, et aussi pour s'appliquer au cas d'un domaine $\mathscr{M}$ arbitraire, dans la lignée des travaux de Hang F. et Lin F. pour $s = 1$.

Abstract

In this chapter, we consider the strong density problem, corresponding to questions (Q1) and (Q2). We give a complete answer to both these questions in the missing case $s > 1$ noninteger. Our approach relies on the method of good and bad cubes developped by F. Bethuel ($s = 1$), to which we incorporate the new tools suggested by P. Bousquet, A. Ponce, and J. Van Schaftingen ($s = 2, 3, \ldots$) to cope with the rigidity of higher order spaces, and that we supplement with new ideas to handle the nonlocal character of the Gagliardo seminorm. The strategy of proof is sufficiently robust so that it cover the whole range $0 < s < +\infty$ with a unified argument, including the case $0 < s < 1$ that had been handled by H. Brezis and P. Mironescu via a different approach, and also to carry over to a general domain $\mathscr{M}$, in the setting studied by Hang F. and Lin F. when $s = 1$.

2.1 Introduction

The chapter is entirely devoted to the proof of the strong density, namely, Theorem 1.4.5 and 1.4.6 for a general domain $\mathscr{M}$. The organization is as follows. In Sections 2.3 to 2.5, we develop the tools that we need to prove Theorem 1.4.6, following the approach in [BPVS15] and extending the auxiliary results to the noninteger case. With these tools at hand, we proceed in Section 2.6 with the proof of the density of the class $\mathscr{R}$. For the sake of simplicity, we first deal with the model case $\mathscr{M} = \overline{Q^m}$, before explaining how to handle more general domains. In Section 2.8, we present the proof of the density of smooth maps. Here also, we start with the case where $\mathscr{M} = \overline{Q^m}$, where the topological assumption to require is that $\pi_{\lfloor sp \rfloor}(\mathscr{N})$. We then explain what is the appropriate assumption in the case of a general domain $\mathscr{M}$, following the work of Hang F. and Lin F. [HL03a], and we prove the density of smooth maps in this general situation. The proofs rely on an additional tool presented in Section 2.7.

Before delving into technicalities, we start by presenting in Section 2.2 a sketch of the proof of Theorems 1.4.5 and 1.4.6. Our objective is to give an overview of the general strategy of the proof while avoiding giving too much details at this stage. We hope that Section 2.2 will provide the reader with some intuition on the basic ideas behind the different tools that will be used, and show how each of them fits into the big picture of the proof, before we move to a more detailed presentation in the next sections. Section 2.2 also gathers the main useful definitions and basic auxiliary results used throughout the chapter.

2.2 Definitions and sketch of the proof

Throughout the chapter, we make intensive use of decompositions of domains into suitable families of cubes. For this purpose, we introduce some notation. Given $\eta > 0$ and $a \in \mathbb{R}^m$, we denote by $Q^m_\eta(a)$ the cube of center a and radius η in $\mathbb{R}^m$, the radius of a cube being half of the length of its edges. When $a = 0$, we abbreviate $Q^m_\eta(0) = Q^m_\eta$. We also abbreviate $Q^m_1 = Q^m$.

A *cubication* K^m_η of radius $\eta > 0$ is any subset of $Q^m_\eta + 2\eta\mathbb{Z}^m$. Given $l \in \{0, \dots, m\}$, the *l-skeleton* of K^m_η is the set K^l_η of all faces of dimension l of all cubes in K^m_η. A *subskeleton* of dimension l of K^m_η is any subset of K^l_η. Given a skeleton S^l, we denote by $\mathcal{S}^l$ the union of all elements of S^l, that is,

$$\mathcal{S}^l = \bigcup_{\sigma^l \in S^j} \sigma^l.$$

Given a skeleton S^l, the *dual skeleton* of S^l is the skeleton T^{l^*} of dimension $l^* = m - l - 1$

consisting in all cubes of the form $\sigma^{l^*} + a - x$, where $\sigma^{l^*} \in S^{l^*}$, a is the center and x a vertex of a cube of S^m with $x \in \sigma^{l^*}$. The dimension l^* is the largest possible so that $\mathscr{S}^l \cap \mathscr{T}^{l^*} = \varnothing$. Here,

$$\mathscr{T}^{l^*} = \bigcup_{\sigma^{l^*} \in T^{l^*}} \sigma^{l^*}.$$

Illustrations of skeletons (in blue) and their duals (in red) in the unit cube $\overline{Q^3}$ are provided on Figure 2.1. The value of l ranges from 2 on the left to 0 on the right, which corresponds to a value of l^* ranging from 0 to 2.

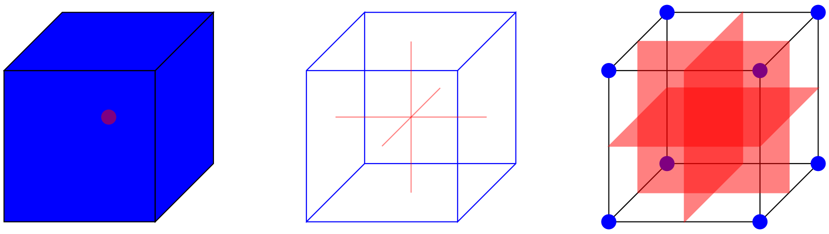

Figure 2.1 – Skeletons and their dual skeletons

For further use, we note that $\mathscr{S}^{l^*}$ is a homotopy retract of $\mathscr{S}^m \setminus \mathscr{T}^{l^*}$; see e.g. [Whi86, Section 1] or [BPVS13, Lemma 2.3].

Given a map $\Phi\colon \mathbb{R}^m \to \mathbb{R}^m$, the *geometric support of* Φ is defined by

$$\operatorname{Supp} \Phi = \overline{\{x \in \mathbb{R}^m \colon \Phi(x) \neq x\}}.$$

This should not be confused with the *analytic support* of a map $\varphi\colon \mathbb{R}^m \to \mathbb{R}$, defined by

$$\operatorname{supp} \varphi = \overline{\{x \in \mathbb{R}^m \colon \varphi(x) \neq 0\}}.$$

We now present the sketch of the proof of the density of the class $\mathscr{R}$. We also include graphical illustrations of the various constructions involved in the proof, with $m = 2$ and $\lfloor sp \rfloor = 1$. As we explained in the introduction, we follow the approach of P. Bousquet, A. Ponce, and J. Van Schaftingen [BPVS15], and we provide the necessary tools and ideas to adapt their method to the fractional setting. Let $u \in W^{s,p}(Q^m; \mathscr{N})$. For the sake of simplicity, we assume that u is defined in a neighborhood of $\overline{Q^m}$. The starting point is Bethuel's concept of *good cubes* and *bad cubes* that we now present. Let K^m_η be a cubication of Q^m, that is, $\mathscr{K}^m_\eta = Q^m$. Here, $\eta > 0$ is such that $1/\eta \in \mathbb{N}_*$. (Actually, for technical reasons, we will need to work on a cubication of a slightly larger cube than Q^m, but for this informal exposition, let us stick to a cubication of Q^m for the sake of simplicity.) We fix $0 < \rho < \frac{1}{2}$ and define the family E^m_η of all *bad cubes* as the set of cubes $\sigma^m \in K^m_\eta$ such that

$$\frac{1}{\eta^{m-sp}} \|Du\|^{sp}_{L^{sp}(\sigma^m + Q^m_{2\rho\eta})} > c \quad \text{if } s \geq 1, \tag{2.2.1}$$

or

$$\frac{1}{\eta^{m-sp}} |u|^{p}_{W^{s,p}(\sigma^m + Q^m_{2\rho\eta})} > c \quad \text{if } 0 < s < 1, \tag{2.2.2}$$

where $c > 0$ is a small parameter to be determined later on. The remaining cubes are the *good cubes*. From now on, we shall assume that we are in the case $s \geq 1$, in order to avoid having to distinguish two cases. (The case $0 < s < 1$ is similar.) The condition defining the good cubes ensures that *u does not oscillate too much* on such cubes. On the contrary, one cannot control the behavior of u on bad cubes, but we can show that there are not too many of them. Indeed, each bad cube contributes with at least $c\eta^{\frac{m}{sp}-1}$ to the energy of u, which limits the number of such cubes.

On Figure 2.2, one finds a possible decomposition of Q^2 in 16 cubes, which corresponds to $\eta = \frac{1}{4}$. Here, the three cubes in red are bad cubes, while green cubes are good cubes. For technical reasons that will become clear later on, it is useful to work on a set slightly larger than the union of bad cubes. We therefore let U^m_η be the set of all cubes in K^m_η that intersect some bad cube in E^m_η. This fact is ignored in our graphical illustrations, which are drawn as if $U^m_\eta = K^m_\eta$. This allows us to keep readable pictures with large cubes. Nevertheless, the reader should keep in mind that all constructions explained below are actually performed not only on the red cubes, but also on all green cubes adjacent to them, and that decompositions could possibly consist in many small cubes.

We now turn to the construction of the maps in u in the class $\mathscr{R}_{m-\lfloor sp \rfloor-1}$ approximating u. The first tool is the *opening*, which is explained in Section 2.3. This technique originates in the work of H. Brezis and Li Y. [BL01] about the topology of Sobolev spaces of maps between manifolds. We *open* the map u in order to obtain a map u^{op}_η which, on a neighborhood of the $\lfloor sp \rfloor$-skeleton $\mathscr{U}^{\lfloor sp \rfloor}_\eta$, is constant on the $(m - \lfloor sp \rfloor)$-dimensional cubes orthogonal to cubes in $\mathscr{U}^{\lfloor sp \rfloor}_\eta$. Therefore, on this neighborhood, the map u^{op}_η behaves locally as a function of $\lfloor sp \rfloor$-variables. But since $sp \geq \lfloor sp \rfloor$, this means that, on this region, u^{op}_η is actually a VMO function. The map u^{op}_η is obtained by modifying u on a slightly larger neighborhood of $\mathscr{U}^{\lfloor sp \rfloor}_\eta$, and the construction does not increase too much the energy of u on this neighborhood.

On Figure 2.3, one finds an illustration of the opening procedure when $\lfloor sp \rfloor = 1$. The

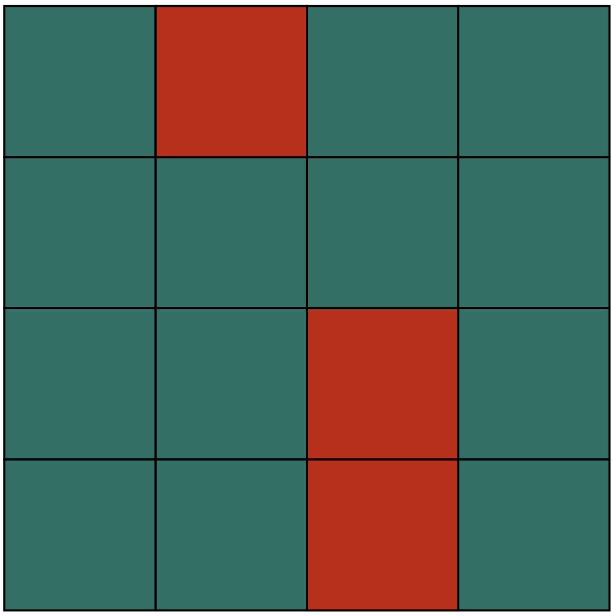

Figure 2.2 – Good and bad cubes

map u is opened on the blue region, where it therefore satisfies VMO estimates.

The next step is to smoothen the map u_η^{op}. Given a mollifier $\varphi \in C_c^\infty(\mathbb{B}^m)$ and $r > 0$, the usual convolution product is defined as

$$\varphi_r * u(x) = \int_{\mathbb{B}^m} \varphi(z) u(x + rz) \, \mathrm{d}z.$$

Here we rely on the method of *adaptive smoothing*, whose principle is to allow the convolution parameter to depend on the point where the convolution is evaluated. This technique was made popular by the work of R. Schoen and K. Uhlenbeck [SU82], where it was used in the study of the regularity of harmonic maps with values into a manifold.

More precisely, given $\psi \in C^\infty(Q^m)$, we let

$$\varphi_\psi * u(x) = \int_{\mathbb{B}^m} \varphi(z) u(x + \psi(x) z) \, \mathrm{d}z.$$

To pursue the proof, we choose a suitable map $\psi_\eta \in C^\infty(\mathbb{B}^m)$, whose construction depends on η and will be explained later on, and we define $u_\eta^{\mathrm{sm}} = \varphi_{\psi_\eta} * u_\eta^{\mathrm{op}}$.

This convolution procedure guarantees that the resulting map u_η^{sm} is smooth, but has the drawback that it need no longer take its values into $\mathcal{N}$, since the convolution product is in general not compatible with non convex constraints. We therefore need to estimate

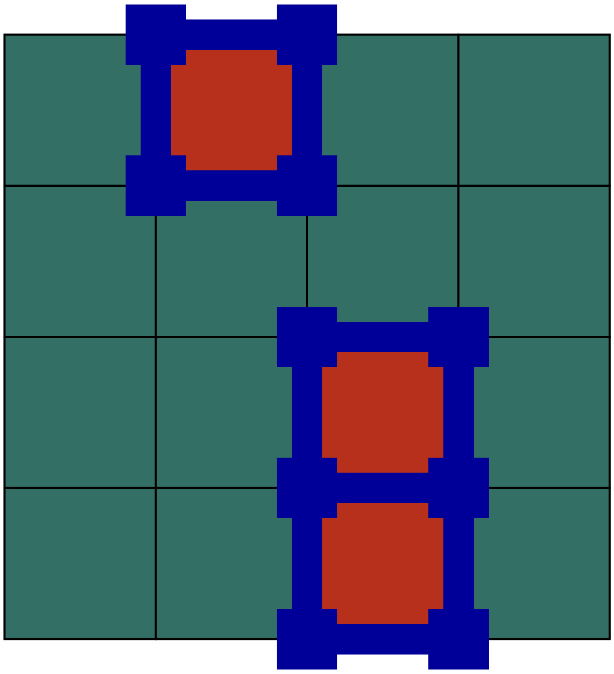

Figure 2.3 – Opening around the 1-skeleton of bad cubes

the distance between u_η^{sm} and $\mathscr{N}$. By straightforward computations, we write

$$|u_\eta^{\mathrm{sm}}(x) - u_\eta^{\mathrm{op}}(z)| \lesssim \fint_{Q^m_{\psi_\eta(x)}(x)} |u_\eta^{\mathrm{op}}(y) - u_\eta^{\mathrm{op}}(z)|\,\mathrm{d}y.$$

Averaging over all $z \in Q^m_{\psi_\eta(x)}(x)$, since $u_\eta^{\mathrm{op}}(z) \in \mathscr{N}$, we deduce that

$$\operatorname{dist}(u_\eta^{\mathrm{sm}}(x), \mathscr{N}) \lesssim \fint_{Q^m_{\psi_\eta(x)}(x)} \fint_{Q^m_{\psi_\eta(x)}(x)} |u_\eta^{\mathrm{op}}(y) - u_\eta^{\mathrm{op}}(z)|\,\mathrm{d}y\,\mathrm{d}z.$$

This computation should remind the reader of the proof of Theorem 1.4.4. Here we see the usefulness of the opening construction performed at the previous step: since u_η^{op} is a VMO function close to $U_\eta^{\lfloor sp \rfloor}$, the right-hand side of the above estimate may be made arbitrarily small in this region provided that we choose $\psi_\eta(x)$ sufficiently small. On the good cubes, we pursue the estimate by invoking the Poincaré–Wirtinger inequality to write

$$
\begin{aligned}
\operatorname{dist}(u_\eta^{\mathrm{sm}}(x), \mathscr{N}) &\lesssim \frac{1}{\psi_\eta(x)^{\frac{m}{sp}-1}} \|\mathrm{D}u_\eta^{\mathrm{op}}\|_{L^{sp}(Q^m_{\psi_\eta(x)}(x))} \\
&\lesssim \frac{1}{\psi_\eta(x)^{\frac{m}{sp}-1}} \|\mathrm{D}u\|_{L^{sp}(Q^m_{\psi_\eta(x)}(x))}.
\end{aligned}
\tag{2.2.3}
$$

If we choose $\psi_\eta(x)$ of order η, then on the right-hand side of (2.2.3), we find precisely the energy of u which is controlled on the good cubes. Therefore, choosing suitably the constant $c > 0$ in (2.2.1), on the good cubes, u_η^{sm} will be δ-close to $\mathscr{N}$, for some given arbitrarily small number $\delta > 0$. To summarize, we are invited to choose the convolution parameter very small on bad cubes, near the $\lfloor sp \rfloor$-skeleton, and of order η on good cubes. Between those two regimes, we need a transition region in order to allow ψ_η to change of magnitude, which is precisely the reason to introduce both families U_η^m and E_η^m instead of working directly on bad cubes. The precise way to perform this construction is explained in Section 2.4, and gathering the estimates on good and bad cubes, we conclude that u_η^{sm} is close to $\mathscr{N}$ on the good cubes, and on the part of bad cubes close to the $\lfloor sp \rfloor$-skeleton.

It therefore remains to deal with the part of bad cubes far from the $\lfloor sp \rfloor$-skeleton, where we have no control on the distance between u_η^{sm} and $\mathscr{N}$ (which corresponds to the red region in Figure 2.3). This is the purpose of the last tool we need, which is called *thickening*. The method is inspired from the use of homogeneous extension by F. Bethuel in the case $s = 1$. We illustrate the idea when $s = 1$ and $m - 1 < p < m$. Given a map $v \in C^\infty(\overline{Q^m})$, we define w on Q^m by

$$
w(x) = v\Big(\frac{x}{|x|_\infty}\Big).
$$

Here we recall that $|\cdot|_\infty$ stands for the ∞-norm in $\mathbb{R}^m$, defined for $x = (x_1, \dots, x_m) \in \mathbb{R}^m$ by $|x|_\infty = \max\limits_{1 \leq i \leq m} |x_i|$. Using radial integration, we see that $w \in W^{1,p}(Q^m)$ and

$$
\|\mathrm{D}w\|_{L^p(Q^m)}^p \lesssim \|\mathrm{D}v\|_{L^p(\partial Q^m)}^p \int_0^1 r^{m-p-1}\,\mathrm{d}r \lesssim \|\mathrm{D}v\|_{L^p(\partial Q^m)}^p.
$$

Here we use the assumption $p < m$. Hence, w is a $W^{1,p}(Q^m)$ map that depends only on the values of v on ∂Q^m. We may iterate this construction on faces by downward induction on the dimension to construct a map which only depends on the values of v on the $\lfloor p \rfloor$-skeleton of Q^m.

Two major difficulties arise when we try to adapt this construction to general Sobolev maps and spaces. First, it requires to work with slices of Sobolev maps on sets of zero measure. But more importantly, gluing such constructions on two cubes sharing a common face is a delicate matter. This is already the case when $s = 1$ if $p < m - 1$, since the resulting maps do not coincide on the whole common face, and gets worse when $s > 1 + \frac{1}{p}$ as the derivatives do not match at the interface. We bypass this difficulty be working with a more involved version of homogeneous extension, the *thickening* procedure.

Let $T_\eta^{\lfloor sp \rfloor^*}$ denote the dual skeleton of $U_\eta^{\lfloor sp \rfloor}$. The homogeneous extension, in the form presented above, associates with a map $v\colon \mathcal{U}_\eta^{\lfloor sp \rfloor} \to \mathbb{R}^\nu$ a map $w\colon \mathcal{U}_\eta^m \setminus \mathcal{T}_\eta^{\lfloor sp \rfloor^*} \to \mathbb{R}^\nu$, and this map is, in general, discontinuous on $\mathcal{T}_\eta^{\lfloor sp \rfloor^*}$. The map w may be written as $w = v \circ \Phi^{\mathrm{he}}$, where $\Phi^{\mathrm{he}}\colon \mathcal{U}_\eta^m \setminus \mathcal{T}_\eta^{\lfloor sp \rfloor^*} \to \mathcal{U}_\eta^{\lfloor sp \rfloor}$ is a Lipschitz map. Instead, the thickening procedure associates with a map $v\colon \mathcal{U}_\eta^{\lfloor sp \rfloor} + Q_\delta^m \to \mathbb{R}^\nu$ (for some $\delta > 0$ sufficiently small) a map $w\colon \mathcal{U}_\eta^m \setminus \mathcal{T}_\eta^{\lfloor sp \rfloor^*} \to \mathbb{R}^\nu$, which, again, is in general singular on the set $\mathcal{T}_\eta^{\lfloor sp \rfloor^*}$. The map w is obtained from v as $w = v \circ \Phi^{\mathrm{th}}$, where $\Phi^{\mathrm{th}}\colon \mathcal{U}_\eta^m \setminus \mathcal{T}_\eta^{\lfloor sp \rfloor^*} \to \mathcal{U}_\eta^{\lfloor sp \rfloor} + Q_\delta^m$ is a *smooth* map. Working with the neighborhood $\mathcal{U}_\eta^{\lfloor sp \rfloor} + Q_\delta^m$ instead of the skeleton $\mathcal{U}_\eta^{\lfloor sp \rfloor}$ is the key idea to avoid working with slices of Sobolev maps, and more importantly, to be able to choose Φ^{th} smooth, which, in turn, is crucial to ensure that composition with Φ^{th} preserves higher order Sobolev regularity.

The detailed construction, devised in [BPVS15, Section 4], is explained in Section 2.5, and we apply it to modify the map u_η^{sm} on $\mathcal{U}_\eta^m$ to a map u_η^{th} whose values on $\mathcal{U}_\eta^m$ only depend on the values of u_η^{sm} near $\mathcal{U}_\eta^{\lfloor sp \rfloor}$, while not increasing too much the energy of the map on $\mathcal{U}_\eta^m$. Therefore, the map u_η^{th} is close to $\mathcal{N}$ on the whole $Q^m \setminus \mathcal{T}_\eta^{\lfloor sp \rfloor^*}$, which makes possible to project it back onto $\mathcal{N}$ relying on the nearest point projection Π. Since the map u_η^{sm} is smooth, the map u_η^{th} is smooth on $Q^m \setminus \mathcal{T}_\eta^{\lfloor sp \rfloor^*}$, and we will show that the singularities created on $\mathcal{T}_\eta^{\lfloor sp \rfloor^*}$ by the thickening are sufficiently mild so that u_η^{th} belongs to the class $\mathcal{R}_{m-\lfloor sp \rfloor-1}(Q^m; \mathbb{R}^\nu)$.

One finds an illustration of the thickening procedure on Figure 2.4. The values of u on the dark blue region are propagated into the light blue region. This process creates point singularities on the centers of bad cubes, which are represented by the intersection of all the black lines.

The maps $u_\eta = \Pi \circ u_\eta^{\mathrm{th}}$ therefore belong to $\mathcal{R}_{m-\lfloor sp \rfloor-1}(Q^m; \mathcal{N})$, and they are actually the approximations of u that we were looking for. The only step that is required to obtain this conclusion is to show the convergence $u_\eta \to u$ in $W^{s,p}$ as $\eta \to 0$. This is done in Section 2.6, and amounts to a careful combination of the estimates obtained at each step of the construction. Except for the adaptive smoothing, all the modifications performed on u are localized in a neighborhood of U_η^m. The main ingredient to reach

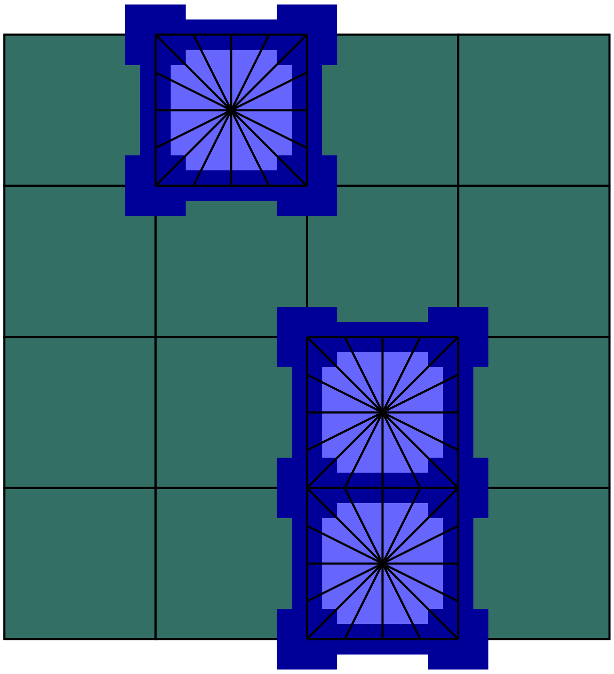

Figure 2.4 – Thickening around the centers of bad cubes

the conclusion $u_\eta \to u$ is therefore the fact that there are not too many bad cubes, and that actually the measure of the union of all bad cubes decays at a sufficiently high rate.

The density of the class $\mathscr{R}$ being established, we may then move to the density of smooth maps under the assumption $\pi_{\lfloor sp \rfloor}(\mathscr{N}) = \{0\}$. For this, it suffices to show that maps u_η of the class $\mathscr{R}_{m-\lfloor sp \rfloor-1}(Q^m;\mathscr{N})$ as constructed in the first part of the proof above may be approximated by smooth maps. Under the assumption $\pi_{\lfloor sp \rfloor}(\mathscr{N}) = \{0\}$, for any given arbitrarily small number $\delta > 0$, one may find a smooth map u^{ex}_δ such that u^{ex}_δ coincides with u_η everywhere on Q^m except on $\mathscr{T}^{\lfloor sp \rfloor^*}_\eta + Q^m_\delta$. This is explained in Section 2.8, in connection with the notion of *extension property* introduced by Hang F. and Lin F. [HL03a].

The map u^{ex}_δ allows us to remove the singularities of u_η, but this topological construction does not allow to conclude that u^{ex}_δ is close to u_η with respect to the $W^{s,p}$ distance, since u^{ex}_δ could have arbitrarily large energy on the set $\mathscr{T}^{\lfloor sp \rfloor^*}_\eta + Q^m_\delta$ where it differs from u_η. To overcome this issue, we use a scaling argument to obtain a better extension. Again, we illustrate the method on the model case where $s = 1$ and $m - 1 < p < m$. Assume that $v \in W^{1,p}(Q^m)$ and that $w \in C^\infty(\overline{Q^m})$ coincides with v on $Q^m \setminus Q^m_\delta$, where

$0 < \delta < \frac{1}{2}$. Given $0 < \tau < 1$, we define w_τ on Q^m by

$$w_\tau(x) = \begin{cases} w(x) & \text{if } x \in Q^m \setminus B_{2\delta}, \\ w\left(\frac{x}{\tau}\right) & \text{if } x \in B_{\tau\delta}, \\ w\left(\frac{x}{|x|}\left(\frac{1}{2-\tau}(|x| - \tau\delta) + \delta\right)\right) & \text{if } x \in B_{2\delta} \setminus B_{2\delta}. \end{cases}$$

This corresponds to shrinking w from B_δ to $B_{\tau\delta}$ while keeping it unchanged on $Q^m \setminus B_{2\delta}$, filling the transition region by linear interpolation. By a change of variable, we estimate

$$\begin{aligned} \|\mathrm{D}w_\tau\|^p_{L^p(Q^m)} &= \|\mathrm{D}w_\tau\|^p_{L^p(B_{\tau\delta})} + \|\mathrm{D}w_\tau\|^p_{L^p(Q^m \setminus B_{\tau\delta})} \\ &\lesssim \tau^{m-p}\|\mathrm{D}w\|^p_{L^p(B_\delta)} + \|\mathrm{D}w\|^p_{L^p(Q^m \setminus B_\delta)}. \end{aligned}$$

Since $v = w$ on $Q^m \setminus Q^m_\delta$, we deduce that

$$\|\mathrm{D}w_\tau\|^p_{L^p(Q^m)} \lesssim \tau^{m-p}\|\mathrm{D}w\|^p_{L^p(B_\delta)} + \|\mathrm{D}v\|^p_{L^p(Q^m \setminus B_\delta)}.$$

Choosing τ sufficiently small — depending on δ and on w — we may therefore make so that

$$\|\mathrm{D}w_\tau\|^p_{L^p(Q^m)} \lesssim \|\mathrm{D}v\|^p_{L^p(Q^m)}.$$

In Section 2.7, we explain the technique of *shrinking*, which is actually a more involved version of this scaling argument, devised in [BPVS15, Section 8] to handle lower order skeletons and higher order regularity.

An illustration of this idea is available on Figure 2.5. The point singularities in Figure 2.4 have been patched with a topological extension, which has been shrinked into the small region in gray to obtain a map with controlled energy.

This allows to proceed with the proof of density of smooth maps in Section 2.8. The strategy is exactly the same as in the model example above: we start with the smooth extension u^{ex}_δ provided by topological arguments, we shrink it to a map $u^{\mathrm{sh}}_{\delta,\tau}$, and we use carefully the estimates available for shrinking to choose the parameter $\tau > 0$ in order to obtain a better extension with control of the energy. As $\delta \to 0$, this provides an approximation of u_η by smooth maps with values into $\mathscr{N}$, which is enough to prove the density of smooth maps, since we already obtained the density of class $\mathscr{R}$.

After this sketch of our proofs, we move to the detailed construction of the different tools that were described above. The proofs being rather long and technical, we hope that this informal presentation will help the reader to identify and keep in mind the purpose and the intuition behind each construction when studying the details of the

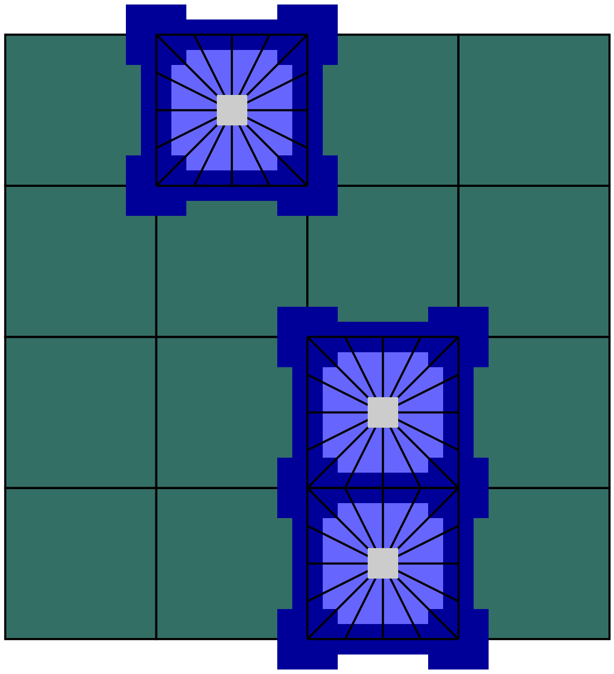

Figure 2.5 – Shrinking around the centers of bad cubes

reasoning.

We end this section with two lemmas that will be used repeatedly in the sequel. Most of our constructions on cubications will be built blockwise: we start from a building block defined on a cube, and we glue copies of this block on each cube of the skeleton to obtain a map defined on the whole skeleton. When establishing Sobolev estimates for such constructions, integer order estimates on the skeleton are readily obtained from corresponding estimates on each cube by additivity of the integral. On the contrary, the Gagliardo seminorm is not additive due to its nonlocal nature. We bypass this obstruction by relying on the lemmas below.

Lemma 2.2.1. *Let $\delta > 0$ and let $\Omega = \bigcup_{i \in I} \Omega_i$, where I is finite or countable and $\Omega_i \subset \mathbb{R}^m$ for every $i \in I$. Set $\Omega_{i,\delta} = \{x \in \Omega \colon \mathrm{dist}(x, \Omega_i) < \delta\}$. For every $u \colon \Omega \to \mathbb{R}$ measurable, one has*

$$|u|^p_{W^{\sigma,p}(\Omega)} \leq \sum_{i \in I} |u|^p_{W^{\sigma,p}(\Omega_{i,\delta})} + C\delta^{-\sigma p} \|u\|^p_{L^p(\Omega)}$$

for some constant $C > 0$ depending on m, σ, and p.

This lemma acts as a replacement for the additivity for the Gagliardo seminorm. Similar kind of results were already present in the work of G. Bourdaud concerning the continuity of the composition operator on Sobolev or Besov spaces; see e.g. [Bou93] and the references therein. The price to pay to have a decomposition of the Gagliardo seminorm is that we need some margin of security between the different parts of the domain on which we split the energy, and that an additional term involving the L^p norm of the map under consideration shows up, which deteriorates as the margin of security shrinks. In the sequel, Lemma 2.2.1 will often be employed by taking the Ω_i to be rectangles, which therefore suggests to have at our disposal estimates on rectangles slightly larger than the Ω_i. Here we use the term *rectangle* to denote any product of m intervals with non-empty interior. We reserve the word *cube* for the case where all the intervals have the same length.

Proof. Let x, $y \in \Omega$. By assumption, either x, $y \in \Omega_{i,\delta}$ for some $i \in I$, or $|x-y| \geq \delta$. Otherwise stated,

$$\Omega \times \Omega \subset \{(x,y) \in \Omega \times \Omega \colon |x-y| \geq \delta\} \cup \bigcup_{i \in I} \Omega_{i,\delta} \times \Omega_{i,\delta}.$$

Therefore,

$$|u|^p_{W^{\sigma,p}(\Omega)} \leq \sum_{i \in I} |u|^p_{W^{\sigma,p}(\Omega_{i,\delta})} + \int_{\{(x,y) \in \Omega \times \Omega \colon |x-y| \geq \delta\}} \frac{|u(x)-u(y)|^p}{|x-y|^{m+\sigma p}} \,\mathrm{d}x\,\mathrm{d}y.$$

We estimate

$$\begin{aligned}\int_{\{(x,y) \in \Omega \times \Omega \colon |x-y| \geq \delta\}} &\frac{|u(x)-u(y)|^p}{|x-y|^{m+\sigma p}} \,\mathrm{d}x\,\mathrm{d}y \\ &\leq 2^p \int_{\Omega} |u(x)|^p \left(\int_{\mathbb{R}^m \setminus B^m_\delta(x)} \frac{1}{|x-y|^{m+\sigma p}} \,\mathrm{d}y \right) \mathrm{d}x \lesssim \delta^{-\sigma p} \int_{\Omega} |u(x)|^p \,\mathrm{d}x.\end{aligned}$$

This completes the proof of the lemma. □

It is also possible to obtain a replacement for the additivity for the Gagliardo seminorm without a term involving the L^p norm of the map under consideration. The price to pay is that such an estimate only applies on finite decompositions, hence not covering the case where an infinite number of sets is involved. If $Q \subset \mathbb{R}^m$ is a rectangle, then λQ is the rectangle having the same center as Q and sidelengths multiplied by λ.

Lemma 2.2.2. *Let $0 < \lambda < 1$ and $Q \subset \mathbb{R}^m$ be a rectangle. For every $\Omega \subset \mathbb{R}^m$ such that $Q \setminus \lambda Q \subset \Omega$ and every $u \colon \Omega \to \mathbb{R}$ measurable, we have*

$$|u|_{W^{\sigma,p}(\Omega)} \leq C\Big(|u|_{W^{\sigma,p}(\Omega \cap Q)} + |u|_{W^{\sigma,p}(\Omega \setminus \lambda Q)} \Big)$$

for some constant $C > 0$ depending on m, σ, p, λ, and the ratio between the largest and the smallest side of Q.

Lemma 2.2.2 is inspired from [MVS19, Lemma 2.2], and we follow their proof. At the core of the argument lies a very classical averaging argument, which was already present in the proof of Besov's lemma; see e.g. [Ada75, Proof of Lemma 7.44]. A similar idea is also used in the proof of Morrey's embedding. This type of argument will be used in multiple occasions in this chapter.

We note that the constant C necessarily diverges to $+\infty$ as $\lambda \to 1$. Moreover, one cannot deduce an improved version of Lemma 2.2.1 without the L^p norm by applying Lemma 2.2.2 inductively, since the constant C is actually larger than 1. Hence, we may iterate the lemma to obtain an estimate for a decomposition into a finite number of sets, but the constant depends on the number of sets.

Proof. We start by writing

$$|u|^p_{W^{\sigma,p}(\Omega)} \leq |u|^p_{W^{\sigma,p}(\Omega\cap Q)} + |u|^p_{W^{\sigma,p}(\Omega\setminus\lambda Q)} + 2\int_{\Omega\cap\lambda Q}\int_{\Omega\setminus Q} \frac{|u(x)-u(y)|^p}{|x-y|^{m+\sigma p}}\,\mathrm{d}y\,\mathrm{d}x.$$

Now we use the average estimate

$$\begin{aligned}
&\int_{\Omega\cap\lambda Q}\int_{\Omega\setminus Q} \frac{|u(x)-u(y)|^p}{|x-y|^{m+\sigma p}}\,\mathrm{d}y\,\mathrm{d}x \\
&\leq 2^{p-1}\biggl(\int_{\Omega\cap\lambda Q}\int_{\Omega\setminus Q}\fint_{Q\setminus\lambda Q} \frac{|u(x)-u(z)|^p}{|x-y|^{m+\sigma p}}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\
&\qquad + \int_{\Omega\cap\lambda Q}\int_{\Omega\setminus Q}\fint_{Q\setminus\lambda Q} \frac{|u(z)-u(y)|^p}{|x-y|^{m+\sigma p}}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\biggr). \qquad (2.2.4)
\end{aligned}$$

Let $c > 0$ be the length of the smallest side of Q. Since $|x-y| \geq c(1-\lambda)$ whenever $x \in \lambda Q$ and $y \in \Omega\setminus Q$, first integrating with respect to y in the first term on the right-hand side of (2.2.4), we find

$$\begin{aligned}
&\int_{\Omega\cap\lambda Q}\int_{\Omega\setminus Q}\fint_{Q\setminus\lambda Q} \frac{|u(x)-u(z)|^p}{|x-y|^{m+\sigma p}}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\
&\qquad\qquad \lesssim \frac{1}{c^{\sigma p}(1-\lambda)^{\sigma p}}\int_{\Omega\cap\lambda Q}\fint_{Q\setminus\lambda Q}|u(x)-u(z)|^p\,\mathrm{d}z\,\mathrm{d}x.
\end{aligned}$$

Now we observe that $|Q\setminus\lambda Q| \geq (1-\lambda^m)c^m$ and that $|x-z| \lesssim c$ for $x \in \lambda Q$ and $z \in Q\setminus\lambda Q$. Here, the hidden constant depends on the ratio between the largest and the smallest side of Q. This allows us to conclude that

$$\int_{\Omega\cap\lambda Q}\int_{\Omega\setminus Q}\fint_{Q\setminus\lambda Q}\frac{|u(x)-u(z)|^p}{|x-y|^{m+\sigma p}}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\ \lesssim \frac{1}{(1-\lambda)^{\sigma p}(1-\lambda^m)}\int_{\Omega\cap\lambda Q}\int_{Q\setminus\lambda Q}\frac{|u(x)-u(z)|^p}{|x-z|^{m+\sigma p}}\,\mathrm{d}z\,\mathrm{d}x \lesssim |u|^p_{W^{\sigma,p}(\Omega\cap Q)}.$$

For the second term in the right-hand side of (2.2.4), we start by noting that if $x \in \lambda Q$ and $y \in \partial(rQ)$ for some $r \geq 1$, then $|x-y| \geq c(r-\lambda)$. On the other hand, if $y \in \partial(rQ)$ and $z \in Q \setminus \lambda Q$, then

$$|y-z| \lesssim c(r+1) = c\frac{r+1}{r-\lambda}(r-\lambda) \lesssim c(r-\lambda),$$

where the hidden constant in the first inequality depends on the ratio between the largest and the smallest side of Q. Therefore, for any $x \in \lambda Q$, $y \in \Omega \setminus Q$, and $z \in Q \setminus \lambda Q$, we have $|y-z| \lesssim |x-y|$. Hence, we obtain

$$\int_{\Omega\cap\lambda Q}\int_{\Omega\setminus Q}\fint_{Q\setminus\lambda Q}\frac{|u(z)-u(y)|^p}{|x-y|^{m+\sigma p}}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\ \lesssim \frac{\lambda^m}{1-\lambda^m}\int_{\Omega\setminus Q}\int_{Q\setminus\lambda Q}\frac{|u(z)-u(y)|^p}{|y-z|^{m+\sigma p}}\,\mathrm{d}z\,\mathrm{d}y \lesssim |u|^p_{W^{\sigma,p}(\Omega\setminus\lambda Q)}.$$

Gathering the estimates for both terms in the right-hand side of (2.2.4) yields the conclusion. □

2.3 Opening

This section is devoted to the opening procedure. We follow the approach of P. Bousquet, A. Ponce, and J. Van Schaftingen [BPVS15, Section 2], who adapted to higher order regularity a construction of H. Brezis and Li Y. [BL01]. The main result of this section is the following fractional counterpart of [BPVS15, Proposition 2.1], which contains the opening construction. Recall that we write $s = k + \sigma$, with $k \in \mathbb{N}$ and $\sigma \in [0, 1)$. Note carefully that the map Φ constructed below *depends on* the map $u \in W^{s,p}$ it is composed with.

Proposition 2.3.1. *Let $\Omega \subset \mathbb{R}^m$ be open, $d \in \{0, \dots, m-1\}$, $\eta > 0$, $0 < \rho < \frac{1}{2}$, and U^d be a subskeleton of $\mathbb{R}^m$ of radius η such that $\mathcal{U}^d + Q^m_{2\rho\eta} \subset \Omega$. For every $u \in W^{s,p}(\Omega; \mathbb{R}^\nu)$, there exists a smooth map $\Phi\colon \mathbb{R}^m \to \mathbb{R}^m$ such that*

(*i*) *for every $l \in \{0, \dots, d\}$ and for every $\sigma^l \in U^l$, Φ is constant on the $(m-l)$-dimensional cubes of radius $\rho\eta$ which are orthogonal to σ^l;*

(*ii*) $\operatorname{Supp}\Phi \subset \mathscr{U}^d + Q^m_{2\rho\eta}$ *and* $\Phi(\mathscr{U}^d + Q^m_{2\rho\eta}) \subset \mathscr{U}^d + Q^m_{2\rho\eta}$;

(*iii*) $u \circ \Phi \in W^{s,p}(\Omega; \mathbb{R}^\nu)$, *and moreover, for every* $\omega \subset \Omega$ *such that* $\mathscr{U}^d + Q^m_{2\rho\eta} \subset \omega$, *the following estimates hold*:

(*a*) *if* $0 < s < 1$, *then*

$$\eta^s |u \circ \Phi|_{W^{s,p}(\omega)} \leq C\Big(\eta^s |u|_{W^{s,p}(\omega)} + \|u\|_{L^p(\omega)}\Big);$$

(*b*) *if* $s \geq 1$, *then for every* $j \in \{1, \dots, k\}$,

$$\eta^j \|D^j(u \circ \Phi)\|_{L^p(\omega)} \leq C \sum_{i=1}^{j} \eta^i \|D^i u\|_{L^p(\omega)};$$

(*c*) *if* $s \geq 1$ *and* $\sigma \neq 0$, *then for every* $j \in \{1, \dots, k\}$,

$$\eta^{j+\sigma} |D^j(u \circ \Phi)|_{W^{\sigma,p}(\omega)} \leq C \sum_{i=1}^{j} \Big(\eta^i \|D^i u\|_{L^p(\omega)} + \eta^{i+\sigma} |D^i u|_{W^{\sigma,p}(\omega)}\Big);$$

(*d*) *for every* $0 < s < +\infty$,

$$\|u \circ \Phi\|_{L^p(\omega)} \leq C \|u\|_{L^p(\omega)};$$

(*iv*) *for every* $\omega \subset \Omega$ *such that* $\mathscr{U}^d + Q^m_{2\rho\eta} \subset \omega$, *the following estimates hold*:

(*a*) *if* $0 < s < 1$, *then*

$$\eta^s |u \circ \Phi - u|_{W^{s,p}(\omega)} \leq C\Big(\eta^s |u|_{W^{s,p}(U^\ell + Q^m_{2\rho\eta})} + \|u\|_{L^p(\mathscr{U}^d + Q^m_{2\rho\eta})}\Big);$$

(*b*) *if* $s \geq 1$, *then for every* $j \in \{1, \dots, k\}$,

$$\eta^j \|D^j(u \circ \Phi) - D^j u\|_{L^p(\omega)} \leq C \sum_{i=1}^{j} \eta^i \|D^i u\|_{L^p(\mathscr{U}^d + Q^m_{2\rho\eta})};$$

(*c*) *if* $s \geq 1$ *and* $\sigma \neq 0$, *then for every* $j \in \{1, \dots, k\}$,

$$\eta^{j+\sigma} |D^j(u \circ \Phi) - D^j u|_{W^{\sigma,p}(\omega)} \leq C \sum_{i=1}^{j} \Big(\eta^i \|D^i u\|_{L^p(\mathscr{U}^d + Q^m_{2\rho\eta})} + \eta^{i+\sigma} |D^i u|_{W^{\sigma,p}(\mathscr{U}^d + Q^m_{2\rho\eta})}\Big);$$

(*d*) *for every* $0 < s < +\infty$,

$$\|u \circ \Phi - u\|_{L^p(\omega)} \le C\|u\|_{L^p(\mathscr{U}^d + Q^m_{2\rho\eta})};$$

for some constant $C > 0$ *depending on* m, s, p, *and* ρ.

Recall that $\operatorname{Supp}\Phi$ denotes the geometric support of Φ, defined as

$$\operatorname{Supp}\Phi = \overline{\{x \in \mathbb{R}^m : \Phi(x) \ne x\}}.$$

Crucial to the proof of the strong density theorem are the estimates in (iii) with $\omega = U^\ell + Q^m_{2\rho\eta}$. They imply that the opening procedure does not increase too much the energy of the map u where it is modified. Proposition 2.3.1 will be used in the proof of the density of the class $\mathscr{R}$ in order to prove that a map can be opened by paying the price of an arbitrarily small increase of the norm.

The map Φ will be constructed blockwise: for every $l \in \{0, \dots, d\}$ and every $\sigma^l \in U$, we construct an opening map Φ_{σ^l} around the face σ^l, and then we suitably combine those maps together to yield the desired map Φ. The construction of the building block Φ_{σ_l} is performed in Proposition 2.3.2 below. Before giving a precise statement, we first introduce, for the convenience of the reader, some additional notation.

The construction of the map Φ provided by Proposition 2.3.2 involves four parameters $0 < \underline{\rho} < \underline{r} < \overline{r} < \overline{\rho} < 1$. These parameters being fixed, we introduce the rectangles

$$\begin{aligned} Q_1 &= Q_{1,\eta} = Q^l_{(1-\overline{\rho})\eta} \times Q^{m-l}_{\underline{\rho}\eta}, \quad Q_2 = Q_{2,\eta} = Q^l_{(1-\overline{r})\eta} \times Q^{m-l}_{\underline{r}\eta},\\ Q_3 &= Q_{3,\eta} = Q^l_{(1-\underline{r})\eta} \times Q^{m-l}_{\overline{r}\eta}, \quad \text{and} \quad Q_4 = Q_{4,\eta} = Q^l_{(1-\underline{\rho})\eta} \times Q^{m-l}_{\overline{\rho}\eta}. \end{aligned} \tag{2.3.1}$$

The rectangle Q_1 is the place where the opening construction is actually performed: the map Φ only depends on the first d variables on Q_1. The rectangle Q_2 contains the support of the map Φ, that is, Φ coincides with the identity outside of Q_2. The region between Q_1 and the exterior of Q_2 serves as a transition region between both regimes.

From now on we shall keep using the notation $Q_1, \dots, Q_4$ for the sake of conciseness and because it makes more apparent the inclusion relations between the four rectangles: observe that $Q_1 \subset Q_2 \subset Q_3 \subset Q_4$. The dependence with respect to the parameters $\underline{\rho}$, $\underline{r}$, $\overline{r}$, $\overline{\rho}$, and η will be implicit.

Proposition 2.3.2. *Let* $l \in \{0, \dots, m-1\}$, $\eta > 0$, *and* $0 < \underline{\rho} < \underline{r} < \overline{r} < \overline{\rho} < 1$. *For every* $u \in W^{s,p}(Q_4; \mathbb{R}^\nu)$, *there exists a smooth map* $\Phi\colon Q_4 \to Q_4$ *such that*

(*i*) $\Phi(x', x'') = (x', \zeta(x))$ *for every* $x = (x', x'') \in Q_4$, *where* $\zeta\colon Q_4 \to Q^{m-l}_{\overline{\rho}\eta}$ *is smooth;*

(*ii*) *for every* $x' \in Q^{m-l}_{\overline{\rho}\eta}$, Φ *is constant on* $\{x'\} \times Q^{m-l}_{\underline{\rho}\eta}$;

(iii) $\operatorname{Supp}\Phi \subset Q_2$ *and* $\Phi(Q_2) \subset Q_2$;

(iv) $u \circ \Phi \in W^{s,p}(Q_3; \mathbb{R}^\nu)$, *and moreover, the following estimates hold*:

a) *if* $0 < s < 1$, *then*

$$|u \circ \Phi|_{W^{s,p}(Q_3)} \leq C|u|_{W^{s,p}(Q_4)};$$

b) *if* $s \geq 1$, *then for every* $j \in \{1, \ldots, k\}$,

$$\eta^j \|D^j(u \circ \Phi)\|_{L^p(Q_3)} \leq C \sum_{i=1}^{j} \eta^i \|D^i u\|_{L^p(Q_4)};$$

c) *if* $s \geq 1$ *and* $\sigma \neq 0$, *then for every* $j \in \{1, \ldots, k\}$,

$$\eta^{j+\sigma} |D^j(u \circ \Phi)|_{W^{\sigma,p}(Q_3)} \leq C \sum_{i=1}^{j} \Big(\eta^i \|D^i u\|_{L^p(Q_4)} + \eta^{i+\sigma} |D^i u|_{W^{\sigma,p}(Q_4)}\Big);$$

d) *for every* $0 < s < +\infty$,

$$\|u \circ \Phi\|_{L^p(Q_3)} \leq C\|u\|_{L^p(Q_4)};$$

for some constant $C > 0$ *depending on* m, s, p, $\underline{\rho}$, $\underline{r}$, $\overline{r}$, *and* $\overline{\rho}$.

We comment on the domains involved in the estimates of item (iv) in Proposition 2.3.2 above. We need estimates on the rectangle Q_3 instead of the smaller rectangle Q_2 containing the support of Φ, in order to have enough room to apply Lemmas 2.2.1 and 2.2.2 as substitutes for the additivity of the integral when proving the fractional estimates in Proposition 2.3.1. Moreover, we only control the energy on Q_3 by the energy on the larger rectangle Q_4 due to the averaging process involved in the proof of Proposition 2.3.2, as we will see later on.

Taking Proposition 2.3.2 granted for the moment, we proceed with the proof of Proposition 2.3.1. Before providing a detailed rigorous proof, we sketch the argument.

We first open the map u around each vertex of U^0 by applying Proposition 2.3.2 with $d = 0$ and using parameters $\overline{\rho} = 2\rho$ and $\underline{\rho} = \rho_0 < 2\rho$. This produces a map u^0 which is constant on cubes of radius $\rho_0\eta$ around each vertex of U^0. We next open the map u^0 around each edge of U^1 using Proposition 2.3.2 with $l = 1$, $\overline{\rho} = \rho_0$, and $\underline{\rho} = \rho_1 < \rho_0$. One may see that the geometric supports of the building blocks around each face do not overlap, so that we may glue them together to obtain a well-defined map on the whole Ω. This construction yields a map u^1 which is constant on all $(m-1)$-cubes of radius $\rho_1\eta$ which are orthogonal to the edges of U^1, provided that they lie at distance at least

$\rho_0\eta$ from the endpoints of the edges. But the map u^1 is constructed from the map u^0 which was constant on the cubes of radius $\rho_0\eta$ centered at the vertices of U^0. Hence we conclude that the map u^1 is constant on all $(m-1)$-cubes of radius $\rho_1\eta$ which are orthogonal to the edges of U^1. We then pursue this construction by induction until we reach the desired dimension, which yields a map Φ as in Proposition 2.3.1.

An illustration of this construction on one cube for $m = 2$ and $\ell = 1$ is presented in Figure 2.6. On the left part of the figure, one sees the result of opening around vertices. The map u becomes constant on the dark blue squares, and is left unchanged on the white region, the light blue region serving as a transition. The central part of the figure shows the opening step around edges. The map u becomes constant on the segments orthogonal to the edges of U^1 that are sufficiently far from the vertices, some of which being represented in black. The regions involved in the construction at the previous step, when opening around the vertices, are depicted in light colors, to show how all the regions are located relatively to each other. One sees that the opening regions around vertices and edges connect perfectly. The right part of the figure shows the combination of both steps. The map u becomes constant on all segments orthogonal to the edges of U^1.

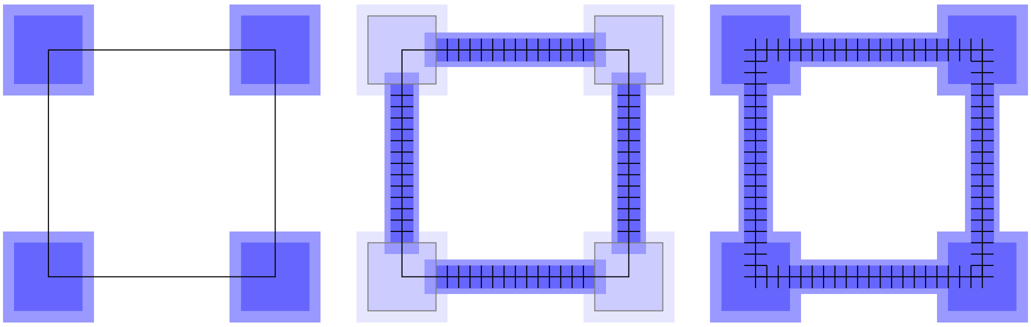

Figure 2.6 – Opening for $m = 2$ and $d = 1$

The construction sketched above is strongly inspired by [BPVS15, Proposition 2.1], but nevertheless significantly different from [BPVS15, Proposition 2.1]. Indeed, in our approach, at each step of the iterative process, the sets on which we apply opening around each face (the building blocks of our global construction) do not overlap. Hence, gathering the constructions made around each face yields a globally well-defined map on the whole Ω, regardless of the map we started with at the beginning of the step. For instance, the map u^1 described in the above sketch is well-defined on the whole Ω, regardless of the form of the map u^0. On the other hand, the construction in [BPVS15] relies on the fact that, at the l-th step of the iterative process, we work with a map that has already been opened around the i-faces for $i < l$. Indeed the constructions made at step l are not compatible near the lower dimensional faces, where they overlap. Our approach simplifies the proof of Sobolev estimates, especially in the fractional case where one needs some margin of security to apply Lemmas 2.2.1 and 2.2.2, but also for the case of integer order estimates (already treated in [BPVS15]).

Proof of Proposition 2.3.1. As announced, we construct a family of maps $(\Phi_l)_{0\leq l\leq d}$ by induction. For the convenience of notation, we set $\Phi_{-1} = \mathrm{id}$. Assuming that the maps $\Phi_{-1}, \dots, \Phi_{l-1}$ have already been constructed, we set $u^l = u \circ \Phi_{-1} \circ \dots \circ \Phi_{l-1}$. Let $(\rho_l)_{0\leq l\leq d}$, $(\underline{r}_l)_{0\leq l\leq d}$ and $(\overline{r}_l)_{0\leq l\leq d}$ be decreasing sequences such that

$$\rho = \rho_d < \underline{r}_d < \overline{r}_d < \rho_{d-1} < \dots < \rho_l < \underline{r}_l < \overline{r}_l < \rho_{l-1} < \dots < \rho_0 < \underline{r}_0 < \overline{r}_0 < 2\rho.$$

For every $l \in \{0, \dots, d\}$ and every $\sigma^l \in \mathscr{U}^l$, there is an isometry T_{σ^l} of $\mathbb{R}^m$ mapping $Q^l_\eta \times \{0\}^{m-d}$ onto σ^l. Via this isometry, we apply Proposition 2.3.2 to u^l around σ^l with parameters $\underline{\rho} = \rho_l$, $\underline{r} = \underline{r}_l$, $\overline{r} = \overline{r}_l$ and $\overline{\rho} = \rho_{l-1}$ — with the convention that $\rho_{-1} = 2\rho$ — in order to obtain a map $\Phi^{\sigma^l}\colon T_{\sigma^l}(Q_4) \to T_{\sigma^l}(Q_4)$ such that, for every $x' \in \sigma^l$ with $\operatorname{dist}(x', \partial\sigma^l) > \rho_{l-1}$, Φ^{σ^l} is constant on the cube orthogonal to σ^l of radius $\rho_l\eta$ passing through x'. We then define $\Phi_l\colon \mathbb{R}^m \to \mathbb{R}^m$ by

$$\Phi_l(x) = \begin{cases} \Phi^{\sigma^l}(x) & \text{if } x \in T_{\sigma^l}(Q_4), \\ x & \text{otherwise.} \end{cases}$$

This map is well-defined since $\operatorname{Supp} \Phi^{\sigma^l} \subset T_{\sigma^l}(Q_2)$ and $T_{\sigma^l_1}(Q_2) \cap T_{\sigma^l_2}(Q_2) = \varnothing$ if $\sigma^l_1 \neq \sigma^l_2$. Finally, we set $\Phi = \Phi_0 \circ \dots \circ \Phi_d$.

By induction and using the definition of the maps Φ^{σ^l} provided by Proposition 2.3.2, we observe that Φ satisfies properties (i) and (ii). We now turn to properties (iii) and (iv). Let $\mathscr{U}^d + Q^m_{2\rho\eta} \subset \omega \subset \Omega$. Notice that it suffices to prove property (iii) with Φ replaced by Φ_l and u replaced by u^l, as one may then conclude by induction.

We start with the estimates for integer order derivatives. Let $j \in \{1, \dots, k\}$ and $l \in \{0, \dots, d\}$. By the additivity of the integral, we have

$$\|D^j(u^l \circ \Phi_l)\|^p_{L^p(\omega)} \leq \sum_{\sigma^l \in \mathscr{U}^l} \|D^j(u^l \circ \Phi_l)\|^p_{L^p(T_{\sigma^l}(Q_3))} + \|D^j(u^l \circ \Phi_l)\|^p_{L^p\left(\omega \setminus \bigcup\limits_{\sigma^l \in \mathscr{U}^l} T_{\sigma^l}(Q_3)\right)}.$$

Since $\operatorname{Supp} \Phi^{\sigma^l} \subset T_{\sigma^l}(Q_2) \subset T_{\sigma^l}(Q_3)$, we find

$$\|D^j(u^l \circ \Phi_l)\|_{L^p\left(\omega \setminus \bigcup\limits_{\sigma^l \in \mathscr{U}^l} T_{\sigma^l}(Q_3)\right)} = \|D^j u^l\|_{L^p\left(\omega \setminus \bigcup\limits_{\sigma^l \in \mathscr{U}^l} T_{\sigma^l}(Q_3)\right)},$$

while the estimate given by Proposition 2.3.2 yields

$$\eta^j \|\mathrm{D}^j (u^l \circ \Phi_l)\|_{L^p(T_{\sigma^l}(Q_3))} = \eta^j \|\mathrm{D}^j (u^l \circ \Phi^{\sigma^l})\|_{L^p(T_{\sigma^l}(Q_3))} \lesssim \sum_{i=1}^{j} \eta^i \|\mathrm{D}^i u^l\|_{L^p(T_{\sigma^l}(Q_4))}.$$

Combining both above estimates and using the fact that the number of overlaps between one given set of the form $T_{\sigma^d}(Q_4)$ and all the other such sets is bounded from above by a number depending only on m, we deduce that

$$\eta^j \|\mathrm{D}^j (u^l \circ \Phi_l)\|_{L^p(\omega)} \lesssim \sum_{i=1}^{j} \eta^i \|\mathrm{D}^i u^l\|_{L^p(\omega)} \quad \text{for every } j \in \{1, \dots, k\}.$$

Since $\operatorname{Supp} \Phi \subset U^d + Q^m_{2\rho\eta}$, the estimate (b) of point (iv) follows directly from estimate (b) of point (iii) using again the additivity of the integral. The estimates for the L^p norm of $u \circ \Phi$ (estimates (d)) are proven similarly.

The estimates for the Gagliardo seminorm are proved similarly, replacing the additivity of the integral by Lemma 2.2.1. Indeed, if $k \geq 1$, this lemma ensures that

$$\begin{aligned} |\mathrm{D}^j (u^l \circ \Phi_l)|^p_{W^{\sigma,p}(\omega)} &\lesssim \sum_{\sigma^l \in \mathscr{U}^l} |\mathrm{D}^j (u^l \circ \Phi_l)|^p_{W^{\sigma,p}(T_{\sigma^l}(Q_3))} \\ &\quad + |\mathrm{D}^j (u^l \circ \Phi_l)|^p_{L^p(\omega \setminus \operatorname{Supp} \Phi_l)} + \eta^{-\sigma p} \|\mathrm{D}^j (u^l \circ \Phi_l)\|^p_{L^p(\omega)} \quad \text{for every } j \in \{1, \dots, k\}. \end{aligned}$$

Note that here, we made use of the fact that the distance between the support of the map provided by Proposition 2.3.2 and the complement of Q_3 is bounded from below by a constant multiple of η. Estimate (c) of point (iii) then follows as for the integer order estimate. To obtain the estimate (c) for point (iv), we observe that actually $\operatorname{dist}(\operatorname{Supp} \Phi, \omega \setminus (\mathscr{U}^d + Q^m_{2\rho\eta}))$ is bounded from below by a constant multiple of η. We conclude by making again use of Lemma 2.2.1 along with the integer order estimate that we already obtained. Indeed, we have

$$\begin{aligned} |\mathrm{D}^j (u \circ \Phi) - \mathrm{D}^j u|^p_{W^{\sigma,p}(\omega)} &\leq |\mathrm{D}^j (u \circ \Phi) - \mathrm{D}^j u|^p_{W^{\sigma,p}(U^d + Q^m_{2\rho\eta})} \\ &\quad + |\mathrm{D}^j (u \circ \Phi) - \mathrm{D}^j u|^p_{W^{\sigma,p}(\omega \setminus \operatorname{Supp} \Phi)} + \eta^{-\sigma p} \|\mathrm{D}^j (u \circ \Phi) - \mathrm{D}^j u\|^p_{L^p(\omega)}. \end{aligned}$$

The first term is upper bounded using the triangle inequality and estimate (c) of (iii), the second one vanishes by definition of the geometric support and the third one is the integer order term that we already estimated (item (b) of (iv)). The case $0 < s < 1$ is handled in the same way, replacing $\mathrm{D}^j u$ by u. □

We now turn to the proof of Proposition 2.3.2. Consider some fixed Borel map $u\colon Q^m \to \mathscr{N}$ such that $u \in W^{s,p}(Q^m)$. In order to prove that there exists some map Φ (*depending on* u) such that $u \circ \Phi \in W^{s,p}(Q^m; \mathscr{N})$ along with the corresponding estimates, it will be convenient to rely on some genericity arguments using the framework of *Fuglede maps*, in a formalism recently developed by P. Bousquet, A. Ponce, and J. Van Schaftingen in [BPVS25]; our presentation is limited to the tools instrumental in our proofs. The results below are taken from [BPVS25], sometimes with slight modifications. Nevertheless, we reproduce the proofs here for the convenience of the reader.

We start with the following lemma [BPVS25], suited for L^p regularity, which gives a criterion to detect a family of maps γ such that composition with γ is compatible with L^p convergence.

Lemma 2.3.3. *Let $(X, \mathcal{X}, \mu)$ be a measure space, $u\colon X \to \mathbb{R}$ a measurable map which does not vanish μ-almost everywhere, and $(u_n)_{n\in\mathbb{N}}$ a sequence of maps in $L^p(X,\mu)$ such that $u_n \to u$ in $L^p(X,\mu)$. There exists a summable function $w\colon X \to [0,+\infty]$ satisfying $\int_X w \,\mathrm{d}\mu > 0$ and a subsequence $(u_{n_i})_{i\in\mathbb{N}}$ such that for every measure space $(Y, \mathcal{Y}, \lambda)$ and every measurable map $\gamma\colon Y \to X$ satisfying $w \circ \gamma \in L^1(Y,\lambda)$, we have $u_{n_i} \circ \gamma \in L^p(Y,\lambda)$,*

$$u_{n_i} \circ \gamma \to u \circ \gamma \quad \text{in } L^p(Y,\lambda),$$

and

$$\int_Y |u \circ \gamma|^p \,\mathrm{d}\lambda \leq 2\frac{\int_Y w \circ \gamma \,\mathrm{d}\lambda}{\int_X w \,\mathrm{d}\mu} \int_X |u|^p \,\mathrm{d}\mu.$$

Lemma 2.3.3 essentially corresponds to [BPVS25, Proposition 2.1 and Lemma 2.15]. We insist on the fact that the map w depends on u. Even modifying u on a null set may change the map w given by Lemma 2.3.3.

Proof. We choose a sequence $(\kappa_i)_{i\in\mathbb{N}}$ diverging to $+\infty$ such that $\kappa_i \geq 1$ for every $i \in \mathbb{N}$. We then extract a subsequence $(u_{n_i})_{i\in\mathbb{N}}$ so that

$$\|u\|_{L^p(X,\mu)} + \sum_{i\in\mathbb{N}} \kappa_i \|u_{n_i} - u\|_{L^p(X,\mu)} < 2^{\frac{1}{p}} \|u\|_{L^p(X,\mu)}$$

and define $w\colon X \to [0,+\infty]$ by

$$w = \Big(|u| + \sum_{i\in\mathbb{N}} \kappa_i |u_{n_i} - u|\Big)^p.$$

We deduce from the triangle inequality and Fatou's lemma that w is summable with

$$\Big(\int_X w \,\mathrm{d}\mu\Big)^{\frac{1}{p}} = \|w^{\frac{1}{p}}\|_{L^p(X,\mu)} \leq \|u\|_{L^p(X,\mu)} + \sum_{i\in\mathbb{N}} \kappa_i \|u_{n_i} - u\|_{L^p(X,\mu)} < 2^{\frac{1}{p}}\|u\|_{L^p(X,\mu)}. \tag{2.3.2}$$

Since $\kappa_i \geq 1$, we have

$$|u_{n_i}|^p \leq \Big(|u| + |u_{n_i} - u|\Big)^p \leq w.$$

Hence, if $\gamma\colon Y \to X$ satisfies $w \circ \gamma \in L^1(Y, \lambda)$, we find that $u_{n_i} \circ \gamma \in L^p(Y, \lambda)$, and moreover, we have

$$\int_Y |u_{n_i} \circ \gamma - u \circ \gamma|^p \,\mathrm{d}\lambda \leq \frac{1}{\kappa_i^p} \int_Y w \circ \gamma \,\mathrm{d}\lambda.$$

Letting $i \to +\infty$ allows us to conclude that $u_{n_i} \circ \gamma \to u \circ \gamma$ in $L^p(Y, \lambda)$. Furthermore, since $|u \circ \gamma|^p \leq w \circ \gamma$, we obtain

$$\int_Y |u \circ \gamma|^p \,\mathrm{d}\lambda \leq \int_Y w \circ \gamma \,\mathrm{d}\lambda.$$

Combining the above inequality with (2.3.2) provides us with the desired estimate, and therefore concludes the proof. □

Using the previous lemma, we may now obtain a criterion to detect a family of maps γ such that composition with γ is compatible with $W^{k,p}$ regularity, along with the corresponding estimates; see [BPVS25, Proposition 2.14]. Once again, we note that the map w given by the lemma below depends on u.

Lemma 2.3.4. *Let $\Omega \subset \mathbb{R}^m$ be an open set and $u \in W^{k,p}(\Omega)$. There exists a summable map $w\colon \Omega \to [0, +\infty]$ such that*

$$\int_\Omega w = 1$$

and such that for every open set $\omega \subset \mathbb{R}^M$ and every map $\gamma \in C^\infty(\omega; \Omega)$ with bounded derivatives, if $w \circ \gamma$ is summable, then we have $u \circ \gamma \in W^{k,p}(\omega)$, the derivatives $\mathrm{D}^j(u \circ \gamma)$ are given by the classical Faà di Bruno formula, and

$$\|\mathrm{D}^j u \circ \gamma\|_{L^p(\omega)} \leq C\Big(\int_\omega w \circ \gamma\Big)^{\frac{1}{p}} \|\mathrm{D}^j u\|_{L^p(\Omega)} \quad \textit{for every } j \in \{0, \dots, k\},$$

for some constant $C > 0$ depending on m, M, k, and p.

We note for further use that, under the assumptions of Lemma 2.3.4, applying the Faà di Bruno formula, we may estimate $\mathrm{D}^j(u \circ \gamma)$ as follows:

$$\|\mathrm{D}^j(u \circ \gamma)\|_{L^p(\omega)} \leq C\biggl(\int_\omega w \circ \gamma\biggr)^{\frac{1}{p}} \sum_{i=1}^{j} \sum_{\substack{1 \leq t_1 \leq \cdots \leq t_i \\ t_1 + \cdots + t_i = j}} \|\mathrm{D}^{t_1}\gamma\|_{L^\infty(\omega)} \cdots \|\mathrm{D}^{t_i}\gamma\|_{L^\infty(\omega)} \|\mathrm{D}^i u\|_{L^p(\Omega)}. \tag{2.3.3}$$

We also make an important remark about a measurability issue. In Lemma 2.3.3, we worked with arbitrary measure spaces. On the other hand, here we implicitly assume that $\mathbb{R}$ and $\mathbb{R}^m$ are endowed with the Borel σ-algebra (and not the Lebesgue σ-algebra) in order to ensure that continuous maps are measurable.

Proof. We may assume, without loss of generality, that u and its k first derivatives are not almost everywhere equal to 0. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of smooth maps converging to u in $W^{k,p}(\Omega)$. We apply inductively Lemma 2.3.3 to $\mathrm{D}^i u$ for $i \in \{0, \ldots, k\}$ to obtain summable maps $w_i \colon \Omega \to [0, +\infty]$ satisfying $\int_\Omega w_i > 0$ and a subsequence $(u_{n_l})_{l\in\mathbb{N}}$ such that, for every measurable map $\gamma \colon \omega \to \Omega$ such that $w_i \circ \gamma$ is summable, $\mathrm{D}^i u_{n_l} \circ \gamma \to \mathrm{D}^i u \circ \gamma$ in $L^p(\omega)$, and

$$\int_\omega |\mathrm{D}^i u \circ \gamma|^p \leq 2\frac{\int_\omega w_i \circ \gamma}{\int_\Omega w_i} \int_\Omega |\mathrm{D}^i u|^p.$$

Let

$$w = \frac{1}{k+1} \sum_{i=0}^{k} \frac{w_i}{\int_\Omega w_i}.$$

It is readily seen that

$$\int_\Omega w = 1.$$

Observe also that $w_i \leq (k+1) w \int_\Omega w_i$. Therefore, if $w \circ \gamma$ is summable, we find that $\mathrm{D}^i u \circ \gamma \in L^p(\omega)$ with

$$\int_\omega |\mathrm{D}^i u \circ \gamma|^p \leq 2(k+1)\biggl(\int_\omega w \circ \gamma\biggr) \int_\Omega |\mathrm{D}^i u|^p. \tag{2.3.4}$$

If in addition γ is smooth and has bounded derivatives, since $\mathrm{D}^i u_{n_l} \circ \gamma \to \mathrm{D}^i u \circ \gamma$ in $L^p(\omega)$, $\mathrm{D}^i(u_{n_l} \circ \gamma)$ converges in $L^p(\omega)$ to a map which coincides with the function one would obtain by applying the Faà di Bruno formula to compute $\mathrm{D}^i(u \circ \gamma)$. Hence, the closure property for Sobolev spaces ensures that $u \circ \gamma \in W^{k,p}(\omega)$ and that the Faà di Bruno formula actually applies. The estimates for $\mathrm{D}^j u \circ \gamma$ are already contained in inequality (2.3.4), and therefore the proof is complete. □

After dealing with integer order Sobolev spaces, we present the next lemma, which contains the construction of a detector for maps preserving fractional Sobolev regularity under composition; see [BPVS25, Proposition 2.12].

Lemma 2.3.5. *Let $\Omega \subset \mathbb{R}^m$ be an open set and $u \in W^{\sigma,p}(\Omega)$. Define $w \colon \Omega \to [0, +\infty]$ by*

$$w(x) = \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{m+\sigma p}}\,\mathrm{d}y.$$

Assume moreover that there exists $c > 0$ such that $|B^m_\lambda(z) \cap \Omega| \geq c\lambda^m$ for every $z \in \Omega$ and $0 < \lambda \leq \frac{1}{2}\operatorname{diam}\Omega$. For every open set $\omega \subset \mathbb{R}^M$ and every Lipschitz map $\gamma \colon \omega \to \Omega$, if $w \circ \gamma$ is summable, then we have $u \circ \gamma \in W^{\sigma,p}(\omega)$ with

$$|u \circ \gamma|_{W^{\sigma,p}(\omega)} \leq C|\gamma|^\sigma_{C^{0,1}(\omega)}\biggl(\int_\omega w \circ \gamma\biggr)^{\frac{1}{p}},$$

for some constant $C > 0$ depending on m, M, σ, p, and c.

We recall that $|\gamma|_{C^{0,1}(\omega)}$ denotes the Lipschitz seminorm of γ, defined by

$$|\gamma|_{C^{0,1}(\omega)} = \sup_{\substack{x,y \in \omega \\ x \neq y}} \frac{|\gamma(x) - \gamma(y)|}{|x-y|}.$$

In contrast to what happens for integer order Sobolev spaces, here we have an explicit expression for w depending on u. It is useful to observe that

$$\int_\Omega w = |u|^p_{W^{\sigma,p}(\Omega)}.$$

We also comment on the assumption on the volume of balls in Ω, which will be crucial during the proof. It is in particular satisfied if Ω is a cube. Indeed, in this case, any ball centered at a point of Ω with radius less that $\frac{1}{2}\operatorname{diam}\Omega$ has at least one quadrant in Ω, which implies that Ω satisfies the assumptions of Lemma 2.3.5. To prove Proposition 2.3.2, we only need to apply Lemma 2.3.5 on cubes, but later on in Section 2.7, we will need to use a similar technique on more general domains, whence our motivation for already presenting a more general statement here.

Proof. For every $x, y \in \omega$, we let $\mathscr{B}_{x,y} = B^m_{|\gamma(x)-\gamma(y)|}\Big(\frac{\gamma(x)+\gamma(y)}{2}\Big) \cap \Omega$. We write

$$|u \circ \gamma(x) - u \circ \gamma(y)|^p \lesssim \fint_{\mathscr{B}_{x,y}} |u \circ \gamma(x) - u(z)|^p \,\mathrm{d}z + \fint_{\mathscr{B}_{x,y}} |u(z) - u \circ \gamma(y)|^p \,\mathrm{d}z.$$

We note that

$$B^m_{\frac{|\gamma(x)-\gamma(y)|}{2}}(\gamma(x)) \cap \Omega \subset \mathscr{B}_{x,y}.$$

Since $\frac{|\gamma(x)-\gamma(y)|}{2} \le \frac{1}{2} \operatorname{diam} \Omega$, we deduce that $|\mathscr{B}_{x,y}| \gtrsim |\gamma(x) - \gamma(y)|^m$. Moreover, we observe that for every $z \in \mathscr{B}_{x,y}$, we have

$$|\gamma(x) - z| \le \left|\frac{\gamma(x) + \gamma(y)}{2} - z\right| + \frac{1}{2}|\gamma(x) - \gamma(y)| \le \frac{3}{2}|\gamma(x) - \gamma(y)|,$$

and similarly $|\gamma(y) - z| \le \frac{3}{2}|\gamma(x) - \gamma(y)|$. Hence,

$$|u \circ \gamma(x) - u \circ \gamma(y)|^p \lesssim \int_{\mathscr{B}_{x,y}} \frac{|u \circ \gamma(x) - u(z)|^p}{|\gamma(x) - z|^m} \,\mathrm{d}z + \int_{\mathscr{B}_{x,y}} \frac{|u(z) - u \circ \gamma(y)|^p}{|\gamma(y) - z|^m} \,\mathrm{d}z.$$

Dividing by $|x - y|^{M+\sigma p}$ and integrating over $\omega \times \omega$, we deduce that

$$|u \circ \gamma|^p_{W^{\sigma,p}(\omega)} \lesssim \int_\omega \int_\omega \int_{\mathscr{B}_{x,y}} \frac{|u \circ \gamma(x) - u(z)|^p}{|x - y|^{M+\sigma p}|\gamma(x) - z|^m} \,\mathrm{d}z \,\mathrm{d}y \,\mathrm{d}x.$$

We use Tonelli's theorem to deduce that

$$|u \circ \gamma|^p_{W^{\sigma,p}(\omega)} \lesssim \int_\Omega \int_\omega \int_{\mathscr{Y}_{x,z}} \frac{|u \circ \gamma(x) - u(z)|^p}{|x - y|^{M+\sigma p}|\gamma(x) - z|^m} \,\mathrm{d}y \,\mathrm{d}x \,\mathrm{d}z,$$

where $\mathscr{Y}_{x,z}$ is the set of all $y \in \omega$ such that $z \in \mathscr{B}_{x,y}$, that is,

$$\mathscr{Y}_{x,z} = \{y \in \omega : |\gamma(x) + \gamma(y) - 2z| < 2|\gamma(x) - \gamma(y)|\}.$$

Observe that

$$\mathscr{Y}_{x,z} \subset \left\{y \in \mathbb{R}^M : |\gamma(x) - z| < \frac{3}{2}|\gamma(x) - \gamma(y)|\right\}$$
$$\subset \{y \in \mathbb{R}^M : |\gamma(x) - z| < \frac{3}{2}|\gamma|_{C^{0,1}(\omega)}|x - y|\} = \mathbb{R}^M \setminus B^M_r(x),$$

where

$$r = r(x,z) = \frac{2|\gamma(x) - z|}{3|\gamma|_{C^{0,1}(\omega)}}.$$

Hence,

$$|u \circ \gamma|^p_{W^{\sigma,p}(\omega)} \lesssim \int_\Omega \int_\omega \int_{\mathbb{R}^M \setminus B^M_r(x)} \frac{|u \circ \gamma(x) - u(z)|^p}{|x - y|^{M+\sigma p}|\gamma(x) - z|^m} \,\mathrm{d}y \,\mathrm{d}x \,\mathrm{d}z$$
$$\lesssim |\gamma|^{\sigma p}_{C^{0,1}(\omega)} \int_\Omega \int_\omega \frac{|u \circ \gamma(x) - u(z)|^p}{|\gamma(x) - z|^{m+\sigma p}} \,\mathrm{d}x \,\mathrm{d}z,$$

which concludes the proof. □

Now that we have at our disposal a criterion to detect a family of maps that preserve membership in Sobolev spaces after composition, it would be useful to know if, given a detector w associated to a fixed map $u \in W^{s,p}$, we may actually construct many smooth maps γ such that $w \circ \gamma$ is summable. This is based on a genericity argument, and is the purpose of the next lemma, whose proof relies on an averaging argument initially due to H. Federer and W. Fleming [FF60]. Our presentation and proof are taken from [BPVS15, Lemma 2.5].

Lemma 2.3.6. *Let ω, Ω, and $P \subset \mathbb{R}^m$ be measurable sets, with $0 < |P| < +\infty$. Let $\Phi\colon \omega + P \to \Omega$ and $w\colon \Omega \to [0, +\infty]$ be measurable maps. For every $a \in P$, let $\Phi_a\colon \omega \to \mathbb{R}^m$ be given by $\Phi_a(x) = \Phi(x - a) + a$. Assume that, for every $a \in P$ and $x \in \omega$, $\Phi_a(x) \in \Omega$. There exists a subset $A \subset P$ of positive measure such that, for every $a \in A$, we have*

$$\int_\omega w \circ \Phi_a \leq C \frac{|\omega + P|}{|P|} \int_\Omega w,$$

for some constant $C > 0$.

The constant C in the above estimate does not depend on the different parameters involved in the statement of the lemma. However, as we shall see in the proof, the measure of the set A may be taken arbitrarily close to $|P|$ provided that we enlarge C accordingly.

Proof. We are going to estimate the average

$$\fint_P \biggl(\int_\omega w \circ \Phi_a\biggr)\,\mathrm{d}a.$$

By a change of variable by translation and Tonelli's theorem, we compute that

$$\begin{aligned}\int_P \biggl(\int_\omega w \circ \Phi_a\biggr)\,\mathrm{d}a &= \int_P \biggl(\int_{\omega+a} w(\Phi(y)+a)\,\mathrm{d}y\biggr)\,\mathrm{d}a\\ &\leq \int_{\omega+P}\biggl(\int_{P\cap(y-\omega)} w(\Phi(y)+a)\,\mathrm{d}a\biggr)\,\mathrm{d}y \leq \int_{\omega+P}\biggl(\int_\Omega w(x)\,\mathrm{d}x\biggr)\,\mathrm{d}y = |\omega+P|\int_\Omega w.\end{aligned}$$

Therefore,

$$\fint_P \biggl(\int_\omega w \circ \Phi_a\biggr)\,\mathrm{d}a \leq \frac{|\omega+P|}{|P|}\int_\Omega w.$$

Hence, for every $0 < \theta < 1$, there exists a subset $A \subset P$ with measure $|A| \geq \theta|P|$ such that, for every $a \in A$, we have

$$\int_\omega w \circ \Phi_a \leq \frac{1}{1-\theta}\frac{|\omega+P|}{|P|}\int_\Omega w,$$

and the proof of the lemma is complete. □

With all these tools at our disposal, we are now ready to prove Proposition 2.3.2. We start by constructing one model map Φ satisfying the geometric properties in the conclusion of the proposition. Then we use the previous lemmas to show that Φ_a satisfies all the conclusions of Proposition 2.3.2 for some $a \in \mathbb{R}^m$.

Proof of Proposition 2.3.2. We use the notation introduced in (2.3.1). We start with the construction of the model map Φ. Let $\lambda > 0$ be such that

$$\lambda < \min\biggl(\frac{\underline{r}-\underline{\rho}}{2}, \frac{\overline{r}-\underline{r}}{2}, \frac{\overline{\rho}-\overline{r}}{2}\biggr).$$

We define $\Phi\colon Q_{4,1} \to Q_{4,1}$ by

$$\Phi(x', x'') = \biggl(x', \phi\Bigl(\frac{x'}{\eta}, \frac{x''}{\eta}\Bigr)x''\biggr),$$

where $\phi\colon Q_{4,1} \to [0,1]$ is a smooth function such that

(a) for $x \in Q_{1,1} + B^m_\lambda$, $\phi(x) = 0$;

(b) for $x \in (Q_{4,1} \setminus Q_{2,1}) + B^m_\lambda$, $\phi(x) = 1$.

Recall that the $Q_{i,1}$ are the rectangles defined in (2.3.1) with parameter $\eta = 1$. By scaling, we have

$$\|D^j \Phi\|_{L^\infty(Q_4)} \lesssim \eta^{1-j} \quad \text{for every } j \in \{1, \dots, k+1\}.$$

Now we set $\Phi_a(x) = \Phi(x - a) + a$ for every $a \in B^m_{\lambda\eta}$. By construction, Φ_a satisfies the geometric properties (i) to (iii) for every $a \in B^m_{\lambda\eta}$.

We now turn to the Sobolev estimates (iv). In the case where $k = 0$, we apply Lemma 2.3.5 to u, with $\Omega = Q_4$. Let $w\colon Q_4 \to [0, +\infty]$ be the corresponding detector. By Lemma 2.3.6 with $\omega = Q_3$, $\Omega = Q_4$, and $P = B^m_{\lambda\eta}$, there exists $a \in B^m_{\lambda\eta}$ such that

$$\int_{Q_3} w \circ \Phi_a \lesssim \frac{|Q_3 + B^m_\lambda|}{|B^m_{\lambda\eta}|} \int_{Q_4} w.$$

Since Q_3 has sides whose length is proportional to η, this implies that

$$\int_{Q_3} w \circ \Phi_a \lesssim \int_{Q_4} w. \tag{2.3.5}$$

Therefore, $u \circ \Phi_a \in W^{s,p}(Q_4)$ and

$$|u \circ \Phi_a|_{W^{s,p}(Q_3)} \lesssim |\Phi_a|^s_{C^{0,1}(Q_3)} \biggl(\int_{Q_3} w \circ \Phi_a\biggr)^{\frac{1}{p}}.$$

Combining the estimate on the derivative of Φ_a, equation (2.3.5), and the remark following Lemma 2.3.5, we conclude that

$$|u \circ \Phi_a|_{W^{s,p}(Q_3)} \lesssim |u|_{W^{s,p}(Q_4)}.$$

The L^p estimate is obtained as in the case $k \geq 1$ below, and this concludes the proof when $0 < s < 1$.

If now $k \geq 1$, we apply Lemma 2.3.4 to u to obtain a detector $w_0\colon Q_4 \to [0, +\infty]$ and we apply Lemma 2.3.5 to $D^j u$ for every $j \in \{1, \dots, k\}$ to obtain a detector $w_j\colon Q_4 \to [0, +\infty]$. (In the case where $\sigma = 0$, we skip this second step and only construct w_0. In the sequel we continue to speak about w_j for $j \in \{0, \dots, k\}$, it is implicit that when $\sigma = 0$ we only consider w_0.) Then we invoke Lemma 2.3.6 to find some $a \in B^m_{\lambda\eta}$ such that

$$\int_{Q_3} w_j \circ \Phi_a \lesssim \frac{|Q_3 + B^m_{\lambda\eta}|}{|B^m_{\lambda\eta}|} \int_{Q_4} w_j \quad \text{for every } j \in \{0, \dots, k\}. \tag{2.3.6}$$

It is indeed possible to choose the same a simultaneously for each w_j since the set A in Lemma 2.3.6 can be chosen of measure arbitrarily close of $|B^m_{\lambda\eta}|$.

For the integer order derivatives, using the estimates on the derivatives of Φ_a and the fact that $\int_{Q_4} w_0 = 1$, we immediately deduce that $u \circ \Phi_a \in W^{k,p}(Q_3)$,

$$\|\mathrm{D}^j(u \circ \Phi_a)\|_{L^p(Q_3)} \lesssim \sum_{i=1}^{j} \eta^{i-j} \|\mathrm{D}^i u\|_{L^p(Q_4)},$$

and

$$\|u \circ \Phi_a\|_{L^p(Q_3)} \leq \|u\|_{L^p(Q_4)}.$$

In the case $k = 0$, we may still obtain the L^p estimate at order 0 above, constructing the detector w_0 with the help of Lemma 2.3.3 instead of Lemma 2.3.4 and using again the fact that we may choose a suitable a for several detectors simultaneously.

Dealing with fractional order derivatives requires additional computations. We continue to work with a as in (2.3.6). Using the Faà di Bruno formula — which is indeed valid for $u \circ \Phi_a$ by Lemma 2.3.4 — and the multilinearity of the differential, we find

$$|\mathrm{D}^j(u \circ \Phi_a)(x) - \mathrm{D}^j(u \circ \Phi_a)(y)|^p \lesssim \sum_{i=1}^{j} \Big(|\mathrm{D}^i u \circ \Phi_a(x) - \mathrm{D}^i u \circ \Phi_a(y)|^p \eta^{(i-j)p} + \sum_{t=1}^{j} |\mathrm{D}^i u \circ \Phi_a(x)|^p \eta^{(i-1-j+t)p} |\mathrm{D}^t \Phi_a(x) - \mathrm{D}^t \Phi_a(y)|^p \Big). \tag{2.3.7}$$

When dividing (2.3.7) by $|x-y|^{m+\sigma p}$ and integrating over $Q_3 \times Q_3$, the first term on the right-hand side gives $\eta^{(i-j)p} |\mathrm{D}^i u \circ \Phi_a|^p_{W^{\sigma,p}(Q_3)}$. As in the case $0 < s < 1$, we may estimate it as

$$\eta^{(i-j)p} |\mathrm{D}^i u \circ \Phi_a|^p_{W^{\sigma,p}(Q_3)} \lesssim \eta^{(i-j)p} |\mathrm{D}^i u|^p_{W^{\sigma,p}(Q_4)}.$$

For the second term on the right-hand side of (2.3.7), we use an optimization argument. For every $r > 0$, we write

$$\int_{Q_3} \frac{|\mathrm{D}^t \Phi_a(x) - \mathrm{D}^t \Phi_a(y)|^p}{|x-y|^{m+\sigma p}} \,\mathrm{d}y \lesssim \int_{B^m_r(x)} \eta^{-tp} \frac{1}{|x-y|^{m+\sigma p - p}} \,\mathrm{d}y + \int_{\mathbb{R}^m \setminus B^m_r(x)} \eta^{(1-t)p} \frac{1}{|x-y|^{m+\sigma p}} \,\mathrm{d}y \lesssim \eta^{-tp} r^{p-\sigma p} + \eta^{(1-t)p} r^{-\sigma p}.$$

Letting $r = \eta$, we find

$$\int_{Q_3} \frac{|D^t\Phi_a(x) - D^t\Phi_a(y)|^p}{|x-y|^{m+\sigma p}}\,dy \lesssim \eta^{(1-t-\sigma)p}.$$

Therefore,

$$\begin{aligned}\int_{Q_3}\int_{Q_3} &\frac{|D^i u \circ \Phi_a(x)|^p \eta^{(i-1-j+t)p} |D^t\Phi_a(x) - D^t\Phi_a(y)|^p}{|x-y|^{m+\sigma p}}\,dx\,dy \\ &\lesssim \eta^{(i-j-\sigma)p}\int_{Q_3} |D^i u\circ\Phi_a(x)|^p\,dx \lesssim \eta^{(i-j-\sigma)p}\int_{Q_4}|D^i u|^p,\end{aligned}$$

where the last inequality follows from Lemma 2.3.4. Gathering the estimates for both terms in (2.3.7) yields the desired fractional estimate and concludes the proof. □

We conclude this section with two additional results which are the counterparts of [BPVS15, Addendum 1 and 2 to Proposition 2.1] in the context of fractional order estimates. From now on, we place ourselves under the assumptions of Proposition 2.3.1. The first proposition ensures that the opening procedure does not increase too much the energy on one given cube.

Proposition 2.3.7. *Let K^m be a cubication containing U^m.*

(*a*) *If $s \geq 1$ and if $u \in W^{1,sp}(\mathscr{K}^m + Q^m_{2\rho\eta}; \mathbb{R}^\nu)$, then the map $\Phi\colon \mathbb{R}^m \to \mathbb{R}^m$ provided by Proposition 2.3.1 can be chosen with the additional property that $u\circ\Phi \in W^{1,sp}(\mathscr{K}^m + Q^m_{\rho\eta}; \mathbb{R}^\nu)$, and for every $\sigma^m \in K^m$,*

$$\|D(u\circ\Phi)\|_{L^{sp}(\sigma^m + Q^m_{\rho\eta})} \leq C'\|Du\|_{L^{sp}(\sigma^m + Q^m_{2\rho\eta})}$$

for some constant $C' > 0$ depending on m, s, p, and ρ.

(*b*) *If $0 < s < 1$, then the map $\Phi\colon \mathbb{R}^m \to \mathbb{R}^m$ provided by Proposition 2.3.1 can be chosen with the additional property that $u\circ\Phi \in W^{s,p}(\mathscr{K}^m + Q^m_{\rho\eta}; \mathbb{R}^\nu)$, and for every $\sigma^m \in K^m$,*

$$|u\circ\Phi|_{W^{s,p}(\sigma^m + Q^m_{\rho\eta})} \leq C'|u|_{W^{s,p}(\sigma^m + Q^m_{2\rho\eta})}$$

for some constant $C' > 0$ depending on m, s, p, and ρ.

Proof. (a) In the case $s \geq 1$, since the choice of the parameter a involved in the construction of the map provided by Proposition 2.3.2 is made over a set of positive measure, according to the remark following Lemma 2.3.6, we may assume that the maps Φ^{σ^l} involved in the construction of the map Φ satisfy in addition the conclusion of Proposition 2.3.2 with parameters 1 and sp.

We keep the notation used in the proof of Proposition 2.3.1. Let $l \in \{0, \dots, d\}$. We are going to prove that

$$\|D(u^l \circ \Phi_l)\|_{L^{sp}(\sigma^m + Q^m_{\rho_l \eta})} \lesssim \|Du^l\|_{L^{sp}(\sigma^m + Q^m_{\rho_{l-1}\eta})},$$

and the conclusion will follow by induction. By our additional assumption on the maps Φ^{σ^l}, we have

$$\|D(u^l \circ \Phi_l)\|_{L^{sp}(T_{\sigma^l}(Q_3))} \leq \|D(u^l \circ \Phi_l)\|_{L^{sp}(T_{\sigma^l}(Q_4))} \lesssim \|Du^l\|_{L^{sp}(T_{\sigma^l}(Q_4))}$$

for every $\sigma^l \in U^l$. We conclude by using the fact that

$$\operatorname{Supp} \Phi_l \subset \bigcup_{\sigma^l \in U^l} T_{\sigma^l}(Q_2) \subset \bigcup_{\sigma^l \in \mathcal{U}^l} T_{\sigma^l}(Q_3)$$

along with the additivity of the integral.

(b) The proof of the case $0 < s < 1$ is identical, except that we replace the additivity of the integral by Lemma 2.2.2. Here we use the fact that the number of l-faces of a given cube depends only on l and m, and that the geometric support of Φ^{σ^l} is contained in $T_{\sigma^l}(Q_2)$, which is slightly smaller than $T_{\sigma^l}(Q_3)$. This justifies the application of Lemma 2.2.2. □

The second proposition gives VMO-type estimates for the opened map. As we mentioned in our informal presentation, such estimates are one of the main features of the opening procedure, and they follow from the fact that $u \circ \Phi$ behaves locally as a map of d variables in $\mathcal{U}^d + Q^m_{\rho\eta}$.

Proposition 2.3.8. *Under the assumptions of Propositions 2.3.1 and 2.3.7, the map $\Phi\colon \mathbb{R}^m \to \mathbb{R}^m$ satisfies the following estimates:*

(*a*) *if $s \geq 1$, then*

(*i*) *it holds that*

$$\lim_{r \to 0} \sup_{Q^m_r(a) \subset \mathcal{U}^d + Q^m_{\rho\eta}} r^{\frac{d}{sp} - 1} \fint_{Q^m_r(a)} \fint_{Q^m_r(a)} |u \circ \Phi(x) - u \circ \Phi(y)| \,\mathrm{d}x \,\mathrm{d}y = 0;$$

(*ii*) *for every $\sigma^m \in U^m$ and every $Q^m_r(a) \subset \mathcal{U}^d + Q^m_{\rho\eta}$ with $a \in \sigma^m$,*

$$\fint_{Q^m_r(a)} \fint_{Q^m_r(a)} |u \circ \Phi(x) - u \circ \Phi(y)| \,\mathrm{d}x \,\mathrm{d}y \leq C'' \frac{r^{1 - \frac{d}{sp}}}{\eta^{\frac{m-d}{sp}}} \|Du\|_{L^{sp}(\sigma^m + Q^m_{2\rho\eta})};$$

(*b*) *if* $0 < s < 1$, *then*

(*i*) *it holds that*

$$\lim_{r \to 0} \sup_{Q_r^m(a) \subset \mathscr{U}^d + Q_{\rho\eta}^m} r^{\frac{d}{p} - s} \fint_{Q_r^m(a)} \fint_{Q_r^m(a)} |u \circ \Phi(x) - u \circ \Phi(y)| \, \mathrm{d}x \, \mathrm{d}y = 0;$$

(*ii*) *for every* $\sigma^m \in U^m$ *and every* $Q_r^m(a) \subset \mathscr{U}^d + Q_{\rho\eta}^m$ *with* $a \in \sigma^m$,

$$\fint_{Q_r^m(a)} \fint_{Q_r^m(a)} |u \circ \Phi(x) - u \circ \Phi(y)| \, \mathrm{d}x \, \mathrm{d}y \le C'' \frac{r^{s - \frac{d}{p}}}{\eta^{\frac{m-d}{p}}} |u|_{W^{s,p}(\sigma^m + Q_{2\rho\eta}^m)};$$

for some constant $C'' > 0$ *depending on* m, s, p, *and* ρ.

Proof. We start with the proof of items (i). Let $a \in \mathbb{R}^m$ and $r > 0$ be such that $Q_r^m(a) \subset \mathscr{U}^d + Q_{\rho\eta}^m$, and write $Q_r^m(a) = Q_r^d(a') \times Q_r^{m-d}(a'')$. Then, $a \in \mathscr{U}^d + Q_{\rho\eta - r}^m$, and hence there exists $\tau^d \in U^d$ such that $Q_r^m(a) \subset \tau^d + Q_{\rho\eta}^m$. We may assume that $\tau^d = Q_\eta^m \times \{0\}^{m-d}$. Recall that the map Φ is constant on each $(m-d)$-dimensional cube orthogonal to $Q_{(1+\rho)\eta}^d \times \{0\}^{m-d}$. Hence we may define $v \colon Q_{(1+\rho)\eta}^d \to \mathbb{R}^\nu$ by

$$v(x') = u \circ \Phi(x', a'').$$

Using Proposition 2.3.7, we deduce that $v \in W^{1,sp}(Q_{(1+\rho)\eta}^d; \mathbb{R}^\nu)$ in the $s \ge 1$ case, respectively $v \in W^{s,p}(Q_{(1+\rho)\eta}^d; \mathbb{R}^\nu)$ in the $0 < s < 1$ case. We next note that

$$\fint_{Q_r^m(a)} \fint_{Q_r^m(a)} |u \circ \Phi(x) - u \circ \Phi(y)| \, \mathrm{d}x \, \mathrm{d}y = \fint_{Q_r^d(a')} \fint_{Q_r^d(a')} |v(x') - v(y')| \, \mathrm{d}x' \, \mathrm{d}y'.$$

The Poincaré–Wirtinger inequality implies that

$$\fint_{Q_r^d(a')} \fint_{Q_r^d(a')} |v(x') - v(y')| \, \mathrm{d}x' \, \mathrm{d}y' \lesssim r^{1 - \frac{d}{sp}} \|\mathrm{D}v\|_{L^{sp}(Q_r^d(a'))}$$

if $s \ge 1$, respectively

$$\fint_{Q_r^d(a')} \fint_{Q_r^d(a')} |v(x') - v(y')| \, \mathrm{d}x' \, \mathrm{d}y' \lesssim r^{s - \frac{d}{p}} |v|_{W^{s,p}(Q_r^d(a'))}$$

if $0 < s < 1$. It then suffices to invoke Lebesgue's lemma to obtain both items (i).

We now turn to the proof of items (ii). When $s \ge 1$, we observe that

$$\|\mathrm{D}(u \circ \Phi)\|_{L^{sp}(Q_r^d(a') \times Q_{\rho\eta}^{m-d}(a''))} = (2\rho\eta)^{\frac{m-d}{sp}} \|\mathrm{D}v\|_{L^{sp}(Q_r^d(a'))}, \tag{2.3.8}$$

and hence, assuming in addition that $a \in \sigma^m$ with $\sigma^m \in \mathcal{U}^m$, we have

$$\|\mathrm{D}v\|_{L^{sp}(Q_r^d(a'))} = \frac{1}{(2\rho\eta)^{\frac{m-d}{sp}}}\|\mathrm{D}(u\circ\Phi)\|_{L^{sp}(Q_r^d(a')\times Q_{\rho\eta}^d(a''))} \leq \frac{1}{(2\rho\eta)^{\frac{m-\ell}{sp}}}\|\mathrm{D}(u\circ\Phi)\|_{L^{sp}(\sigma^m+Q_{\rho\eta}^m)}.$$

Thus,

$$\fint_{Q_r^m(a)}\fint_{Q_r^m(a)}|u\circ\Phi(x)-u\circ\Phi(y)|\,\mathrm{d}x\,\mathrm{d}y \lesssim \frac{r^{1-\frac{d}{sp}}}{(2\rho\eta)^{\frac{m-d}{sp}}}\|\mathrm{D}(u\circ\Phi)\|_{L^{sp}(\sigma^m+Q_{\rho\eta}^m)}.$$

Proposition 2.3.7 implies that

$$\|\mathrm{D}(u\circ\Phi)\|_{L^{sp}(\sigma^m+Q_{\rho\eta}^m)} \lesssim \|\mathrm{D}u\|_{L^{sp}(\sigma^m+Q_{2\rho\eta}^m)},$$

which yields the desired conclusion.

For the case $0 < s < 1$, we follow the same path, replacing inequality (2.3.8) by the fact that

$$\begin{aligned}|v|_{W^{s,p}(Q_r^d(a'))} &= \frac{1}{(2\rho\eta)^{\frac{m-d}{p}}}\left(\int_{Q_{\rho\eta}^{m-d}(a'')}|u\circ\Phi(\cdot,x'')|^p_{W^{s,p}(Q_r^d(a'))}\,\mathrm{d}x''\right)^{\frac{1}{p}}\\ &\lesssim \frac{1}{\eta^{\frac{m-d}{p}}}|u|_{W^{s,p}(Q_r^d(a')\times Q_{\rho\eta}^{m-d}(a''))}.\end{aligned}$$

This concludes the proof of the proposition. □

2.4 Adaptive smoothing

In this section, we present the adaptive smoothing, which consists in a regularization by convolution, $x \mapsto \varphi_{\psi(x)} * u(x)$, where the parameter ψ of convolution is allowed to depend on the point x where the regularized map is calculated. Already implicit in the proof of the $H = W$ theorem [MS64], this method was made popular by R. Schoen and K. Uhlenbeck [SU82, Section 3]. The approach we follow here is an adaptation, suited to fractional Sobolev spaces, of the one in [BPVS15, Section 3].

We now become more specific. Let φ be a *mollifier*, i.e.,

$$\varphi \in C_c^\infty(\mathbb{B}^m), \quad \varphi \geq 0 \quad \text{in } \mathbb{B}^m, \quad \varphi \text{ is radial,} \quad \text{and} \quad \int_{\mathbb{B}^m}\varphi = 1.$$

Let $u \in L^1_{\mathrm{loc}}(\Omega)$, and consider a map $\psi \in C^\infty(\Omega;(0,+\infty))$. For every $x \in \Omega$ satisfying

$\operatorname{dist}(x, \partial\Omega) \geq \psi(x)$, we may define

$$\varphi_\psi * u(x) = \int_{\mathbb{B}^m} \varphi(z) u(x + \psi(x)z)\,\mathrm{d}z.$$

A change of variable yields

$$\varphi_\psi * u(x) = \frac{1}{\psi(x)^m} \int_{B^m_{\psi(x)}(x)} \varphi\Big(\frac{y-x}{\psi(x)}\Big) u(y)\,\mathrm{d}y. \tag{2.4.1}$$

In particular, $\varphi_\psi * u$ is smooth.

Let us first note a straightforward inequality. Let $\omega \subset \{x \in \Omega \colon \operatorname{dist}(x, \partial\Omega) \geq \psi(x)\}$, so that $\varphi_\psi * u$ is well-defined on ω. For any $x \in \omega$, we write

$$\varphi_\psi * u(x) - u(x) = \int_{B^m_1} \varphi(z)(u(x + \psi(x)z) - u(x))\,\mathrm{d}z.$$

Therefore, by Minkowski's inequality, we find

$$\begin{aligned} \|\varphi_\psi * u - u\|_{L^p(\omega)} &\leq \int_{\mathbb{B}^m} \varphi(z) \Big(\int_\omega |u(x + \psi(x)z) - u(x)|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} \mathrm{d}z \\ &\leq \sup_{v \in \mathbb{B}^m} \|\tau_{\psi v}(u) - u\|_{L^p(\omega)}, \end{aligned} \tag{2.4.2}$$

where $\tau_{\psi v}(u)(x) = u(x + \psi(x)v)$. Our main task in this section will be to obtain estimates in the spirit of 2.4.2 for maps in $W^{s,p}(\Omega; \mathbb{R}^\nu)$.

Before stating the main result of this section, we pause to explain the role of an important assumption. In the sequel, we will assume that $\|\mathrm{D}\psi\|_{L^\infty(\Omega)} < 1$. We illustrate the usefulness of this condition in the simpler context of L^p estimates. We start by using Minkowski's inequality to write

$$\|\varphi_\psi * u\|_{L^p(\omega)} \leq \int_{\mathbb{B}^m} \varphi(z) \Big(\int_\omega |u(x + \psi(x)z)|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} \mathrm{d}z.$$

Next we use the change of variable $w = x + \psi(x)z$. We note that the map $\Psi \colon \omega \to \Omega$ defined by $\Psi(x) = x + \psi(x)z$ satisfies $\mathrm{D}\Psi(x) = \mathrm{id} + \mathrm{D}\psi(x) \otimes z$. Therefore, by rank-one perturbation of the identity (see e.g. [Ser10, Section 3.8]), we deduce that

$$\mathcal{J}\Psi = |\det(\mathrm{id} + \mathrm{D}\psi \otimes z)| = |1 + \mathrm{D}\psi \cdot z| \geq 1 - \|\mathrm{D}\psi\|_{L^\infty(\Omega)} \quad \text{for } z \in \mathbb{B}^m$$

(where $\mathcal{J}u = |\det(\mathrm{D}u)|$ is the Jacobian of u). Thanks to the assumption $\|\mathrm{D}\psi\|_{L^\infty(\Omega)} < 1$, the linear map $D\Psi(x)$ is invertible for $z \in \mathbb{B}^m$, so that the above change of variable is well-defined with Jacobian less than $\frac{1}{1-\|D\psi\|_{L^\infty(\omega)}}$. We conclude that

$$\|\varphi_\psi * u\|_{L^p(\omega)} \leq \frac{1}{(1-\|D\psi\|_{L^\infty(\omega)})^{\frac{1}{p}}}\|u\|_{L^p(\Omega)}. \tag{2.4.3}$$

We now state the main result of this section, which is the counterpart of [BPVS15, Proposition 3.2] in the context of fractional Sobolev spaces.

Proposition 2.4.1. *Let $\varphi \in C_c^\infty(\mathbb{B}^m)$ be a mollifier and let $\psi \in C^\infty(\Omega)$ be a nonnegative function such that $\|D\psi\|_{L^\infty(\Omega)} < 1$. For every $u \in W^{s,p}(\Omega;\mathbb{R}^\nu)$ and every open set $\omega \subset \{x \in \Omega\colon \operatorname{dist}(x,\partial\Omega) > \psi(x)\}$, we have $\varphi_\psi * u \in W^{s,p}(\omega;\mathbb{R}^\nu)$, and moreover, the following estimates hold*:

(*i*) (*a*) *if $0 < s < 1$, then*

$$|\varphi_\psi * u|_{W^{s,p}(\omega)} \leq C\frac{1}{(1-\|D\psi\|_{L^\infty(\omega)})^{\frac{2}{p}}}|u|_{W^{s,p}(\Omega)};$$

(*b*) *if $s \geq 1$, then for every $j \in \{1,\ldots,k\}$,*

$$\eta^j\|D^j(\varphi_\psi * u)\|_{L^p(\omega)} \leq C\frac{1}{(1-\|D\psi\|_{L^\infty(\omega)})^{\frac{1}{p}}}\sum_{i=1}^{j}\eta^i\|D^i u\|_{L^p(\Omega)};$$

(*c*) *if $s \geq 1$ and $\sigma \neq 0$, then for every $j \in \{1,\ldots,k\}$,*

$$\begin{aligned}\eta^{j+\sigma}&|D^j(\varphi_\psi * u)|_{W^{\sigma,p}(\omega)}\\ &\leq C\frac{1}{(1-\|D\psi\|_{L^\infty(\omega)})^{\frac{2}{p}}}\sum_{i=1}^{j}\Big(\eta^i\|D^i u\|_{L^p(\Omega)} + \eta^{i+\sigma}|D^i u|_{W^{\sigma,p}(\Omega)}\Big);\end{aligned}$$

(*ii*) (*a*) *if $0 < s < 1$, then*

$$|\varphi_\psi * u - u|_{W^{s,p}(\omega)} \leq \sup_{v\in B_1^m}|\tau_{\psi v}(u) - u|_{W^{s,p}(\omega)};$$

(*b*) *if* $s \geq 1$, *then for every* $j \in \{1, \dots, k\}$,

$$\eta^j \|D^j(\varphi_\psi * u) - D^j u\|_{L^p(\omega)} \leq \sup_{v \in \mathbb{B}^m} \eta^j \|\tau_{\psi v}(D^j u) - D^j u\|_{L^p(\omega)} + C \frac{1}{(1 - \|D\psi\|_{L^\infty(\omega)})^{\frac{1}{p}}} \sum_{i=1}^{j} \eta^i \|D^i u\|_{L^p(A)};$$

(*c*) *if* $s \geq 1$ *and* $\sigma \neq 0$, *then for every* $j \in \{1, \dots, k\}$,

$$\eta^{j+\sigma} |D^j(\varphi_\psi * u) - D^j u|_{W^{\sigma,p}(\omega)} \leq \sup_{v \in \mathbb{B}^m} \eta^{j+\sigma} |\tau_{\psi v}(D^j u) - D^j u|_{W^{\sigma,p}(\omega)} + C \frac{1}{(1 - \|D\psi\|_{L^\infty(\omega)})^{\frac{2}{p}}} \sum_{i=1}^{j} \Big(\eta^i \|D^i u\|_{L^p(A)} + \eta^{i+\sigma} |D^i u|_{W^{\sigma,p}(A)} \Big);$$

for some constant $C > 0$ *depending on* m, s, *and* p, *where*

$$A = \bigcup_{x \in \omega \cap \operatorname{supp} D\psi} B^m_{\psi(x)}(x)$$

and $\eta > 0$ *satisfies*

$$\eta^j \|D^j \psi\|_{L^\infty} \leq \eta \quad \textit{for every } j \in \{2, \dots, k+1\}.$$

Proof. The proof of item (i) is completely analogous to the proof of item (ii) and uses the same ingredients. Hence we focus on item (ii), and we explain in the end what should be changed in order to get (i).

We start with the integer order estimate in the case $s \geq 1$. By the Faà di Bruno formula, for every $x \in \omega$, we have

$$\begin{aligned} D^j(\varphi_\psi * u)(x) = &\int_{\mathbb{B}^m} \varphi(z) D^j u(x + \psi(x)z)[(\mathrm{id} + D\psi(x) \otimes z)^j] \, dz \\ &+ \sum_{i=1}^{j-1} \sum_{l=1}^{n(i,j)} \int_{\mathbb{B}^m} \varphi(z) D^i u(x + \psi(x)z)[L_{i,l,j}(x,z)] \, dz, \end{aligned} \tag{2.4.4}$$

where $n(i,j) \in \mathbb{N}_*$ and $L_{i,l,j}(x,z)$ is a linear mapping $\mathbb{R}^{j \times m} \to \mathbb{R}^{i \times m}$ depending on ψ and its derivatives. More precisely, each entry of $L_{i,l,j}(x,z)$ is either $\mathrm{id} + D\psi(x) \otimes z$ or $D^t \psi(x) \otimes z$ for some $t \in \{2, \dots, j\}$, and the sum over all i components of $L_{i,l,j}(x,z)$ of the order of the derivative of ψ appearing in this component is j. Moreover, since $i < j$, at least one entry of $L_{i,j,l}(x,z)$ has the form $D^t \psi(x)$, and thus the second integral in (2.4.4) lives only on $\operatorname{supp} \mathrm{D}\psi$. Therefore, taking into account the assumption $\|\mathrm{D}^t u\|_{L^\infty} \leq \eta^{1-t}$, we deduce that

$$|\mathrm{D}^i u(x+\psi(x)z)[L_{i,l,j}(x,z)]| \lesssim |\mathrm{D}^i u(x+\psi(x)z)|\eta^{i-j}\chi_{\operatorname{supp}\mathrm{D}\psi}(x).$$

On the other hand, we note that, by j-linearity of $\mathrm{D}^j u$, we may write $\mathrm{D}^j u(x+\psi(x)z)[(\mathrm{id} + \mathrm{D}\psi(x)\otimes z)^j]$ as the sum of $\mathrm{D}^j u(x+\psi(x)z)$ and $2^j - 1$ terms which are the composition of $\mathrm{D}^j u(x+\psi(x)z)$ with a j-linear map $\mathbb{R}^{j\times m} \to \mathbb{R}^{j\times m}$ whose entries are either id or $\mathrm{D}\psi(x)\otimes z$, with at least one of them being the latter. Hence, since $\|\mathrm{D}\psi\|_{L^\infty} < 1$, each of those $2^j - 1$ last terms is bounded by $|\mathrm{D}^j u(x+\psi(x)z)|\chi_{\operatorname{supp}\mathrm{D}\psi}(x)$. For instance, if $j = 2$, then

$$\begin{aligned}\mathrm{D}^2 u(x+\psi(x)z)[(\mathrm{id} + \mathrm{D}\psi(x)\otimes z)^2] &= \mathrm{D}^2 u(x+\psi(x)z) + \mathrm{D}^2 u(x+\psi(x)z)[\mathrm{id}, \mathrm{D}\psi(x)\otimes z]\\ &\quad + \mathrm{D}^2 u(x+\psi(x)z)[\mathrm{D}\psi(x)\otimes z, \mathrm{id}] + \mathrm{D}^2 u(x+\psi(x)z)[\mathrm{D}\psi(x)\otimes z, \mathrm{D}\psi(x)\otimes z].\end{aligned}$$

We observe that indeed, the three last terms are obtained by composition of $\mathrm{D}^2 u(x+\psi(x)z)$ with a bilinear map, at least one of whose entries being $\mathrm{D}\psi(x)\otimes z$.

As a consequence, for every $x \in \omega$, we may write

$$\begin{aligned}|\mathrm{D}^j(\varphi_\psi * u)(x) - \mathrm{D}^j u(x)| \leq &\int_{\mathbb{B}^m} \varphi(z)|\mathrm{D}^j u(x+\psi(x)z) - D^j u(x)|\,\mathrm{d}z\\ &+ C\sum_{i=1}^{j} \eta^{i-j}\chi_{\operatorname{supp}\mathrm{D}\psi}(x)\int_{\mathbb{B}^m}\varphi(z)|\mathrm{D}^i u(x+\psi(x)z)|\,\mathrm{d}z.\end{aligned}$$

Minkowski's inequality ensures that

$$\begin{aligned}\|\mathrm{D}^j(\varphi_\psi * u) - \mathrm{D}^j u\|_{L^p(\omega)} \leq &\int_{\mathbb{B}^m}\varphi(z)\Bigg[\bigg(\int_\omega |\mathrm{D}^j u(x+\psi(x)z) - \mathrm{D}^j u(x)|^p\,\mathrm{d}x\bigg)^{\frac{1}{p}}\\ &+ C\sum_{i=1}^{j}\eta^{i-j}\bigg(\int_{\omega\cap\operatorname{supp}\mathrm{D}\psi}|\mathrm{D}^i u(x+\psi(x)z)|^p\,\mathrm{d}x\bigg)^{\frac{1}{p}}\Bigg]\,\mathrm{d}z.\end{aligned}$$

For the first term, we note that

$$\bigg(\int_\omega |\mathrm{D}^j u(x+\psi(x)z) - \mathrm{D}^j u(x)|^p\,\mathrm{d}x\bigg)^{\frac{1}{p}} \leq \sup_{v\in\mathbb{B}^m}\|\tau_{\psi v}(\mathrm{D}^j u) - \mathrm{D}^j u\|_{L^p(\omega)}.$$

For the second term, we use the change of variable $w = x + \psi(x)z$ that we considered before. Taking into account the definition of the set A, we have $w \in B^m_{\psi(x)}(x) \subset A$, and therefore

$$\left(\int_{\omega\cap\operatorname{supp} D\psi} |D^i u(x+\psi(x)z)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \le \frac{1}{(1-\|D\psi\|_{L^\infty(\omega)})^{\frac{1}{p}}}\left(\int_A |D^i u(w)|^p \,\mathrm{d}w\right)^{\frac{1}{p}}.$$

We obtain the desired estimate by using the fact that φ has integral equal to 1.

The estimate in the fractional case $0 < s < 1$ is straightforward. Indeed, we first write

$$\begin{aligned}&|\varphi_\psi * u(x) - u(x) - \varphi_\psi * u(y) + u(y)|\\ &\qquad\qquad \le \int_{\mathbb{B}^m} \varphi(z)|u(x+\psi(x)z) - u(x) - u(y+\psi(y)z) + u(y)|\,\mathrm{d}z.\end{aligned}$$

Minkowski's inequality then implies that

$$\begin{aligned}&|\varphi_\psi * u - u|_{W^{s,p}(\omega)}\\ &\quad \le \int_{\mathbb{B}^m} \varphi(z)\left(\int_\omega\int_\omega \frac{|u(x+\psi(x)z) - u(x) - u(y+\psi(y)z) + u(y)|^p}{|x-y|^{m+sp}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{p}}\mathrm{d}z\\ &\quad \le \sup_{v\in\mathbb{B}^m} |\tau_{\psi v}(u) - u|_{W^{s,p}(\omega)}.\end{aligned}$$

For the fractional estimate when $s \ge 1$, we again use equation (2.4.4) and the observations following this equation. Let $x, y \in \omega$. We proceed by distinction of cases, using the multilinearity of the differential.

Case 1. — We assume that $x, y \in \operatorname{supp} D\psi$. For the terms with $i < j$, using the j-linearity of $D^j u$, we estimate

$$\begin{aligned}&|D^i u(x+\psi(x)z)[L_{i,l,j}(x,z)] - D^i u(y+\psi(y)z)[L_{i,l,j}(y,z)]|\\ &\quad \lesssim \sum_{t=1}^{j} \eta^{i-1-j+t}|D^t\psi(x) - D^t\psi(y)||D^i u(x+\psi(x)z)|\\ &\qquad + \eta^{i-j}|D^i u(x+\psi(x)z) - D^i u(y+\psi(y)z)|.\end{aligned}$$

On the other hand, for the term involving the derivative of order j of u, we have

$$\begin{aligned}&|D^j u(x+\psi(x)z)[(\mathrm{id}+D\psi(x)\otimes z)^j] - D^j u(x) - D^j u(y+\psi(y)z)[(\mathrm{id}+D\psi(y)\otimes z)^j] + D^j u(y)|\\ &\qquad \le A_j(x,y,z) + C_1 B_j(x,y,z) + C_2|D^j u(x+\psi(x)z)||D\psi(x) - D\psi(y)|,\end{aligned}$$

where

$$A_j(x,y,z) = |\mathrm{D}^j u(x+\psi(x)z) - \mathrm{D}^j u(x) - \mathrm{D}^j u(y+\psi(y)z) + \mathrm{D}^j u(y)|$$

and

$$B_j(x,y,z) = |\mathrm{D}^j u(x+\psi(x)z) - \mathrm{D}^j u(y+\psi(y)z)|.$$

Now, for $t \in \{1,\dots,j\}$, we estimate

$$\begin{aligned}
&\int_\omega \frac{|\mathrm{D}^t\psi(x) - \mathrm{D}^t\psi(y)|^p}{|x-y|^{m+\sigma p}}\,\mathrm{d}y \\
&\quad\lesssim \eta^{-tp}\int_{B^m_r(x)} \frac{1}{|x-y|^{m+(\sigma-1)p}}\,\mathrm{d}y + \eta^{(1-t)p}\int_{\mathbb{R}^m\setminus B^m_r(x)} \frac{1}{|x-y|^{m+\sigma p}}\,\mathrm{d}y \\
&\quad\lesssim \eta^{-tp}r^{(1-\sigma)p} + \eta^{(1-t)p}r^{-\sigma p}
\end{aligned}$$

for every $r > 0$. Letting $r = \eta$ yields

$$\int_\omega \frac{|\mathrm{D}^t\psi(x) - \mathrm{D}^t\psi(y)|^p}{|x-y|^{m+\sigma p}}\,\mathrm{d}y \lesssim \eta^{(1-t-\sigma)p}. \tag{2.4.5}$$

Therefore, using Minkowski's inequality on the expression obtained from (2.4.4), we deduce that

$$\begin{aligned}
&\biggl(\int_{\omega\cap\operatorname{supp}\mathrm{D}\psi}\int_{\omega\cap\operatorname{supp}\mathrm{D}\psi} \frac{|\mathrm{D}^j(\varphi_\psi * u)(x) - \mathrm{D}^j u(x) - \mathrm{D}^j(\varphi_\psi * u)(y) + \mathrm{D}^j u(y)|^p}{|x-y|^{m+\sigma p}}\,\mathrm{d}x\,\mathrm{d}y\biggr)^{\frac{1}{p}} \\
&\le \int_{\mathbb{B}^m}\varphi(z)\biggl[\biggl(\int_{\omega\cap\operatorname{supp}\mathrm{D}\psi}\int_{\omega\cap\operatorname{supp}\mathrm{D}\psi} \frac{A_j(x,y,z)^p}{|x-y|^{m+\sigma p}}\,\mathrm{d}x\,\mathrm{d}y\biggr)^{\frac{1}{p}} \\
&\quad + C_3\sum_{i=1}^{j}\eta^{i-j}\biggl(\int_{\omega\cap\operatorname{supp}\mathrm{D}\psi}\int_{\omega\cap\operatorname{supp}\mathrm{D}\psi} \frac{B_i(x,y,z)^p}{|x-y|^{m+\sigma p}}\,\mathrm{d}x\,\mathrm{d}y\biggr)^{\frac{1}{p}} \\
&\quad + C_4\sum_{i=1}^{j}\eta^{i-j-\sigma}\biggl(\int_{\omega\cap\operatorname{supp}\mathrm{D}\psi}|\mathrm{D}^i u(x+\psi(x)z)|^p\,\mathrm{d}x\biggr)^{\frac{1}{p}}\biggr]\,\mathrm{d}z.
\end{aligned}$$

Case 2. — Without loss of generality, we assume that $x \in \operatorname{supp}\mathrm{D}\psi$ and $y \notin \operatorname{supp}\mathrm{D}\psi$. In this case, since each $L_{i,l,j}(y,z)$ has at least one entry equal to $\mathrm{D}^t\psi(y)$, we find

$$\mathrm{D}^i u(y+\psi(y)z)[L_{i,l,j}(y,z)] = 0 = \mathrm{D}^i u(x+\psi(x)z)[L_{i,l,j}(y,z)].$$

Hence,

$$
\begin{aligned}
&|\mathrm{D}^i u(x + \psi(x)z)[L_{i,l,j}(x,z)] - \mathrm{D}^i u(y + \psi(y)z)[L_{i,l,j}(y,z)]| \\
&\qquad \leq \sum_{t=1}^{j} \eta^{i-1-j+t} |\mathrm{D}^t \psi(x) - \mathrm{D}^t \psi(y)| |\mathrm{D}^i u(x + \psi(x)z)|.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&|\mathrm{D}^j u(x+\psi(x)z)[(\mathrm{id} + \mathrm{D}\psi(x)\otimes z)^j] - \mathrm{D}^j u(x) - \mathrm{D}^j u(y+\psi(y)z)[(\mathrm{id} + \mathrm{D}\psi(y)\otimes z)^j] + \mathrm{D}^j u(y)| \\
&\qquad \leq A_j(x,y,z) + C_5 |\mathrm{D}^j u(x + \psi(x)z)| |\mathrm{D}\psi(x) - \mathrm{D}\psi(y)|.
\end{aligned}
$$

We then argue as in Case 1, using (2.4.5) to deal with the terms containing $|\mathrm{D}^t \psi(x) - \mathrm{D}^t \psi(y)|$, and we deduce that

$$
\begin{aligned}
&\biggl(\int_{\omega \setminus \operatorname{supp} \mathrm{D}\psi} \int_{\omega \cap \operatorname{supp} \mathrm{D}\psi} \frac{|\mathrm{D}^j(\varphi_\psi * u)(x) - \mathrm{D}^j u(x) - \mathrm{D}^j(\varphi_\psi * u)(y) + \mathrm{D}^j u(y)|^p}{|x-y|^{m+\sigma p}} \,\mathrm{d}x \,\mathrm{d}y \biggr)^{\frac{1}{p}} \\
&\leq \int_{\mathbb{B}^m} \varphi(z) \Biggl[\biggl(\int_{\omega \setminus \operatorname{supp} \mathrm{D}\psi} \int_{\omega \cap \operatorname{supp} \mathrm{D}\psi} \frac{A_j(x,y,z)^p}{|x-y|^{m+\sigma p}} \,\mathrm{d}x \,\mathrm{d}y \biggr)^{\frac{1}{p}} \\
&\quad + C_6 \sum_{i=1}^{j} \eta^{i-j-\sigma} \biggl(\int_{\omega \cap \operatorname{supp} \mathrm{D}\psi} |\mathrm{D}^i u(x + \psi(x)z)|^p \,\mathrm{d}x \biggr)^{\frac{1}{p}} \Biggr] \mathrm{d}z.
\end{aligned}
$$

Case 3. — We assume that $x, y \notin \operatorname{supp} \mathrm{D}\psi$. In this case, for $i < j$, we observe that

$$
|\mathrm{D}^i u(x + \psi(x)z)[L_{i,l,j}(x,z)] - \mathrm{D}^i u(y + \psi(y)z)[L_{i,l,j}(y,z)]| = 0.
$$

Moreover,

$$
\begin{aligned}
&|\mathrm{D}^j u(x+\psi(x)z)[(\mathrm{id} + \mathrm{D}\psi(x)\otimes z)^j] - \mathrm{D}^j u(x) - \mathrm{D}^j u(y+\psi(y)z)[(\mathrm{id} + \mathrm{D}\psi(y)\otimes z)^j] + \mathrm{D}^j u(y)| \\
&\qquad = A_j(x,y,z).
\end{aligned}
$$

Hence, unlike in the previous cases, estimate (2.4.5) is not needed, and a simple application of Minkowski's inequality yields

$$\begin{aligned}\Big(\int_{\omega\setminus\operatorname{supp}\mathrm{D}\psi}\int_{\omega\setminus\operatorname{supp}\mathrm{D}\psi}&\frac{|\mathrm{D}^j(\varphi_\psi * u)(x)-\mathrm{D}^j u(x)-\mathrm{D}^j(\varphi_\psi * u)(y)+\mathrm{D}^j u(y)|^p}{|x-y|^{m+\sigma p}}\,\mathrm{d}x\,\mathrm{d}y\Big)^{\frac{1}{p}}\\&\le\int_{B_1^m}\varphi(z)\Big[\Big(\int_{\omega\setminus\operatorname{supp}\mathrm{D}\psi}\int_{\omega\setminus\operatorname{supp}\mathrm{D}\psi}\frac{A_j(x,y,z)^p}{|x-y|^{m+\sigma p}}\,\mathrm{d}x\,\mathrm{d}y\Big)^{\frac{1}{p}}\,\mathrm{d}z.\end{aligned}$$

Gathering the estimates obtained in Cases 1, 2, and 3, we deduce that

$$\begin{aligned}\Big(\int_\omega\int_\omega&\frac{|\mathrm{D}^j(\varphi_\psi * u)(x)-\mathrm{D}^j u(x)-\mathrm{D}^j(\varphi_\psi * u)(y)+\mathrm{D}^j u(y)|^p}{|x-y|^{m+\sigma p}}\,\mathrm{d}x\,\mathrm{d}y\Big)^{\frac{1}{p}}\\&\le\int_{\mathbb{B}^m}\varphi(z)\Big[\Big(\int_\omega\int_\omega\frac{A_j(x,y,z)^p}{|x-y|^{m+\sigma p}}\,\mathrm{d}x\,\mathrm{d}y\Big)^{\frac{1}{p}}\\&\quad+C_7\sum_{i=1}^{j}\eta^{i-j}\Big(\int_{\omega\cap\operatorname{supp}\mathrm{D}\psi}\int_{\omega\cap\operatorname{supp}\mathrm{D}\psi}\frac{B_i(x,y,z)^p}{|x-y|^{m+\sigma p}}\,\mathrm{d}x\,\mathrm{d}y\Big)^{\frac{1}{p}}\\&\quad+C_8\sum_{i=1}^{j}\eta^{i-j-\sigma}\Big(\int_{\omega\cap\operatorname{supp}\mathrm{D}\psi}|\mathrm{D}^i u(x+\psi(x)z)|^p\,\mathrm{d}x\Big)^{\frac{1}{p}}\Big]\,\mathrm{d}z.\end{aligned}$$

For the first term, we observe once again that

$$\Big(\int_\omega\int_\omega\frac{A_j(x,y,z)^p}{|x-y|^{m+\sigma p}}\,\mathrm{d}x\,\mathrm{d}y\Big)^{\frac{1}{p}}\le\sup_{v\in\mathbb{B}^m}|\tau_{\psi v}(\mathrm{D}^j u)-\mathrm{D}^j u|_{W^{\sigma,p}(\omega)}.$$

For the third term, we use the change of variable $w=x+\psi(x)z$, and we find

$$\begin{aligned}\int_{\omega\cap\operatorname{supp}\mathrm{D}\psi}|\mathrm{D}^i u(x+\psi(x)z)|^p\,\mathrm{d}x&\le\frac{1}{(1-\|\mathrm{D}\psi\|_{L^\infty(\omega)})}\int_A|\mathrm{D}^i u(w)|^p\,\mathrm{d}w\\&\le\frac{1}{(1-\|\mathrm{D}\psi\|_{L^\infty(\omega)})^2}\|\mathrm{D}^i u\|^p_{L^p(A)}.\end{aligned}$$

For the second term, we make use of the change of variable $w=x+\psi(x)z$ and $\tilde{w}=y+\psi(y)z$. Observing that $|w-\tilde{w}|\le 2|x-y|$, we obtain

$$\begin{aligned}\int_{\omega\cap\operatorname{supp}\mathrm{D}\psi}\int_{\omega\cap\operatorname{supp}\mathrm{D}\psi}&\frac{B_j(x,y,z)^p}{|x-y|^{m+\sigma p}}\,\mathrm{d}x\,\mathrm{d}y\\&\lesssim\frac{1}{(1-\|\mathrm{D}\psi\|_{L^\infty(\omega)})^2}\int_A\int_A\frac{|\mathrm{D}^i u(w)-\mathrm{D}^i u(\tilde{w})|^p}{|w-\tilde{w}|^{m+\sigma p}}\,\mathrm{d}w\,\mathrm{d}\tilde{w}.\end{aligned}$$

Using the fact that φ has integral equal to 1, this concludes the proof of the fractional estimate when $s \geq 1$.

The proof of assertion (i) follows the same strategy. The only change is that, instead of grouping the term $D^j u(x + \psi(x)z)$ coming from the first term in (2.4.4) with the $D^j u$, we have to estimate it as all the other terms. Unlike the $2^j - 1$ terms involving a derivative of order j of u, this term does not vanish outside of the support of $D\psi$. This explains the presence of the norm on the whole Ω in estimates (i). □

Adaptive smoothing is a very useful tool to approximate a $W^{s,p}$ map by smooth maps, but it has a major drawback in the context of Sobolev spaces with values into manifolds. Indeed, if $u \in W^{s,p}(\Omega; \mathscr{N})$, in general $\varphi_\psi * u$ does not take values into $\mathscr{N}$, since the convolution product is in general not compatible with the constraint. Therefore, it will be crucial in the proof of the density theorem to be able to estimate the distance between the smoothed maps and the manifold. We close this section by a discussion devoted to this purpose, which also sheds light on how to use the estimates obtained during the opening procedure in the previous section.

We work in a slightly more general setting, assuming that $u \in W^{s,p}(\Omega; \mathbb{R}^\nu)$ is such that $u(x) \in F$ for almost every $x \in \Omega$, where $F \subset \mathbb{R}^\nu$ is an arbitrary closed set. We place ourselves under the assumptions of Propositions 2.3.1, 2.3.7, and 2.3.8. We denote by Φ^{op}_η the map provided by Proposition 2.3.1 and we set $u^{\mathrm{op}}_\eta = u \circ \Phi^{\mathrm{op}}_\eta$. Let $u^{\mathrm{sm}}_\eta = \varphi_{\psi_\eta} * u^{\mathrm{op}}_\eta$, where φ is a fixed mollifier, and the variable regularization parameter ψ_η is to be chosen later on, depending on η.

Let $0 < \underline{\rho} < \rho$ be fixed, and assume that U^m_η is a subskeleton of some skeleton K^m_η such that $\mathscr{K}^m_\eta \subset \omega$. To fix the ideas, one may keep in mind that $\mathscr{K}^m_\eta = \omega$ in the case where ω can be decomposed as a finite union of cubes of radius η. We consider a subskeleton E^m_η of U^m_η such that

$$\mathscr{E}^m_\eta \subset \operatorname{int} \mathscr{U}^m_\eta \tag{2.4.6}$$

in the relative topology of $\mathscr{K}^m_\eta$. (Later on in the proof of the density of the class $\mathscr{R}$, E^m_η will be the class of all bad cubes.)

Given a set $S \subset \mathbb{R}^\nu$, the *directed Hausdorff distance from S to F* is defined as

$$\operatorname{Dist}_F(S) = \sup\{\operatorname{dist}(x, F) \colon x \in S\}.$$

Our objective is to show that, for a suitable choice of ψ_η and $r > 0$, we have

$$\begin{aligned}\operatorname{Dist}_F(u_\eta^{\mathrm{sm}}((\mathscr{K}^m \setminus \mathscr{U}_\eta^m) \cup (\mathscr{U}_\eta^d + Q_{\underline{\rho}\eta}^m))) \leq \max\Bigg\{ &\max_{\sigma^m \in K_\eta^m \setminus E_\eta^m} C_1 \frac{1}{\eta^{\frac{m}{sp}-1}} \|\mathrm{D}u\|_{L^{sp}(\sigma^m + Q_{2\rho\eta}^m)}, \\ &\sup_{x \in \mathscr{U}_\eta^d + Q_{\underline{\rho}\eta}^m} C_2 \fint_{Q_r^m(x)} \fint_{Q_r^m(x)} |u_\eta^{\mathrm{op}}(y) - u_\eta^{\mathrm{op}}(z)| \,\mathrm{d}y \,\mathrm{d}z \Bigg\} \end{aligned} \tag{2.4.7}$$

if $s \geq 1$, respectively

$$\begin{aligned}\operatorname{Dist}_F(u_\eta^{\mathrm{sm}}((\mathscr{K}^m \setminus \mathscr{U}_\eta^m) \cup (\mathscr{U}_\eta^d + Q_{\underline{\rho}\eta}^m))) \leq \max\Bigg\{ &\max_{\sigma^m \in K_\eta^m \setminus E_\eta^m} C_1 \frac{1}{\eta^{\frac{m}{p}-s}} |u|_{W^{s,p}(\sigma^m + Q_{2\rho\eta}^m)}, \\ &\sup_{x \in \mathscr{U}_\eta^d + Q_{\underline{\rho}\eta}^m} C_2 \fint_{Q_r^m(x)} \fint_{Q_r^m(x)} |u_\eta^{\mathrm{op}}(y) - u_\eta^{\mathrm{op}}(z)| \,\mathrm{d}y \,\mathrm{d}z \Bigg\} \end{aligned} \tag{2.4.8}$$

if $0 < s < 1$. We note that, in order to make the right-hand side of (2.4.7), respectively (2.4.8), small, we need to take r sufficiently small, and also to have control on the L^{sp} norm of $\mathrm{D}u$, respectively the $W^{s,p}$ norm of u, on the cubes in $K_\eta^m \setminus E_\eta^m$. This will be our motivation for the choice of good and bad cubes in the proof of the density of the class $\mathscr{R}$.

We proceed with the proof of (2.4.7), respectively (2.4.8). Since u_η^{op} takes its values into F, for almost every $z \in Q_{\psi_\eta(x)}^m(x)$, we have

$$\operatorname{dist}(u_\eta^{\mathrm{sm}}(x), F) \leq |u_\eta^{\mathrm{sm}}(x) - u_\eta^{\mathrm{op}}(z)|.$$

Averaging over $Q_{\psi_\eta(x)}^m(x)$, we find

$$\operatorname{dist}(u_\eta^{\mathrm{sm}}(x), F) \leq \fint_{Q_{\psi_\eta(x)}^m(x)} |u_\eta^{\mathrm{sm}}(x) - u_\eta^{\mathrm{op}}(z)| \,\mathrm{d}z.$$

Using the rewriting (2.4.1), we deduce that, for every $x \in \omega$,

$$\begin{aligned}\operatorname{dist}(u_\eta^{\mathrm{sm}}(x), F) &\leq \fint_{Q_{\psi_\eta(x)}^m(x)} \fint_{Q_{\psi_\eta(x)}^m(x)} \varphi\Big(\frac{y-x}{\psi(x)}\Big) |u_\eta^{\mathrm{op}}(y) - u_\eta^{\mathrm{op}}(z)| \,\mathrm{d}y \,\mathrm{d}z \\ &\lesssim \fint_{Q_{\psi_\eta(x)}^m(x)} \fint_{Q_{\psi_\eta(x)}^m(x)} |u_\eta^{\mathrm{op}}(y) - u_\eta^{\mathrm{op}}(z)| \,\mathrm{d}y \,\mathrm{d}z. \end{aligned} \tag{2.4.9}$$

If $Q_{\psi_\eta(x)}^m(x) \subset \mathscr{U}_\eta^d + Q_{\rho\eta}^m$ and $d \leq sp$, Proposition 2.3.8 ensures that the right-hand side of (2.4.9) can be made arbitrarily small if we take $\psi_\eta(x)$ sufficiently small. This invites us to choose ψ_η to be very small in a neighborhood of $\mathscr{U}^d$.

On the other hand, the Poincaré–Wirtinger inequality ensures that

$$\operatorname{dist}(u_\eta^{\mathrm{sm}}(x), F) \lesssim \frac{1}{\psi_\eta(x)^{\frac{m}{sp}-1}} \|\mathrm{D}u_\eta^{\mathrm{op}}\|_{L^{sp}(Q^m_{\psi_\eta(x)}(x))} \tag{2.4.10}$$

if $s \geq 1$, respectively

$$\operatorname{dist}(u_\eta^{\mathrm{sm}}(x), F) \lesssim \frac{1}{\psi_\eta(x)^{\frac{m}{p}-s}} |u_\eta^{\mathrm{op}}|_{W^{s,p}(Q^m_{\psi_\eta(x)}(x))} \tag{2.4.11}$$

if $0 < s < 1$. These estimates are only useful in the region where we can control the L^{sp} norm of $\mathrm{D}u$ or the $W^{s,p}$ norm of u, that is, on the good cubes. On the other hand, since $sp < m$, (2.4.10) and (2.4.11) suggest that ψ_η should not be too small.

We now pause to explain the construction of a function ψ_η suited for our approximation results. As explained in Section 2.2, we distinguish between three regimes. In $\mathscr{U}^d_\eta + Q^m_{\rho\eta}$, we take ψ_η very small, according to Proposition 2.3.8. On the good cubes, we take ψ_η of order η, in order to apply (2.4.10), respectively (2.4.11). Between these two regimes, we need a transition region in order for ψ_η to change of magnitude. Here the second part of Proposition 2.3.8 comes into play. Indeed, if $x \in \sigma^m$ for some $\sigma^m \in U^m_\eta$ and $Q^m_{\psi_\eta(x)}(x) \subset \mathscr{U}^d_\eta + Q^m_{\rho\eta}$, we have

$$\operatorname{dist}(u_\eta^{\mathrm{sm}}(x), F) \lesssim \frac{\psi_\eta(x)^{1-\frac{d}{sp}}}{\eta^{\frac{m-d}{sp}}} \|\mathrm{D}u\|_{L^{sp}(\sigma^m + Q^m_{2\rho\eta})} \tag{2.4.12}$$

if $s \geq 1$, respectively

$$\operatorname{dist}(u_\eta^{\mathrm{sm}}(x), F) \lesssim \frac{\psi_\eta(x)^{s-\frac{d}{p}}}{\eta^{\frac{m-d}{p}}} |u|_{W^{s,p}(\sigma^m + Q^m_{2\rho\eta})} \tag{2.4.13}$$

if $0 < s < 1$. Again, this inequality is only useful on good cubes, but now it requires an upper bound on ψ_η instead if we take $d \leq sp$.

Hence, we proceed with the following construction. Assumption (2.4.6) ensures that we have enough room for the transition region for ψ_η: we have $\operatorname{dist}(\mathscr{E}^m_\eta, \mathscr{K}^m_\eta \setminus \mathscr{U}^m_\eta) \geq \eta$. Therefore, we may find $\zeta_\eta \in C^\infty(\Omega)$ such that

(a) $0 \leq \zeta_\eta \leq 1$ in Ω;

(b) $\zeta_\eta = 1$ in $\mathscr{K}^m_\eta \setminus \mathscr{U}^m_\eta$;

(c) $\zeta_\eta = 0$ in $\mathscr{E}^m_\eta$;

(d) for every $j \in \{1, \dots, k+1\}$,

$$\eta^j \|D^j \zeta_\eta\|_{L^\infty} \leq C_3,$$

for some constant $C_3 > 0$ depending only on m.

Now we pick $0 < r < t$ and we let

$$\psi_\eta = t\zeta_\eta + r(1 - \zeta_\eta).$$

Therefore, $\psi_\eta = r$ on $\mathscr{E}_\eta^m$ and $\psi_\eta = t$ on $\mathscr{K}_\eta^m \setminus \mathscr{U}_\eta^m$. As we observed, we will need to take r very small, while keeping t of order η. We choose

$$t = \min\left\{\frac{\kappa}{C_3}, \rho - \underline{\rho}\right\}\eta \tag{2.4.14}$$

for some fixed $0 < \kappa < 1$. Therefore,

$$\eta^j \|D^j \psi_\eta\|_{L^\infty} \leq \kappa\eta \quad \text{for every } j \in \{1, \dots, k+1\},$$

which ensures that the assumptions of Proposition 2.4.1 are satisfied. Moreover, we have $0 < \psi_\eta \leq (\rho - \underline{\rho})\eta$, which implies that, if $x \in \mathscr{U}_\eta^d + Q_{\underline{\rho}\eta}^m$, then $Q_{\psi_\eta(x)}^m(x) \subset \mathscr{U}_\eta^d + Q_{\rho\eta}^m$. Estimate (2.4.7), respectively (2.4.8), is a straightforward consequence of estimate (2.4.9) for $x \in \mathscr{E}_\eta^m \cap (\mathscr{U}^d + Q_{\underline{\rho}\eta})$, estimate (2.4.10), respectively (2.4.11), for $x \in \mathscr{K}_\eta^m \setminus \mathscr{U}_\eta^m$, and estimate (2.4.12), respectively (2.4.13), for $x \in (\mathscr{U}_\eta^m \setminus \mathscr{E}_\eta^m) \cap (\mathscr{U}_\eta^d + Q_{\underline{\rho}\eta}^m)$.

Before closing this section, we summarize what we have obtained so far. Given a map $u \in W^{s,p}(\Omega; F)$, we have constructed a smooth map u_η^{sm} for which we may estimate its distance to u in $W^{s,p}$. Moreover, even though u_η^{sm} does not necessarily take values into F, we are able to control the distance between u_η^{sm} and F everywhere on the cubication $\mathscr{K}_\eta^m$, except on the cubes in U_η^m, far from their d-skeleton. Therefore, our next step is to be able to modify u_η^{sm} into a new map which, on the cubes in U_η^m, depends only on the values of u_η^{sm} near the d-skeleton of the cubes, while controlling the $W^{s,p}$ distance between u_η^{sm} and this new map.

2.5 Thickening

This section is devoted to the thickening procedure. As we explained in Section 2.2, this technique is reminiscent of the homogeneous extension method, which was used by F. Bethuel to deal with the case $s = 1$; see [Bet91]. This approach is valid for $W^{s,p}$ maps with $s < 1 + \frac{1}{p}$ (but not beyond $s = 1 + \frac{1}{p}$). In order to deal with $W^{s,p}$ maps with arbitrary s, a new tool, thickening, is needed. Its construction was performed by P. Bousquet, A.

Ponce, and J. Van Schaftingen in [BPVS15, Section 4], which also contains the analytic estimates that make thickening instrumental in the proof of the density of the class $\mathscr{R}$ for integer s. In this section, we establish the fractional counterparts of the estimates in [BPVS15]. The main feature of this section is the need for new techniques, taking into account the geometry of the thickening maps, that we develop in order to obtain fractional estimates. This will become transparent, e.g., in the proof of estimates (a) and (c) in Proposition 2.5.3, relying crucially on estimate (2.5.4).

Proposition 2.5.1. *Let $\Omega \subset \mathbb{R}^m$ be open, $d \in \{0, \dots, m-1\}$, $\eta > 0$, $0 < \rho < 1$, K^m be a cubication in $\mathbb{R}^m$ of radius η, U^m be a subskeleton of K^m such that $\mathscr{U}^m + Q^m_{\rho\eta} \subset \Omega$, and T^{d^*} be the dual skeleton of U^d. There exists a smooth map $\Phi\colon \mathbb{R}^m \setminus \mathscr{T}^{d^*} \to \mathbb{R}^m$ such that*

(*i*) *Φ is injective;*

(*ii*) *for every $\sigma^m \in K^m$, $\Phi(\sigma^m \setminus \mathscr{T}^{d^*}) \subset \sigma^m \setminus \mathscr{T}^{d^*}$;*

(*iii*) $\operatorname{Supp}\Phi \subset \mathscr{U}^m + Q^m_{\rho\eta}$ *and* $\Phi(\mathscr{U}^m \setminus \mathscr{T}^{d^*}) \subset \mathscr{U}^d + Q^m_{\rho\eta}$;

(*iv*) *for every $j \in \mathbb{N}_*$ and for every $x \in (\mathscr{U}^m + Q^m_{\rho\eta}) \setminus \mathscr{T}^{d^*}$,*

$$|\mathrm{D}^j\Phi(x)| \le C\frac{\eta}{\operatorname{dist}(x, \mathscr{T}^{d^*})^j},$$

for some constant $C > 0$ depending on j, m and ρ.

If in addition $d + 1 > sp$, then for every $u \in W^{s,p}(\Omega; \mathbb{R}^\nu)$, we have $u \circ \Phi \in W^{s,p}(\Omega; \mathbb{R}^\nu)$, and moreover, the following estimates hold:

(*a*) *if $0 < s < 1$, then*

$$\eta^s|u \circ \Phi - u|_{W^{s,p}(\Omega)} \le C'\Big(\eta^s|u|_{W^{s,p}(\mathscr{U}^m+Q^m_{\rho\eta})} + \|u\|_{L^p(\mathscr{U}^m+Q^m_{\rho\eta})}\Big);$$

(*b*) *if $s \ge 1$, then for every $j \in \{1, \dots, k\}$,*

$$\eta^j\|\mathrm{D}^j(u \circ \Phi) - \mathrm{D}^j u\|_{L^p(\Omega)} \le C'\sum_{i=1}^{j}\eta^i\|\mathrm{D}^i u\|_{L^p(\mathscr{U}^m+Q^m_{\rho\eta})};$$

(*c*) *if $s \ge 1$ and $\sigma \ne 0$, then for every $j \in \{1, \dots, k\}$,*

$$\eta^{j+\sigma}|\mathrm{D}^j(u\circ\Phi) - \mathrm{D}^j u|_{W^{\sigma,p}(\Omega)} \le C'\sum_{i=1}^{j}\Big(\eta^i\|\mathrm{D}^i u\|_{L^p(\mathscr{U}^m+Q^m_{\rho\eta})} + \eta^{i+\sigma}|\mathrm{D}^i u|_{W^{\sigma,p}(\mathscr{U}^m+Q^m_{\rho\eta})}\Big);$$

(*d*) *for every $0 < s < +\infty$,*

$$\|u \circ \Phi - u\|_{L^p(\Omega)} \le C'\|u\|_{L^p(\mathscr{U}^m+Q^m_{\rho\eta})};$$

for some constant $C' > 0$ *depending on* m*,* s*,* p*, and* ρ*.*

We emphasize that, unlike for opening in Section 2.3, the map Φ constructed in Proposition 2.5.1 above is independent of the map $u \in W^{s,p}$ it shall be composed with.

Similarly to opening, crucial to the proof of the strong density theorem is the fact that the thickening procedure increases the energy of the map u at most by a constant factor in the region where u is modified. This, in turn, implies that the distance between u and $u \circ \Phi$ is controlled by the energy of u on $\mathscr{U}^m + Q^m_{\rho\eta}$, as stated in the conclusion of Proposition 2.5.1. In the proof of the density of the class $\mathscr{R}$, this will be used in combination with the fact that the measure of the set $\mathscr{U}^m + Q^m_{\rho\eta}$ tends to 0 as $\eta \to 0$ in order to ensure that $u \circ \Phi$ is close to u when η is sufficiently small.

As for the opening, Proposition 2.5.1 is proved blockwise: we first construct, in Proposition 2.5.2, a map, still denoted Φ, which thickens each face of $\mathscr{T}^{\ell^*}$. We then suitably glue those maps to obtain a thickening map as in Proposition 2.5.1. Before giving the description of the building blocks used in the proof of Proposition 2.5.1, we introduce some additional notation similarly to what we did for opening. The construction of the map in Proposition 2.5.2 below involves three parameters $0 < \underline{\rho} < \rho < \overline{\rho} < 1$. These parameters being fixed, and given $l \in \{1, \ldots, m\}$, we define the rectangles

$$Q_1 = Q^l_{(1-\overline{\rho})\eta} \times Q^{m-l}_{\underline{\rho}\eta}, \quad Q_2 = Q^l_{(1-\rho)\eta} \times Q^{m-l}_{\underline{\rho}\eta}, \quad Q_3 = Q^l_{(1-\rho)\eta} \times Q^{m-l}_{\rho\eta}. \tag{2.5.1}$$

We note that $Q_1 \subset Q_2 \subset Q_3$. We also set $\mathscr{T} = \{0\}^l \times Q^{m-l}_{\rho\eta}$, the part of the dual skeleton contained in Q_3. The rectangle Q_3 contains the geometric support of Φ, that is, $\Phi = \mathrm{id}$ outside of Q_3. The rectangle Q_2 is the region where the thickening procedure is fully performed: the set $\mathscr{T} \cap Q_2$ is entirely mapped outside of Q_1, in $Q_2 \setminus Q_1$. The region $Q_3 \setminus Q_2$ serves as a transition, on which the map Φ becomes less and less singular, until it reaches the exterior of Q_3 where it coincides with the identity.

This section is organized as follows. First we describe the geometric construction of the building blocks for thickening. Then we prove the analytic estimates satisfied by the composition of a map $u \in W^{s,p}$ with those building blocks. Finally, we explain the construction of the global thickening map based on the aforementioned building blocks, and we prove all properties stated in the conclusion of Proposition 2.5.1.

We start by stating the geometric properties satisfied by the building blocks, which do not depend on the map u to which thickening is applied. The map Φ constructed in Proposition 2.5.2 is exactly the map given by [BPVS15, Proposition 4.3]. Hence, we shall not give a complete proof of Proposition 2.5.2, but we will limit ourselves to recall, for the convenience of the reader, the main steps in the construction of the map Φ.

The main difference with [BPVS15] is the proof of the Sobolev estimates. In [BPVS15], they were obtained on the whole Ω as a corollary of the geometric properties of the map Φ, by the use of the change of variable theorem. This approach does not seem to work to deal with the Gagliardo seminorm. Hence, we first establish the estimates for the building blocks, and we deduce the global estimates by gluing, as for opening. To do so, we need to take into account some additional features of the map Φ, that are part of its construction in [BPVS15, Proof of Proposition 4.3] but do not appear in the conclusion of Proposition 4.3 in [BPVS15].

The construction of the map Φ involves another map $\zeta\colon \mathbb{R}^m \to \mathbb{R}$, which we describe hereafter. For $(x', x'') \in \mathbb{R}^l \times \mathbb{R}^{m-l}$, we define

$$\zeta(x', x'') = \sqrt{|x'|^2 + \eta^2 \theta\Big(\frac{x''}{\eta}\Big)}. \tag{2.5.2}$$

In [BPVS15], $\theta\colon \mathbb{R}^{m-l} \to [0,1]$ is an arbitrary smooth map such that $\theta(x'') = 0$ if $x'' \in Q^{m-l}_{\underline{\rho}}$ and $\theta(x'') = 1$ if $x'' \in \mathbb{R}^{m-l} \setminus Q^{m-l}_{\rho}$. For our purposes, we need to be more precise in our choice of θ. We would like to choose θ to be nondecreasing with respect to cubes, that is, $\theta(x'')$ depends only on the ∞-norm of x'' and $\theta(x'') \leq \theta(y'')$ if $|x''|_\infty \leq |y''|_\infty$. However, this is not possible if we want θ to be smooth, since the ∞-norm is not differentiable. Nevertheless, we may choose θ sufficiently close to be nondecreasing with respect to cubes for our purposes, by replacing the ∞-norm by some q-norm for q sufficiently large.

More precisely, we take $1 < q < +\infty$ sufficiently large, depending on $\underline{\rho}$ and ρ, so that there exists $0 < r_1 < r_2$ satisfying

$$Q^{m-l}_{\underline{\rho}} \subset \{x'' \in \mathbb{R}^{m-l} \colon |x''|_q < r_1\} \subset \{x'' \in \mathbb{R}^{m-l} \colon |x''|_q < r_2\} \subset Q^{m-l}_{\rho}.$$

This is indeed possible since $Q^{m-l}_{\underline{\rho}}$ and Q^{m-l}_{ρ} are respectively the balls of radius $\underline{\rho}$ and ρ with respect to the ∞-norm in $\mathbb{R}^{m-l}$, and since the q-norm converges uniformly on compact sets to the ∞-norm as $q \to +\infty$. We then pick a nondecreasing smooth map $\tilde{\theta}\colon \mathbb{R}_+ \to [0,1]$ such that $\tilde{\theta}(r) = 0$ if $0 \leq r \leq r_1$ and $\tilde{\theta}(r) = 1$ if $r \geq r_2$. We finally set $\theta(x'') = \tilde{\theta}(|x''|_q)$. Since $1 < q < +\infty$, θ is smooth on $\mathbb{R}^{m-l}$, and, by our choice of q, r_1, and r_2, we indeed have $\theta(x'') = 0$ if $x'' \in Q^{m-l}_{\underline{\rho}}$ and $\theta(x'') = 1$ if $x'' \in \mathbb{R}^{m-l} \setminus Q^{m-l}_{\rho}$.

With the description of the map ζ at our disposal, we are now ready to state Proposition 2.5.2. We recall that the rectangles Q_i in (2.5.1) depend on l and η.

Proposition 2.5.2. *Let $l \in \{1, \dots, m\}$, $\eta > 0$, and $0 < \underline{\rho} < \rho < \overline{\rho}$. There exists a smooth function $\Phi\colon \mathbb{R}^m \setminus \mathcal{T} \to \mathbb{R}^m$ such that*

(*i*) *Φ is injective;*

(*ii*) $\operatorname{Supp} \Phi \subset Q_3$;

(*iii*) $\Phi(Q_2 \setminus \mathscr{T}) \subset Q_2 \setminus Q_1$;

(*iv*) *for every* $x \in Q_3 \setminus T$,

$$|D^j\Phi(x)| \leq C\frac{\eta}{\zeta^j(x)} \quad \textit{for every } j \in \mathbb{N}_*,$$

for some constant $C > 0$ *depending on* j, m, $\underline{\rho}$, ρ, *and* $\overline{\rho}$;

(*v*) *for every* $x \in \mathbb{R}^m \setminus \mathscr{T}$,

$$\mathcal{J}\Phi(x) \geq C'\frac{\eta^\beta}{\zeta^\beta(x)} \quad \textit{for every } 0 < \beta < l,$$

for some constant $C' > 0$ *depending on* β, m, $\underline{\rho}$, ρ, *and* $\overline{\rho}$.

Proof. As we already mentioned, the desired map Φ is provided by [BPVS15, Proposition 4.3]. Hence, we limit ourselves to briefly recall its construction for the convenience of the reader, and we refer to [BPVS15] for the complete proof of its properties. For technical reasons, we start by constructing an intermediate map $\Psi\colon \mathbb{R}^m \setminus T \to \mathbb{R}^m$ as follows; see [BPVS15, Lemma 4.5]. We define

$$B_1 = B^l_{(1-\overline{\rho})\eta} \times Q^{m-l}_{\underline{\rho}\eta}, B_2 = B^l_{(1-\rho)\eta} \times Q^{m-l}_{\underline{\rho}\eta}, B_3 = B^l_{(1-\rho)\eta} \times Q^{m-l}_{\rho\eta}.$$

The map Ψ is constructed to satisfy the conclusion of Proposition 2.5.2 with the rectangles Q_i replaced by the corresponding cylinders B_i for $i \in \{1, 2, 3\}$. Since $B_i \subset Q_i$, it will then suffice to compose Ψ with a suitable diffeomorphism $\Theta\colon \mathbb{R}^m \to \mathbb{R}^m$ that dilates B_1 to a set containing Q_1 in order to obtain the desired map Φ.

We choose a smooth map $\varphi\colon (0, +\infty) \to [1, +\infty)$ such that

(a) for $0 < r \leq 1 - \overline{\rho}$,

$$\varphi(r) = \frac{1-\overline{\rho}}{r}\Bigl(1 + \frac{b}{\ln(\frac{1}{r})}\Bigr);$$

(b) for $r \geq 1 - \rho$, $\varphi(r) = 1$;

(c) the function $r \in (0, +\infty) \mapsto r\varphi(r)$ is increasing.

This is possible provided that we choose $b > 0$ such that

$$(1-\overline{\rho})\Bigl(1 + \frac{b}{\ln\frac{1}{1-\overline{\rho}}}\Bigr) < 1 - \rho.$$

Then, we define $\lambda\colon \mathbb{R}^m \setminus \mathscr{T} \to [1, +\infty)$ by

$$\lambda(x) = \varphi\Big(\frac{\zeta(x)}{\eta}\Big),$$

and finally

$$\Psi(x', x'') = (\lambda(x', x'')x', x'').$$

The injectivity of Ψ is a consequence of assumption (c) on φ. The fact that $\operatorname{Supp}\Psi \subset B_3$ relies on assumption (b) on φ, since we may observe that $\zeta(x) \geq (1-\rho)\eta$ if $x \in \mathbb{R}^m \setminus B_3$, and therefore $\lambda(x) = 1$. Combining the observation that, using (c) again,

$$r\varphi(r) \geq \lim_{r \to 0} r\varphi(r) = 1 - \overline{\rho}$$

with the fact that $\zeta(x) = |x'|$ if $x = (x', x'') \in B_2$, we find that $\Psi(B_2 \setminus \mathscr{T}) \subset B_2 \setminus B_1$. In order to obtain (iv) on $B_3 \setminus \mathscr{T}$, we estimate $|\mathrm{D}^j \lambda(x)|$ with the help of the Faà di Bruno formula, and then conclude using Leibniz's rule. The proof of estimate (v) is more delicate. The Jacobian of Ψ may be explicitly evaluated as the determinant of a rank-one perturbation of a diagonal map, as we did in the proof of (2.4.3), and one then uses the properties of φ and ζ to get the required lower bound on the obtained expression. We refer the reader to [BPVS15, Lemma 4.5] for the details.

It remains to correct the fact that we worked with the cylinders B_i instead of the rectangles Q_i. This essentially amount to construct a suitable deformation of $\mathbb{R}^m$ with bounded derivatives and a suitable lower bound on the Jacobian. We let $\Theta\colon \mathbb{R}^m \to \mathbb{R}^m$ be a diffeomorphism whose geometric support is contained in Q_3, which maps $B_2 \setminus B_1$ on a set contained in $Q_2 \setminus Q_1$ — that is, Θ dilates B_1 on a set containing Q_1 — and satisfies the estimates

$$\eta^{j-1}|\mathrm{D}^j\Theta| \leq C_1 \quad \text{and} \quad 0 < C_2 \leq \mathcal{J}\Theta \leq C_3 \quad \text{on } \mathbb{R}^m.$$

We refer the reader to [BPVS15, Lemma 4.4] for the precise construction of this diffeomorphism.

Finally, we let $\Phi = \Theta \circ \Psi$. We observe that this construction satisfies the geometric properties (i) to (iii). The estimate (v) on the Jacobian readily follows from the composition formula for the Jacobian. To get (iv), we invoke the Faà di Bruno formula to compute

$$
\begin{aligned}
|D^j\Phi(x)| &\lesssim \sum_{i=1}^{j} \sum_{\substack{1\le t_1\le\cdots\le t_i\\ t_1+\cdots+t_i=j}} |D^i\Theta(x)||D^{t_1}\Psi(x)|\cdots|D^{t_i}\Psi(x)|\\
&\lesssim \sum_{i=1}^{j} \sum_{\substack{1\le t_1\le\cdots\le t_i\\ t_1+\cdots+t_i=j}} \eta^{1-i}\frac{\eta}{\zeta^{t_1}(x)}\cdots\frac{\eta}{\zeta^{t_i}(x)} \lesssim \frac{\eta}{\zeta^j(x)}.
\end{aligned}
$$

This concludes the proof of the proposition. □

Now that we have the building block Φ, we move to the Sobolev estimates satisfied by the composition $u\circ\Phi$.

Proposition 2.5.3. *Let $l > sp$. Let Φ be as in Proposition 2.5.2. Let $\omega\subset\mathbb{R}^m$ be such that $Q_3\subset\omega\subset B^m_{c\eta}$ for some $c>0$, and assume that there exists $c'>0$ such that*

$$
|B^m_\lambda(z)\cap\omega| \ge c'\lambda^m \quad \text{for every } z\in\omega \text{ and } 0<\lambda\le\frac{1}{2}\operatorname{diam}\omega. \tag{2.5.3}
$$

For every $u\in W^{s,p}(\omega;\mathbb{R}^\nu)$, we have $u\circ\Phi\in W^{s,p}(\omega;\mathbb{R}^\nu)$, and moreover, the following estimates hold:

(*a*) *if $0<s<1$, then*

$$
|u\circ\Phi|_{W^{s,p}(\omega)} \le C|u|_{W^{s,p}(\omega)};
$$

(*b*) *if $s\ge1$, then for every $j\in\{1,\dots,k\}$,*

$$
\eta^j\|D^j(u\circ\Phi)\|_{L^p(\omega)} \le C\sum_{i=1}^{j}\eta^i\|D^i u\|_{L^p(\omega)};
$$

(*c*) *if $s\ge1$ and $\sigma\ne0$, then for every $j\in\{1,\dots,k\}$,*

$$
\eta^{j+\sigma}|D^j(u\circ\Phi)|_{W^{\sigma,p}(\omega)} \le C\sum_{i=1}^{j}\Big(\eta^i\|D^i u\|_{L^p(\omega)}+\eta^{i+\sigma}|D^i u|_{W^{\sigma,p}(\omega)}\Big);
$$

(*d*) *for every $0<s<+\infty$,*

$$
\|u\circ\Phi\|_{L^p(\omega)} \le C\|u\|_{L^p(\omega)};
$$

for some constant $C>0$ depending on s, m, p, c, c', $\underline{\rho}$, ρ, and $\overline{\rho}$.

We comment on the assumptions (2.5.3) on ω. In this section, ω will be a rectangle whose sidelengths are constant multiples of η. However, in Section 2.7, we will use Proposition 2.7.3, which is very similar to Proposition 2.5.3, with a more complicated ω. This is why we stated Proposition 2.5.3 in a rather general form.

The assumption that ω is contained in some ball having radius of order η is purely technical. It ensures that estimate (iv) of Proposition 2.5.2 still applies for $x \in \omega \setminus \mathscr{T}$, possibly increasing the constant. Indeed, since $\operatorname{Supp} \Phi \subset Q_3$, this estimate clearly still applies when $x \notin Q_3$, but the constant deteriorates as $|x'| \to +\infty$, since then $\Phi = \mathrm{id}$ while $\zeta(x) \to +\infty$. We could bypass this restriction, but it would not be useful since anyway we intend to apply Proposition 2.5.3 to domains ω satisfying this requirement.

To prove Proposition 2.5.3, we need a technical lemma. As it was the case for opening, in the proof of the fractional Sobolev estimates, we will need to estimate terms of the form $|D^t\Phi(x) - D^t\Phi(y)|$. However, unlike for the opening, we cannot upper bound such terms by a simple application of the mean value theorem along a segment connecting x and y, since such a segment could potentially get very close to – or even cross – the dual skeleton $\mathscr{T}$ where Φ is singular. The following lemma provides us with a suitable path along which to apply the mean value theorem.

Lemma 2.5.4. *For every $x, y \in \mathbb{R}^m \setminus \mathscr{T}$, there exists a Lipschitz path $\gamma\colon [0,1] \to \mathbb{R}^m \setminus \mathscr{T}$ from x to y such that*

$$|\gamma|_{C^{0,1}([0,1])} \leq C|x-y|,$$

for some constant $C > 0$ depending only on m, and such that $\zeta \geq \min\{\zeta(x), \zeta(y)\}$ along γ, where ζ is the map defined in (2.5.2)*.*

Proof. We recall the well-known fact that, given x, y on a sphere, there exists a Lipschitz path on this sphere connecting those two points, with Lipschitz constant less that $C_1|x-y|$. Indeed, it suffices to take the shortest arc of great circle joining x to y. The same fact holds for any q-sphere with $1 \leq q \leq +\infty$. This can be deduced from the Euclidean case using the *changing norm projection* defined by $x \mapsto \frac{|x|_q}{|x|_2}x$, which is a Lipschitz map.

The desired path is then obtained as follows. If $x = (x', x'')$ and $y = (y', y'')$, we first go from x to (y', x'') by following successively an arc of great circle and a straight line in the first l components, while keeping the $m-l$ last components fixed. Then we go from (y', x'') to y by following a path on a q-sphere as above, where q is the parameter used in the definition of ζ, followed by a straight line in the $m-l$ last components, while keeping the first l components fixed. By construction, using the observations above, this path has Lipschitz constant less than $C_2|x-y|$. Moreover, since ζ only depends on the 2-norm of the l first components and on the q-norm of the $m-l$ last components and is increasing with respect to both these parameters, we conclude that the constructed path has all the expected properties. $\square$

We may now prove Proposition 2.5.3.

Proof of Proposition 2.5.3. The integer order estimates were obtained in [BPVS15, Corollary 4.2]. Since the proof in the fractional case relies, in part, on the calculations in the integer case, we reproduce here, for the convenience of the reader, the proof in [BPVS15]. When $s \geq 1$, we have $l \geq sp \geq 1$, and hence the dimension of $\mathscr{T}$ is less than $m-l-1 \leq m-2$. Therefore, in order to prove that $u \circ \Phi \in W^{k,p}(\omega; \mathbb{R}^\nu)$, it suffices to prove that

$$\int_{\omega \setminus \mathscr{T}} |\mathrm{D}^j(u \circ \Phi)|^p < +\infty \quad \text{for every } j \in \{0, \dots, k\}.$$

By the Faà di Bruno formula, we estimate for every $j \in \{1, \dots, k\}$ and $x \in \omega \setminus \mathscr{T}$

$$|\mathrm{D}^j(u \circ \Phi)(x)|^p \lesssim \sum_{i=1}^{j} \sum_{\substack{1 \leq t_1 \leq \dots \leq t_i \\ t_1 + \dots + t_i = j}} |\mathrm{D}^i u(\Phi(x))|^p |\mathrm{D}^{t_1}\Phi(x)|^p \cdots |\mathrm{D}^{t_i}\Phi(x)|^p.$$

Let $0 < \beta < l$. Using the estimates on the derivatives and the Jacobian of Φ, we find

$$|\mathrm{D}^t\Phi| \lesssim \frac{(\mathcal{J}\Phi)^{\frac{t}{\beta}}}{\eta^{t-1}},$$

and therefore,

$$\begin{aligned}
|\mathrm{D}^j(u \circ \Phi)(x)|^p &\lesssim \sum_{i=1}^{j} \sum_{\substack{1 \leq t_1 \leq \dots \leq t_i \\ t_1 + \dots + t_i = j}} |\mathrm{D}^i u(\Phi(x))|^p \frac{(\mathcal{J}\Phi(x))^{\frac{t_1 p}{\beta}}}{\eta^{(t_1-1)p}} \cdots \frac{(\mathcal{J}\Phi(x))^{\frac{t_i p}{\beta}}}{\eta^{(t_i-1)p}} \\
&\lesssim \sum_{i=1}^{j} |\mathrm{D}^i u(\Phi(x))|^p \frac{(\mathcal{J}\Phi(x))^{\frac{jp}{\beta}}}{\eta^{(j-i)p}}.
\end{aligned}$$

Since $jp \leq sp < l$, we may choose $\beta = jp$. Hence,

$$|\mathrm{D}^j(u \circ \Phi)(x)|^p \lesssim \sum_{i=1}^{j} |\mathrm{D}^i u(\Phi(x))|^p \frac{\mathcal{J}\Phi(x)}{\eta^{(j-i)p}}.$$

Since Φ is injective and $\operatorname{Supp}\Phi \subset Q_3$, we have $\Phi(\omega \setminus \mathscr{T}) \subset \omega$. Hence, the change of variable theorem ensures that

$$\int_{\omega\setminus\mathcal{T}} \eta^{jp}|\mathrm{D}^j(u\circ\Phi)|^p \lesssim \int_{\omega\setminus\mathcal{T}} \sum_{i=1}^{j} \eta^{ip}|\mathrm{D}^i u(\Phi(x))|^p \mathcal{J}\Phi(x)\,\mathrm{d}x \lesssim \sum_{i=1}^{j} \int_{\omega} \eta^{ip}|\mathrm{D}^i u|^p.$$

The proof of the zero order estimate (valid in the full range $0 < s < 1$) is straightforward using the same change of variable, noting that in particular, $\mathcal{J}\Phi \geq C_1 > 0$. In particular, we have $u \circ \Phi \in W^{k,p}(\omega; \mathbb{R}^\nu)$.

We now turn to the proof of the fractional estimate in the case $0 < s < 1$.

Step 1. — Mean value-type estimate. We prove that, for every $x, y \in \omega \setminus \mathcal{T}$,

$$\frac{|\Phi(x) - \Phi(y)|}{|x - y|} \lesssim \frac{\eta}{\zeta(y)}. \tag{2.5.4}$$

It suffices to consider the case when $\zeta(x) \leq \zeta(y)$. First assume that $\zeta(y) \leq 2\zeta(x)$. In this case, we use the mean value theorem with the path γ provided by Lemma 2.5.4 along with the estimate satisfied by $D\Phi$ to write

$$\frac{|\Phi(x) - \Phi(y)|}{|x - y|} \lesssim \frac{\eta}{\zeta(x)} \lesssim \frac{\eta}{\zeta(y)}.$$

We consider now the case where $2\zeta(x) \leq \zeta(y)$. We observe that we have $\zeta(y) - \zeta(x) \lesssim |x - y|$ — this can be seen as a consequence of the triangle inequality for the Euclidean norm. Hence,

$$|x - y| \geq \zeta(y) - \zeta(x) \gtrsim \frac{1}{2}\zeta(y).$$

On the other hand, since $\omega \subset B^m_{c\eta}$, we have $|\Phi(x) - \Phi(y)| \lesssim \eta$. This concludes the proof of (2.5.4).

Step 2. — Averaging. We write

$$\iint\limits_{\substack{(\omega\setminus\mathcal{T})\times(\omega\setminus\mathcal{T})\\ \zeta(x)\leq\zeta(y)}} \frac{|u\circ\Phi(x) - u\circ\Phi(y)|^p}{|x - y|^{m+sp}}\,\mathrm{d}x\,\mathrm{d}y \lesssim \int_{\omega\setminus\mathcal{T}} \int_{\omega\setminus\mathcal{T}} \fint_{\mathcal{B}_{x,y}} \frac{|u\circ\Phi(x) - u(z)|^p}{|x - y|^{m+sp}}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y,$$

where we have defined

$$\mathcal{B}_{x,y} = B^m_{|\Phi(x)-\Phi(y)|}\left(\frac{\Phi(x) + \Phi(y)}{2}\right) \cap \omega.$$

We observe that

$$B^m_{\frac{|\Phi(x)-\Phi(y)|}{2}}(\Phi(x)) \cap \omega \subset \mathscr{B}_{x,y}.$$

Therefore, since $\frac{|\Phi(x)-\Phi(y)|}{2} \leq \frac{1}{2}\operatorname{diam}\omega$, we find

$$|\mathscr{B}_{x,y}| \gtrsim |\Phi(x)-\Phi(y)|^m.$$

Here, we have used the volume assumption (2.5.3). Hence,

$$\begin{aligned}
&\int_{\omega\setminus\mathscr{T}}\int_{\omega\setminus\mathscr{T}}\fint_{\mathscr{B}_{x,y}} \frac{|u\circ\Phi(x)-u(z)|^p}{|x-y|^{m+sp}}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y \\
&\quad\lesssim \int_{\omega\setminus\mathscr{T}}\int_{\omega\setminus\mathscr{T}}\int_{\mathscr{B}_{x,y}} \frac{|u\circ\Phi(x)-u(z)|^p}{|\Phi(x)-\Phi(y)|^m|x-y|^{m+sp}}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y \\
&\quad\lesssim \int_{\omega\setminus\mathscr{T}}\int_{\omega\setminus\mathscr{T}}\int_{\mathscr{B}_{x,y}} \frac{|u\circ\Phi(x)-u(z)|^p}{|\Phi(x)-z|^m|x-y|^{m+sp}}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}$$

where we made use of the fact that $|\Phi(x)-z| \leq \frac{3}{2}|\Phi(x)-\Phi(y)|$ whenever $z \in \mathscr{B}_{x,y}$. Invoking Tonelli's theorem, we find

$$\begin{aligned}
&\int_{\omega\setminus\mathscr{T}}\int_{\omega\setminus\mathscr{T}}\int_{\mathscr{B}_{x,y}} \frac{|u\circ\Phi(x)-u(z)|^p}{|\Phi(x)-z|^m|x-y|^{m+sp}}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad\qquad = \int_{\omega\setminus\mathscr{T}}\int_{\omega\setminus\mathscr{T}}\int_{\mathscr{Y}_{x,z}} \frac{|u\circ\Phi(x)-u(z)|^p}{|\Phi(x)-z|^m|x-y|^{m+sp}}\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x,
\end{aligned}$$

where $\mathscr{Y}_{x,z}$ is the set of all those $y \in \omega\setminus\mathscr{T}$ such that $z \in \mathscr{B}_{x,y}$, that is,

$$\mathscr{Y}_{x,z} = \{y \in \omega\setminus\mathscr{T} : |\Phi(x)+\Phi(y)-2z| \leq 2|\Phi(x)-\Phi(y)|\}.$$

Since $|\Phi(x)+\Phi(y)-2z| \geq 2|\Phi(x)-z| - |\Phi(x)-\Phi(y)|$, we find, using (2.5.4),

$$\mathscr{Y}_{x,z} \subset \left\{y \in \mathbb{R}^m : |\Phi(x)-z| \leq \frac{3}{2}|\Phi(x)-\Phi(y)|\right\} \subset \left\{y \in \mathbb{R}^m : |\Phi(x)-z| \lesssim \frac{\eta}{\zeta(x)}|x-y|\right\}.$$

Therefore,

$$\int_{\mathscr{Y}_{x,z}} \frac{1}{|x-y|^{m+sp}}\,\mathrm{d}y \lesssim \frac{\eta^{sp}}{\zeta(x)^{sp}|\Phi(x)-z|^{sp}}.$$

We conclude that

$$\iint\limits_{\substack{(\omega\setminus\mathscr{T})\times(\omega\setminus\mathscr{T})\\ \zeta(x)\leq\zeta(y)}} \frac{|u\circ\Phi(x)-u\circ\Phi(y)|^p}{|x-y|^{m+sp}}\,\mathrm{d}x\,\mathrm{d}y \lesssim \int_{\omega\setminus\mathscr{T}}\int_{\omega}\frac{|u\circ\Phi(x)-u(z)|^p}{|\Phi(x)-z|^{m+sp}}\frac{\eta^{sp}}{\zeta(x)^{sp}}\,\mathrm{d}z\,\mathrm{d}x.$$

Step 3. — Change of variable. Since $0<sp<l$, we may apply estimate (v) of Proposition 2.5.2 with $\beta=sp$. Taking into account the fact that Φ is injective and $\Phi(\omega\setminus\mathscr{T})\subset\omega$, we deduce from the change of variable theorem that

$$\begin{aligned}|u\circ\Phi|^p_{W^{s,p}(\omega)} &\lesssim \int_{\omega\setminus\mathscr{T}}\int_{\omega}\frac{|u\circ\Phi(x)-u(z)|^p}{|\Phi(x)-z|^{m+sp}}\mathcal{J}\Phi(x)\,\mathrm{d}z\,\mathrm{d}x\\ &\lesssim \int_{\omega}\int_{\omega}\frac{|u(y)-u(z)|^p}{|y-z|^{m+sp}}\,\mathrm{d}z\,\mathrm{d}y.\end{aligned}$$

This concludes the proof in the case $0<s<1$.

We finish with the fractional estimate in the case $s\geq 1$.

Step 1. — Estimate of $|\mathrm{D}^j u(x)-\mathrm{D}^j u(y)|$. Consider $x,y\in\omega\setminus\mathscr{T}$ such that, without loss of generality, $\zeta(x)\leq\zeta(y)$. As in the previous sections, using the Faà di Bruno formula, the multilinearity of the differential, and the estimates on the derivatives of Φ, we write

$$\begin{aligned}&|\mathrm{D}^j(u\circ\Phi)(x)-\mathrm{D}^j(u\circ\Phi)(y)|\\ &\quad\lesssim\sum_{i=1}^{j}\Big(|\mathrm{D}^i u\circ\Phi(x)-\mathrm{D}^i u\circ\Phi(y)|\frac{\eta^i}{\zeta(y)^j}\\ &\qquad+\sum_{t=1}^{j}|\mathrm{D}^i u\circ\Phi(x)||\mathrm{D}^t\Phi(x)-\mathrm{D}^t\Phi(y)|\frac{\eta^{i-1}}{\zeta(x)^{j-t}}\Big).\end{aligned}\tag{2.5.5}$$

Step 2. — Estimate of the second term in (2.5.5). We proceed as we did in the proofs of Propositions 2.3.2 and 2.4.1, relying on an optimization argument. We split the integral over $B^m_r(x)$ and $\mathbb{R}^m\setminus B^m_r(x)$ and we insert $r=\zeta(x)$ to arrive at

$$\int\limits_{\substack{\omega\setminus\mathscr{T}\\ \zeta(x)\leq\zeta(y)}}\frac{|\mathrm{D}^t\Phi(x)-\mathrm{D}^t\Phi(y)|^p}{|x-y|^{m+\sigma p}}\,\mathrm{d}y\lesssim\frac{\eta^p}{\zeta(x)^{(t+\sigma)p}}.$$

Hence,

$$\iint_{\substack{(\omega\setminus\mathscr{T})\times(\omega\setminus\mathscr{T})\\ \zeta(x)\leq\zeta(y)}} \frac{|\mathrm{D}^i u\circ\Phi(x)|^p|\mathrm{D}^t\Phi(x)-\mathrm{D}^t\Phi(y)|^p}{|x-y|^{m+\sigma p}}\frac{\eta^{(i-1)p}}{\zeta(x)^{(j-t)p}}\,\mathrm{d}x\,\mathrm{d}y \lesssim \int_{\omega\setminus T}|\mathrm{D}^i u\circ\Phi(x)|^p\frac{\eta^{ip}}{\zeta(x)^{(j+\sigma)p}}\,\mathrm{d}x.$$

Now, by the estimate satisfied by $\mathcal{J}\Phi$, we have, for $0<\beta<l$,

$$\int_{\omega\setminus\mathscr{T}}|\mathrm{D}^i u\circ\Phi(x)|^p\frac{\eta^{ip}}{\zeta(x)^{(j+\sigma)p}}\,\mathrm{d}x \lesssim \eta^{ip-(j+\sigma)p}\int_{\omega\setminus\mathscr{T}}|\mathrm{D}^i u\circ\Phi(x)|^p(\mathcal{J}\Theta(x))^{\frac{(j+\sigma)p}{\beta}}\,\mathrm{d}x.$$

Since $l>sp\geq(j+\sigma)p$, we may choose $\beta=(j+\sigma)p$. We conclude by using the change of variable theorem that

$$\int_{\omega\setminus\mathscr{T}}|\mathrm{D}^i u\circ\Phi(x)|^p\frac{\eta^{ip}}{\zeta(x)^{(j+\sigma)p}}\,\mathrm{d}x \lesssim \eta^{(i-j-\sigma)p}\int_{\omega}|\mathrm{D}^i u|^p.$$

Step 3. — Estimate of the first term in (2.5.5): averaging. We use the same methodology as for the case $0<s<1$. Hence, we only write the main steps of the reasoning. We write

$$\begin{aligned}
&\iint_{\substack{(\omega\setminus\mathscr{T})\times(\omega\setminus\mathscr{T})\\ \zeta(x)\leq\zeta(y)}} \frac{|\mathrm{D}^i u\circ\Phi(x)-\mathrm{D}^i u\circ\Phi(y)|^p}{|x-y|^{m+\sigma p}}\frac{\eta^{ip}}{\zeta(y)^{jp}}\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad\leq \iint_{\substack{(\omega\setminus\mathscr{T})\times(\omega\setminus\mathscr{T})\\ \zeta(x)\leq\zeta(y)}} \fint_{\mathscr{B}_{x,y}} \frac{|\mathrm{D}^i u\circ\Phi(x)-\mathrm{D}^i u(z)|^p}{|x-y|^{m+\sigma p}}\frac{\eta^{ip}}{\zeta(y)^{jp}}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad\quad+ \iint_{\substack{(\omega\setminus\mathscr{T})\times(\omega\setminus\mathscr{T})\\ \zeta(x)\leq\zeta(y)}} \fint_{\mathscr{B}_{x,y}} \frac{|\mathrm{D}^i u(z)-\mathrm{D}^i u\circ\Phi(y)|^p}{|x-y|^{m+\sigma p}}\frac{\eta^{ip}}{\zeta(y)^{jp}}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad\leq \int_{\omega\setminus\mathscr{T}}\int_{\omega\setminus\mathscr{T}}\fint_{\mathscr{B}_{x,y}} \frac{|\mathrm{D}^i u\circ\Phi(x)-\mathrm{D}^i u(z)|^p}{|x-y|^{m+\sigma p}}\frac{\eta^{ip}}{\zeta(x)^{jp}}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Observe that here it is important that we wrote estimate (2.5.5) with $\frac{1}{\zeta(y)}$ on the first term in the right-hand side, so that we may further upper bound $\frac{1}{\zeta(y)}$ by $\frac{1}{\zeta(x)}$. We then pursue exactly as in the case $0 < s < 1$. Using the volume assumption (2.5.3), we find

$$\int_{\omega\setminus\mathcal{T}}\int_{\omega\setminus\mathcal{T}}\fint_{\mathcal{B}_{x,y}}\frac{|D^i u\circ\Phi(x)-D^i u(z)|^p}{|x-y|^{m+\sigma p}}\frac{\eta^{ip}}{\zeta(x)^{jp}}\,dz\,dx\,dy \lesssim \int_{\omega\setminus\mathcal{T}}\int_{\omega\setminus\mathcal{T}}\int_{\mathcal{B}_{x,y}}\frac{|D^i u\circ\Phi(x)-D^i u(z)|^p}{|\Phi(x)-z|^m|x-y|^{m+\sigma p}}\frac{\eta^{ip}}{\zeta(x)^{jp}}\,dz\,dx\,dy.$$

Relying on Tonelli's theorem, we deduce that

$$\int_{\omega\setminus\mathcal{T}}\int_{\omega\setminus\mathcal{T}}\int_{\mathcal{B}_{x,y}}\frac{|D^i u\circ\Phi(x)-D^i u(z)|^p}{|\Phi(x)-z|^m|x-y|^{m+\sigma p}}\frac{\eta^{ip}}{\zeta(x)^{jp}}\,dz\,dx\,dy = \int_{\omega\setminus\mathcal{T}}\int_{\omega\setminus\mathcal{T}}\int_{\mathcal{Y}_{x,z}}\frac{|D^i u\circ\Phi(x)-D^i u(z)|^p}{|\Phi(x)-z|^m|x-y|^{m+\sigma p}}\frac{\eta^{ip}}{\zeta(x)^{jp}}\,dy\,dz\,dx.$$

Using the inclusion

$$\mathcal{Y}_{x,z}\subset\mathbb{R}^m\setminus B^m_r(x),$$

where

$$r = r(x,z) = \frac{C_2|\Phi(x)-z|\zeta(x)}{\eta},$$

we conclude that

$$\iint\limits_{\substack{(\omega\setminus\mathcal{T})\times(\omega\setminus\mathcal{T})\\ \zeta(x)\le\zeta(y)}}\frac{|D^i u\circ\Phi(x)-D^i u\circ\Phi(y)|^p}{|x-y|^{m+\sigma p}}\frac{\eta^{ip}}{\zeta(y)^{jp}}\,dx\,dy \lesssim \int_{\omega\setminus\mathcal{T}}\int_{\omega}\frac{|D^i u\circ\Phi(x)-D^i u(z)|^p}{|\Phi(x)-z|^{m+\sigma p}}\frac{\eta^{\sigma p}}{\zeta(x)^{\sigma p}}\frac{\eta^{ip}}{\zeta(x)^{jp}}\,dz\,dx.$$

Step 4. — Estimate of the first term in (2.5.5): change of variable. As previously, we use estimate (v) of Proposition 2.5.2 with $\beta = sp$ and the change of variable theorem to conclude that

$$\iint\limits_{\substack{(\omega\setminus\mathscr{T})\times(\omega\setminus\mathscr{T})\\ \zeta(x)\le\zeta(y)}} \frac{|\mathrm{D}^i u\circ\Phi(x)-\mathrm{D}^i u\circ\Phi(y)|^p}{|x-y|^{m+\sigma p}}\frac{\eta^{ip}}{\zeta(y)^{jp}}\,\mathrm{d}x\,\mathrm{d}y \\ \lesssim \eta^{(i-j)p}\int_\omega\int_\omega \frac{|\mathrm{D}^i u(y)-\mathrm{D}^i u(z)|^p}{|x-y|^{m+\sigma p}}\,\mathrm{d}z\,\mathrm{d}y.$$

Gathering the estimates for both terms in (2.5.5), we obtain the desired conclusion, hence finishing the proof of the proposition. □

Now that we have constructed the building block for the thickening procedure, we are ready to proceed with the proof of Proposition 2.5.1. We start by presenting an informal explanation of the construction to clarify the method.

We first apply thickening around the vertices of the dual skeleton $\mathscr{T}^{d^*}$, which are actually the centers of the cubes in U^m, with parameters $0 < \rho_m < \tau_{m-1} < \rho_{m-1}$. This maps the complement of the center of each cube on a neighborhood of the faces of the cube. Then, we apply thickening around the edges of the dual skeleton, which are segments of lines passing through the center of the $(m-1)$-faces of U^m, with parameters $\rho_{m-1} < \tau_{m-2} < \rho_{m-2}$. This maps the part of the complement of the edges of $\mathscr{T}^{\ell^*}$ lying at distance at most ρ_{m-1} of the $(m-1)$-faces of U^m on a neighborhood of the $(m-2)$-faces of U^m. But, since at the previous step the complement of the centers of the cubes was already mapped in a neighborhood of the faces of width ρ_{m-1}, we deduce that the whole complement of the 1-skeleton of $\mathscr{T}^{\ell^*}$ is mapped in a neighborhood of $\mathscr{U}^{m-2}$. We pursue this procedure by induction until we reach dimension d^* with respect to the dual skeleton — which corresponds to dimension d with respect to U^m — and this produces the required map Φ.

Figures 2.7, 2.8, and 2.9 provide an illustration of this procedure on one cube when $m = 2$ and $d = 0$. This allows us to see the combination of two steps of the induction procedure. Figure 2.7 shows thickening around the vertices of the dual skeleton, which correspond to the centers of the cubes of U^m. The values of u in the blue region on the left part of the figure are propagated into the blue region on the right part of the figure. This creates a point singularity in the center of each cube, depicted in red. Figure 2.8 illustrates thickening around the edges of the dual skeleton. The values of u in the dark blue region on the left part of the figure are propagated into the dark blue region on the right part of the figure, which creates line singularities in red. The map u is left unchanged on the white region, the part in light blue serving as a transition. The boundaries of the regions in Figure 2.7 are shown in light colors, to illustrate how all the different regions involved in the construction combine together. The combination of both steps inside the square is shown in Figure 2.9. The values in the blue regions on the corners are propagated inside of the whole square, which creates line singularities in red, forming a cross.

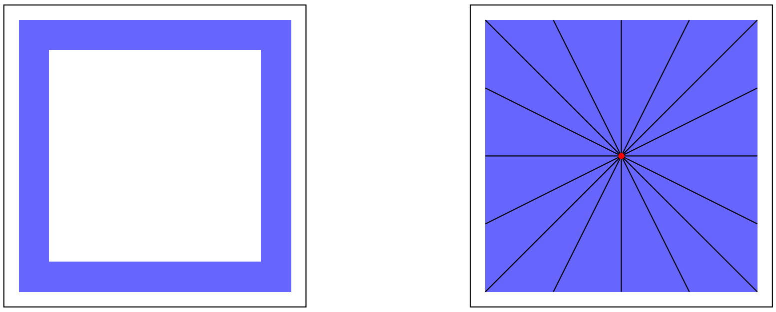

Figure 2.7 – Thickening around vertices

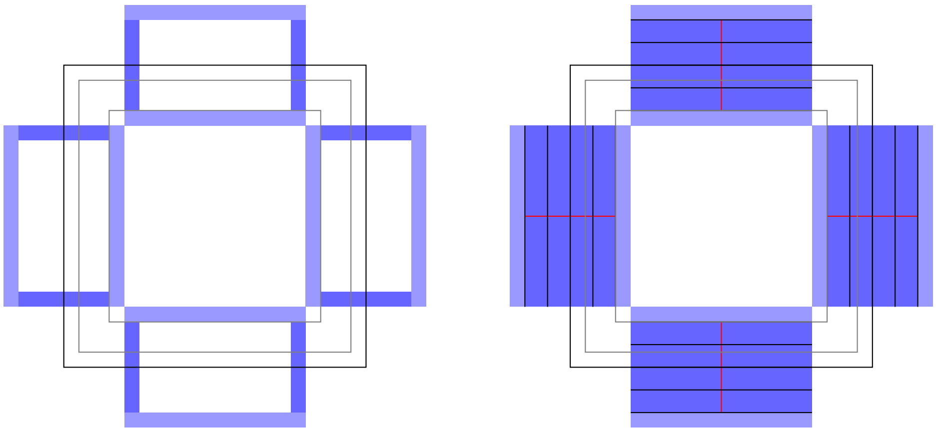

Figure 2.8 – Thickening around edges

Proof of Proposition 2.5.1. The map Φ is constructed as follows. We first take finite sequences $(\rho_i)_{d \leq i \leq m}$ and $(\tau_i)_{d \leq i \leq m}$ such that

$$0 < \rho_m < \tau_{m-1} < \rho_{m-1} < \cdots < \rho_{d+1} < \tau_d < \rho_d = \rho.$$

The map Φ is defined by downward induction. For $l = m$, we let $\Phi^l = \mathrm{id}$. Then, if $l \in \{d+1, \dots, m\}$, given $\sigma^l \in \mathscr{U}^l$, we identify σ^l with $Q^l_\eta \times \{0\}^{m-l}$ and $\mathscr{T}^{(l-1)^*} \cap (\sigma^l + Q^m_{\tau_{l-1}\eta})$ with $\{0\}^l \times Q^{m-l}_{\tau_{l-1}\eta}$, and we let Φ_{σ^l} be the map given by Proposition 2.5.2 applied around σ^l with parameters $\underline{\rho} = \rho_l$, $\rho = \tau_{l-1}$, and $\overline{\rho} = \rho_{l-1}$. We let $\Psi^l \colon \mathbb{R}^m \setminus \mathscr{T}^{(l-1)^*} \to \mathbb{R}^m$ be

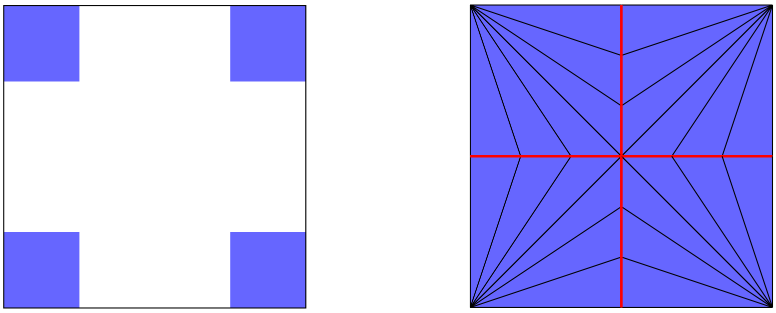

Figure 2.9 – Final thickening at order 1

defined by

$$\Psi^l(x) = \begin{cases} \Phi_{\sigma^l}(x) & \text{if } x \in T_{\sigma^l}(Q_3) \text{ for some } \sigma^l \in U^l, \\ x & \text{otherwise,} \end{cases}$$

where T_{σ^l} is an isometry mapping $Q^l_\eta \times \{0\}^{m-l}$ on σ^l. Finally, we define $\Phi^{l-1} = \Psi^l \circ \Phi^l$. The desired map is given by $\Phi = \Phi^d$.

As we mentioned, properties (i) to (iv) are already contained in [BPVS15, Proposition 4.1], so that we only need to prove estimates (a) to (d). We first prove estimates with Φ replaced by Ψ^l for every $l \in \{d+1, \dots, m\}$. We let $\omega = Q^l_{(1-\rho_m)\eta} \times Q^{m-l}_{\rho\eta}$. We note that ω satisfies the assumptions of Proposition 2.5.3. In particular, we have $Q_3 \subset \omega \subset \mathscr{U}^m + Q^m_{\rho\eta}$ for every $l \in \{d+1, \dots, m\}$. We apply Proposition 2.5.3 to find that, for every $\sigma^l \in U^l$, the following estimates hold:

(a) if $0 < s < 1$, then

$$|u \circ \Phi_{\sigma^l}|_{W^{s,p}(T_{\sigma^l}(\omega))} \lesssim |u|_{W^{s,p}(T_{\sigma^l}(\omega))};$$

(b) if $s \geq 1$, then for every $j \in \{1, \dots, k\}$,

$$\eta^j \|D^j(u \circ \Phi_{\sigma^l})\|_{L^p(T_{\sigma^l}(\omega))} \lesssim \sum_{i=1}^{j} \eta^i \|D^i u\|_{L^p(T_{\sigma^l}(\omega))};$$

(c) if $s \geq 1$ and $\sigma \neq 0$, then for every $j \in \{1, \dots, k\}$,

$$\eta^{j+\sigma} |D^j(u \circ \Phi_{\sigma^l})|_{W^{\sigma,p}(T_{\sigma^l}(\omega))} \lesssim \sum_{i=1}^{j} \Big(\eta^i \|D^i u\|_{L^p(T_{\sigma^l}(\omega))} + \eta^{i+\sigma} |D^i u|_{W^{\sigma,p}(T_{\sigma^l}(\omega))}\Big);$$

(d) for every $0 < s < +\infty$,

$$\|u \circ \Phi_{\sigma^l}\|_{L^p(T_{\sigma^l}(\omega))} \lesssim \|u\|_{L^p(T_{\sigma^l}(\omega))}.$$

Using the additivity of the integral for integer order estimates and Lemma 2.2.1 for fractional order estimates, we find that

(a) if $0 < s < 1$, then

$$\begin{aligned}|u \circ \Psi^l|^p_{W^{s,p}(\mathcal{U}^m + Q^m_{\rho\eta})} \lesssim \sum_{\sigma^l \in U^d} |u \circ \Phi_{\sigma^l}|^p_{W^{s,p}(T_{\sigma^l}(\omega))} \\ + |u \circ \Psi^l|^p_{W^{s,p}((\mathcal{U}^m + Q^m_{\rho\eta}) \setminus \operatorname{Supp} \Psi^l)} + \eta^{-sp} \|u \circ \Psi^l\|^p_{L^p(\mathcal{U}^m + Q^m_{\rho\eta})};\end{aligned}$$

(b) if $s \geq 1$, then for every $j \in \{1, \dots, k\}$,

$$\begin{aligned}\|D^j(u \circ \Psi^l)\|^p_{L^p(\mathcal{U}^m + Q^m_{\rho\eta})} \lesssim \sum_{\sigma^l \in \mathcal{U}^l} \|D^j(u \circ \Phi_{\sigma^l})\|^p_{L^p(T_{\sigma^l}(\omega))} \\ + \|D^j(u \circ \Psi^l)\|^p_{L^p((\mathcal{U}^m + Q^m_{\rho\eta}) \setminus \operatorname{Supp} \Psi^l)};\end{aligned}$$

(c) if $s \geq 1$ and $\sigma \neq 0$, then for every $j \in \{1, \dots, k\}$,

$$\begin{aligned}|D^j(u \circ \Psi^l)|^p_{W^{\sigma,p}(\mathcal{U}^m + Q^m_{\rho\eta})} \lesssim \sum_{\sigma^l \in U^l} |D^j(u \circ \Phi_{\sigma^l})|^p_{W^{\sigma,p}(T_{\sigma^l}(\omega))} \\ + |D^j(u \circ \Psi^l)|^p_{W^{\sigma,p}((\mathcal{U}^m + Q^m_{\rho\eta}) \setminus \operatorname{Supp} \Psi^l)} + \eta^{-\sigma p} \|D^j(u \circ \Psi^l)\|^p_{L^p(\mathcal{U}^m + Q^m_{\rho\eta})};\end{aligned}$$

(d) for every $0 < s < +\infty$,

$$\|u \circ \Psi^l\|^p_{L^p(\mathcal{U}^m + Q^m_{\rho\eta})} \lesssim \sum_{\sigma^l \in U^l} \|u \circ \Phi_{\sigma^l}\|^p_{L^p(T_{\sigma^l}(\omega))} + \|u \circ \Psi^l\|^p_{L^p((\mathcal{U}^m + Q^m_{\rho\eta}) \setminus \operatorname{Supp} \Psi^l)}.$$

Combining both sets of estimates, by downward induction, we deduce that

(a) if $0 < s < 1$, then

$$\eta^s |u \circ \Phi|_{W^{s,p}(\mathcal{U}^m + Q^m_{\rho\eta})} \lesssim \eta^s |u|_{W^{s,p}(\mathcal{U}^m + Q^m_{\rho\eta})} + \|u\|_{L^p(\mathcal{U}^m + Q^m_{\rho\eta})};$$

(b) if $s \geq 1$, then for every $j \in \{1, \dots, k\}$,

$$\eta^j \|D^j(u \circ \Phi)\|_{L^p(\mathcal{U}^m + Q^m_{\rho\eta})} \lesssim \sum_{i=1}^{j} \eta^i \|D^i u\|_{L^p(\mathcal{U}^m + Q^m_{\rho\eta})};$$

(c) if $s \geq 1$ and $\sigma \neq 0$, then for every $j \in \{1, \dots, k\}$,

$$\eta^{j+\sigma}|\mathrm{D}^j(u \circ \Phi)|_{W^{\sigma,p}(\mathscr{U}^m + Q^m_{\rho\eta})} \lesssim \sum_{i=1}^{j}\Big(\eta^i \|\mathrm{D}^i u\|_{L^p(\mathscr{U}^m + Q^m_{\rho\eta})} + \eta^{i+\sigma}|\mathrm{D}^i u|_{W^{\sigma,p}(\mathscr{U}^m + Q^m_{\rho\eta})}\Big);$$

(d) for every $0 < s < +\infty$,

$$\|u \circ \Phi\|_{L^p(\mathscr{U}^m + Q^m_{\rho\eta})} \lesssim \|u\|_{L^p(\mathscr{U}^m + Q^m_{\rho\eta})}.$$

The conclusion follows by an additional application of the additivity of the integral or Lemma 2.2.1, by noting that actually $\operatorname{Supp}\Phi \subset \mathscr{U}^m + Q^m_{\tau_d\eta}$. □

We close this section with a discussion about how the thickening technique that we investigated inserts itself in the proof of the strong density theorem. At the end of Section 2.4, we obtained an estimate on $\operatorname{Dist}_F(u^{\mathrm{sm}}_\eta((\mathscr{K}^m \setminus \mathscr{U}^m_\eta) \cup (\mathscr{U}^d_\eta + Q^m_{\underline{\rho}\eta}))$, where we recall that u^{sm}_η is the map obtained by successively opening and smoothing a map $u \in W^{s,p}(\Omega; F)$, with $F \subset \mathbb{R}^\nu$ being an arbitrary closed set. Informally, we were able to control the distance between u^{sm}_η and F except on the cubes in U^m_η, far from the d-skeleton. We apply now thickening to the map u^{sm}_η. Let Φ^{th}_η be the map provided by Proposition 2.5.1 applied to U^m_η with $K^m = K^m_\eta$ and using parameter $\underline{\rho}$. We set $u^{\mathrm{th}}_\eta = u^{\mathrm{sm}}_\eta \circ \Phi^{\mathrm{th}}_\eta$. To have $u \in W^{s,p}$ along with the estimates provided by Proposition 2.5.1, we need to take $d + 1 > sp$. Since we already required that $d \leq sp$ in Section 2.4, this invites us to work with $d = \lfloor sp \rfloor$.

By inclusion (ii) in Proposition 2.5.1, we have $\Phi^{\mathrm{th}}_\eta(\mathscr{K}^m_\eta \setminus (\mathscr{T}^{d^*}_\eta \cup \mathscr{U}^m_\eta)) \subset \mathscr{K}^m_\eta \setminus \mathscr{U}^m_\eta$. On the other hand, by inclusion (iii) in Proposition 2.5.1, we have $\Phi^{\mathrm{th}}_\eta(\mathscr{U}^m_\eta \setminus \mathscr{T}^{d^*}_\eta) \subset \mathscr{U}^d_\eta + Q^m_{\underline{\rho}\eta}$. Therefore,

$$\Phi^{\mathrm{th}}_\eta(\mathscr{K}^m_\eta \setminus \mathscr{T}^{d^*}_\eta) \subset (\mathscr{K}^m_\eta \setminus \mathscr{U}^m_\eta) \cup (\mathscr{U}^d_\eta + Q^m_{\underline{\rho}\eta}).$$

Combining this observation with estimate (2.4.7), respectively (2.4.8), we deduce that

$$\begin{aligned}\operatorname{Dist}_F(u^{\mathrm{th}}_\eta(\mathscr{K}^m_\eta \setminus \mathscr{T}^{d^*}_\eta)) \leq \max\Bigg\{ &\max_{\sigma^m \in K^m_\eta \setminus E^m_\eta} C\frac{1}{\eta^{\frac{m}{sp}-1}}\|\mathrm{D}u\|_{L^{sp}(\sigma^m + Q^m_{2\rho\eta})}, \\ &\sup_{x \in \mathscr{U}^d_\eta + Q^m_{\underline{\rho}\eta}} C' \fint_{Q^m_r(x)}\fint_{Q^m_r(x)} |u^{\mathrm{op}}_\eta(y) - u^{\mathrm{op}}_\eta(z)|\,\mathrm{d}y\,\mathrm{d}z\Bigg\}\end{aligned} \tag{2.5.6}$$

if $s \geq 1$, respectively

$$\begin{aligned}\operatorname{Dist}_F\left(u^{\mathrm{th}}_\eta(\mathscr{K}^m_\eta \setminus \mathscr{T}^{d^*}_\eta)\right) \leq \max\Bigg\{ & \max_{\sigma^m \in K^m_\eta \setminus E^m_\eta} C\frac{1}{\eta^{\frac{m}{p}-s}} |u|_{W^{s,p}(\sigma^m + Q^m_{2\rho\eta})}, \\ & \sup_{x \in \mathscr{U}^d_\eta + Q^m_{\underline{\rho}\eta}} C' \fint_{Q^m_r(x)} \fint_{Q^m_r(x)} |u^{\mathrm{op}}_\eta(y) - u^{\mathrm{op}}_\eta(z)| \,\mathrm{d}y\,\mathrm{d}z \Bigg\} \end{aligned} \tag{2.5.7}$$

if $0 < s < 1$. Moreover, u^{sm}_η being smooth, the map u^{th}_η is smooth on $\mathscr{K}^m_\eta \setminus \mathscr{T}^{d^*}_\eta$. To summarize, we have obtained a map u^{th}_η which is smooth on $\mathscr{K}^m_\eta \setminus \mathscr{T}^{d^*}_\eta$, and whose distance from F is controlled on the whole $\mathscr{K}^m_\eta \setminus \mathscr{T}^{d^*}_\eta$.

Now let us get back to the case we are interested in, that is, where $F = \mathscr{N}$. In this case, we know that there exists $\iota > 0$ and a smooth retraction $\Pi\colon \mathscr{N} + B^\nu_\iota \to \mathscr{N}$ which is well-defined and smooth on the tubular neighborhood $\mathscr{N} + B^\nu_\iota$; see Proposition 1.4.2 in the introduction. Assume that the right-hand side of (2.5.6) or (2.5.7) is less than ι. Note that this requires both to take r sufficiently small and to choose E^m_η such that, for every $\sigma^m \in K^m_\eta \setminus E^m_\eta$,

$$\|\mathrm{D}u\|_{L^{sp}(\sigma^m + Q^m_{2\rho\eta})} \leq \frac{\eta^{\frac{m}{sp}-1}}{C}\iota, \quad \text{respectively} \quad |u|_{W^{s,p}(\sigma^m + Q^m_{2\rho\eta})} \leq \frac{\eta^{\frac{m}{p}-s}}{C}\iota. \tag{2.5.8}$$

Under this assumption, the map $u_\eta = \Pi \circ u^{\mathrm{th}}_\eta$ is well-defined and smooth on $\mathscr{K}^m_\eta \setminus \mathscr{T}^{d^*}_\eta$, and takes its values into $\mathscr{N}$.

We next prove that the map u_η actually belongs to the class $\mathscr{R}_{m-\lfloor sp \rfloor -1}(\mathscr{K}^m_\eta; \mathscr{N})$. This follows from property (iv) in Proposition 2.5.1. Indeed, since $m - \lfloor sp \rfloor - 1 = d^*$, the singular set of u^{th}_η, and hence of u_η, is as in the definition of $\mathscr{R}_{m-\lfloor sp \rfloor -1}(\mathscr{K}^m_\eta; \mathscr{N})$. Therefore, it only remains to prove the estimates on the derivatives of u_η. Since u^{sm}_η and Π are smooth, we deduce from the Faà di Bruno formula that

$$\begin{aligned}|\mathrm{D}^j u_\eta| &\lesssim \sum_{i=1}^{j} \sum_{\substack{1 \leq t_1 \leq \dots \leq t_i \\ t_1 + \dots + t_i = j}} |\mathrm{D}^i(\Pi \circ u^{\mathrm{sm}}_\eta)| |\mathrm{D}^{t_1}\Phi^{\mathrm{th}}_\eta| \cdots |\mathrm{D}^{t_i}\Phi^{\mathrm{th}}_\eta| \\ &\lesssim \sum_{i=1}^{j} \sum_{\substack{1 \leq t_1 \leq \dots \leq t_i \\ t_1 + \dots + t_i = j}} |\mathrm{D}^{t_1}\Phi^{\mathrm{th}}_\eta| \cdots |\mathrm{D}^{t_i}\Phi^{\mathrm{th}}_\eta|.\end{aligned}$$

By property (iv) in Proposition 2.5.1, we conclude that, for $x \in (\mathscr{U}^m_\eta + Q^m_{\underline{\rho}\eta}) \setminus \mathscr{T}^{d^*}_\eta$,

$$|\mathrm{D}^j u_\eta(x)| \lesssim \sum_{i=1}^{j} \sum_{\substack{1 \le t_1 \le \cdots \le t_i \\ t_1 + \cdots + t_i = j}} \frac{\eta}{\operatorname{dist}(x, \mathscr{T}^{d^*})^{t_1}} \cdots \frac{\eta}{\operatorname{dist}(x, \mathscr{T}^{d^*})^{t_i}} \lesssim \frac{\eta^i}{\operatorname{dist}(x, \mathscr{T}^{d^*})^j}. \tag{2.5.9}$$

Combining (2.5.9) with the fact that, clearly, u_η is smooth outside $\mathscr{U}^m_\eta + Q^m_{\underline{\rho}\eta}$, we find that u_η belongs indeed to $\mathscr{R}_{m-\lfloor sp \rfloor -1}(\mathscr{K}^m_\eta; \mathscr{N})$.

With all these observations and tools at our disposal, we are finally ready to proceed with the proof of the density of the class $\mathscr{R}$. It only remains to explain carefully how to implement the aforementioned steps and to check that the estimates obtained at each step combine to yield $u_\eta \to u$ in $W^{s,p}$ as $\eta \to 0$.

2.6 Density of the class $\mathscr{R}$

This section is devoted to the proof of the density of the class $\mathscr{R}_{m-\lfloor sp \rfloor -1}(\mathscr{M}; \mathscr{N})$ in $W^{s,p}(\mathscr{M}; \mathscr{N})$. For the sake of clarity, we start by proving the result in the local situation, where the domain is topologically trivial, which corresponds to Theorem 1.4.6 stated in the introduction. More precisely, we prove the following result.

Theorem 2.6.1. *If $sp < m$, then $\mathscr{R}_{m-\lfloor sp \rfloor -1}(Q^m; \mathscr{N})$ is dense in $W^{s,p}(Q^m; \mathscr{N})$.*

The only difference with Theorem 1.4.6 in the introduction is that we have replaced the unit ball $\mathbb{B}^m$ by the unit cube Q^m, which will be more convenient from a technical point of view. In a second step, we explain how to deal with more general domains.

As we explained, the major part of the work that remains to be done is to suitably estimate the $W^{s,p}$ distance between the maps u_η and u. As the reader may have noticed, the Sobolev estimates obtained in Sections 2.3 to 2.5 deteriorate as $\eta \to 0$. For instance, the term involving the L^p norm of $\mathrm{D}^i u$ in the estimate of the j-order derivative blows up at rate η^{i-j}. As we shall see in the proof, this blow-up is compensated by the fact that the measure of the set $\mathscr{U}^m_\eta + Q^m_{2\rho\eta}$ decays sufficiently fast as $\eta \to 0$. For the integer order terms, this is exploited by a combination of the Hölder and Gagliardo–Nirenberg inequalities. The treatment of fractional order terms is more involved, and we bring ourselves back to the integer order setting with the help of the following lemma.

Lemma 2.6.2. *Let $\Omega \subset \mathbb{R}^m$ be a convex set and let $\omega \Subset \Omega$. For every q, $r \ge p$, and every $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$,*

$$|u|_{W^{\sigma,p}(\omega)} \le C|\omega|^{\frac{1}{p} - \frac{\sigma}{r} - \frac{1-\sigma}{q}} \|\mathrm{D}u\|^{\sigma}_{L^r(\Omega)} \|u\|^{1-\sigma}_{L^q(\omega)},$$

for some constant $C > 0$ depending only on m.

Proof. By density, we may assume that $u \in C^{\infty}(\Omega)$. We once again rely on an optimization technique. For every $\rho > 0$, we write

$$|u|^p_{W^{\sigma,p}(\omega)} \leq \int_{\omega}\int_{\omega\setminus B^m_\rho(x)} \frac{|u(x)-u(y)|^p}{|x-y|^{m+\sigma p}}\,\mathrm{d}y\,\mathrm{d}x + \int_{\omega}\int_{\omega\cap B^m_\rho(x)} \frac{|u(x)-u(y)|^p}{|x-y|^{m+\sigma p}}\,\mathrm{d}y\,\mathrm{d}x.$$

The first term is readily estimated as

$$\int_{\omega}\int_{\omega\setminus B^m_\rho(x)} \frac{|u(x)-u(y)|^p}{|x-y|^{m+\sigma p}}\,\mathrm{d}y\,\mathrm{d}x \lesssim \rho^{-\sigma p}\int_{\omega}|u|^p \lesssim \rho^{-\sigma p}|\omega|^{1-\frac{p}{q}}\biggl(\int_{\omega}|u|^q\biggr)^{\frac{p}{q}}.$$

For the second term, we start by using the mean value theorem along with Jensen's inequality to find

$$\int_{\omega}\int_{\omega\cap B^m_\rho(x)} \frac{|u(x)-u(y)|^p}{|x-y|^{m+\sigma p}}\,\mathrm{d}y\,\mathrm{d}x \leq \int_0^1\int_{\omega}\int_{\omega\cap B^m_\rho(x)} \frac{|\mathrm{D}u(x+t(y-x))|^p}{|x-y|^{m+(\sigma-1)p}}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t.$$

Here we use the convexity of Ω to ensure that $x + t(y-x) \in \Omega$ for every $x, y \in \omega$ and $t \in [0,1]$. We use the change of variable $h = y - x$ and Tonelli's theorem to deduce that

$$\int_{\omega}\int_{\omega\cap B^m_\rho(x)} \frac{|u(x)-u(y)|^p}{|x-y|^{m+\sigma p}}\,\mathrm{d}y\,\mathrm{d}x \leq \int_0^1\int_{(\omega-\omega)\cap B^m_\rho}\int_{\omega\cap(\omega-h)} \frac{|\mathrm{D}u(x+th)|^p}{|h|^{m+(\sigma-1)p}}\,\mathrm{d}x\,\mathrm{d}h\,\mathrm{d}t.$$

By convexity of Ω, if $x \in \omega\cap(\omega-h)$, we have $x+th \in \Omega$ for every $t \in [0,1]$. Moreover, the measure of the set $(\omega\cap(\omega-h))+th$ is less than $|\omega|$. Hence,

$$\begin{aligned}\int_{\omega}\int_{\omega\cap B^m_\rho(x)} &\frac{|u(x)-u(y)|^p}{|x-y|^{m+\sigma p}}\,\mathrm{d}y\,\mathrm{d}x \\ &\leq |\omega|^{1-\frac{p}{r}}\int_0^1\int_{(\omega-\omega)\cap B^m_\rho}\frac{1}{|h|^{m+(\sigma-1)p}}\biggl(\int_{\Omega}|\mathrm{D}u(z)|^r\,\mathrm{d}z\biggr)^{\frac{p}{r}}\,\mathrm{d}h\,\mathrm{d}t.\end{aligned}$$

We conclude that

$$\int_{\omega}\int_{\omega\cap B^m_\rho(x)} \frac{|u(x)-u(y)|^p}{|x-y|^{m+\sigma p}}\,\mathrm{d}y\,\mathrm{d}x \lesssim \rho^{(1-\sigma)p}|\omega|^{1-\frac{p}{r}}\biggl(\int_{\Omega}|\mathrm{D}u|^r\biggr)^{\frac{p}{r}}.$$

We may assume that $\mathrm{D}u$ does not vanish identically, otherwise there is nothing to prove. We insert

$$\rho = |\omega|^{\frac{1}{r}-\frac{1}{q}} \frac{\|u\|_{L^q(\omega)}}{\|Du\|_{L^p(\Omega)}},$$

and we find

$$|u|_{W^{\sigma,p}(\omega)} \lesssim |\omega|^{\frac{1}{p}-\frac{\sigma}{r}-\frac{1-\sigma}{q}} \|u\|_{L^q(\omega)}^{1-\sigma} \|Du\|_{L^r(\Omega)}^{\sigma}.$$

The proof of the lemma is complete. □

We finally prove Theorem 2.6.1. We note that, in Sections 2.3 to 2.5, no assumptions were required on the domain Ω. During the proof, we shall carefully indicate whenever restrictions on Ω are needed. Then, we shall explain how the proof should be modified to cover the case where the domain is not a cube, but a more general bounded open set $\Omega \subset \mathbb{R}^m$ which will lead to a counterpart of Theorem 2.6.1 for more general open subsets of $\mathbb{R}^m$. Finally, we will explain how to cover the case where the domain can more generally be a submanifold $\mathscr{M}$ of dimension m.

Proof of Theorem 2.6.1. Let $u \in W^{s,p}(Q^m; \mathscr{N})$. We note that, for every $\gamma > 0$, the map u_γ defined by $u_\gamma(x) = u\bigl(\frac{x}{1+2\gamma}\bigr)$ belongs to $W^{s,p}(Q^m_{1+2\gamma})$ and satisfies $u_\gamma \to u$ in $W^{s,p}(Q^m)$ as $\gamma \to 0$. Therefore, we may assume that $u \in W^{s,p}(Q^m_{1+2\gamma}; \mathscr{N})$. Here we used the fact that $\Omega = Q^m$, but we could work instead with any domain on which such a dilation argument may be implemented.

Let $0 < \eta < \gamma$ and $0 < \rho < \frac{1}{2}$, so that $2\rho\eta < \gamma$. Guided by the observations at the end of Sections 2.4 and 2.5, we define the following families of cubes. We let K^m_η be a cubication of $Q^m_{1+\gamma}$, that is, $\mathscr{K}^m_\eta = Q^m_{1+\gamma}$. This uses that $Q^m_{1+\gamma}$ is a cube, but the important fact is that $Q^m \subset \mathscr{K}^m_\eta \subset Q^m_{1+\gamma}$. Then, following (2.5.8), we construct the set of bad cubes E^m_η as the family of all cubes $\sigma^m \in K^m_\eta$ such that

$$\|Du\|_{L^{sp}(\sigma^m + Q^m_{2\rho\eta})} \geq \frac{\eta^{\frac{m}{sp}-1}}{C}\iota \quad \text{if } s \geq 1, \tag{2.6.1}$$

respectively

$$|u|_{W^{s,p}(\sigma^m + Q^m_{2\rho\eta})} \geq \frac{\eta^{\frac{m}{p}-s}}{C}\iota \quad \text{if } 0 < s < 1, \tag{2.6.2}$$

where $\iota > 0$ is the radius of a tubular neighborhood of $\mathscr{N}$. We also define U^m_η to be the set of all cubes in K^m_η intersecting a cube in E^m_η. Doing so, we indeed have $\mathscr{E}^m_\eta \subset \operatorname{int} \mathscr{U}^m_\eta$ in the relative topology of $\mathscr{K}^m_\eta$.

We apply opening to the map u choosing $d = \lfloor sp \rfloor$. Let $\Phi_\eta^{\mathrm{op}} \colon \mathbb{R}^m \to \mathbb{R}^m$ be the smooth map provided by Proposition 2.3.1 applied to u with $\Omega = Q^m_{1+\gamma}$, and define

$$u_\eta^{\mathrm{op}} = u \circ \Phi_\eta^{\mathrm{op}}.$$

Hence, we find that

(a) if $0 < s < 1$, then

$$\eta^s |u_\eta^{\mathrm{op}} - u|_{W^{s,p}(Q^m_{1+\gamma})} \lesssim \eta^s |u|_{W^{s,p}(\mathcal{U}^d_\eta + Q^m_{2\rho\eta})} + \|u\|_{L^p(\mathcal{U}^d_\eta + Q^m_{2\rho\eta})};$$

(b) if $s \geq 1$, then for every $j \in \{1, \dots, k\}$,

$$\eta^j \|D^j u_\eta^{\mathrm{op}} - D^j u\|_{L^p(Q^m_{1+\gamma})} \lesssim \sum_{i=1}^{j} \eta^i \|D^i u\|_{L^p(\mathcal{U}^d_\eta + Q^m_{2\rho\eta})};$$

(c) if $s \geq 1$ and $\sigma \neq 0$, then for every $j \in \{1, \dots, k\}$,

$$\eta^{j+\sigma} |D^j u_\eta^{\mathrm{op}} - D^j u|_{W^{\sigma,p}(Q^m_{1+\gamma})} \lesssim \sum_{i=1}^{j} \Big(\eta^i \|D^i u\|_{L^p(\mathcal{U}^d_\eta + Q^m_{2\rho\eta})} + \eta^{i+\sigma} |D^i u|_{W^{\sigma,p}(\mathcal{U}^d_\eta + Q^m_{2\rho\eta})} \Big);$$

(d) for every $0 < s < +\infty$,

$$\|u_\eta^{\mathrm{op}} - u\|_{L^p(Q^m_{1+\gamma})} \lesssim \|u\|_{L^p(\mathcal{U}^d_\eta + Q^m_{2\rho\eta})}.$$

Then, we apply adaptive smoothing to the map u_η^{op} with $\Omega = Q^m_{1+2\gamma}$. Let $\varphi \in \mathbb{B}^m$ be a fixed mollifier. Since $\mathscr{E}^m_\eta \subset \operatorname{int} \mathcal{U}^m_\eta$, we may define ψ_η as at the end of Section 2.4. Namely, we let

$$\psi_\eta = t\zeta_\eta + r(1 - \zeta_\eta),$$

where ζ_η satisfies assumptions (a) to (d) page 84 and $0 < r < t$ with t defined by (2.4.14). With this choice, ψ_η satisfies the assumptions of Proposition 2.4.1, and moreover $0 < \psi_\eta \leq \rho\eta$. This implies that $Q^m_{1+\gamma} \subset \{x \in Q^m_{1+2\gamma} \colon \operatorname{dist}(x, \partial Q^m_{1+2\gamma}) \geq \psi(x)\}$, and hence $u_\eta^{\mathrm{sm}} = \varphi_{\psi_\eta} * u_\eta^{\mathrm{op}}$ is well-defined and smooth on $Q^m_{1+\gamma}$. Moreover, Proposition 2.4.1 and equation (2.4.2) for the zero order case applied with $\omega = Q^m_{1+\gamma}$ ensure that

(a) if $0 < s < 1$, then

$$|u_\eta^{\mathrm{sm}} - u_\eta^{\mathrm{op}}|_{W^{s,p}(Q^m_{1+\gamma})} \leq \sup_{v \in \mathbb{B}^m} |\tau_{\psi_\eta v}(u_\eta^{\mathrm{op}}) - u_\eta^{\mathrm{op}}|_{W^{s,p}(Q^m_{1+\gamma})};$$

(b) if $s \geq 1$, then for every $j \in \{1, \dots, k\}$,

$$\eta^j \|\mathrm{D}^j u_\eta^{\mathrm{sm}} - \mathrm{D}^j u_\eta^{\mathrm{op}}\|_{L^p(Q^m_{1+\gamma})} \lesssim \sup_{v \in \mathbb{B}^m} \eta^j \|\tau_{\psi_\eta v}(\mathrm{D}^j u_\eta^{\mathrm{op}}) - \mathrm{D}^j u_\eta^{\mathrm{op}}\|_{L^p(Q^m_{1+\gamma})} + \sum_{i=1}^{j} \eta^i \|\mathrm{D}^i u_\eta^{\mathrm{op}}\|_{L^p(A)};$$

(c) if $s \geq 1$ and $\sigma \neq 0$, then for every $j \in \{1, \dots, k\}$,

$$\eta^{j+\sigma} |\mathrm{D}^j u_\eta^{\mathrm{sm}} - \mathrm{D}^j u_\eta^{\mathrm{op}}|_{W^{\sigma,p}(Q^m_{1+\gamma})} \lesssim \sup_{v \in \mathbb{B}^m} \eta^{j+\sigma} |\tau_{\psi_\eta v}(\mathrm{D}^j u_\eta^{\mathrm{op}}) - \mathrm{D}^j u_\eta^{\mathrm{op}}|_{W^{\sigma,p}(Q^m_{1+\gamma})} + \sum_{i=1}^{j} \Big(\eta^i \|\mathrm{D}^i u_\eta^{\mathrm{op}}\|_{L^p(A)} + \eta^{i+\sigma} |\mathrm{D}^i u_\eta^{\mathrm{op}}|_{W^{\sigma,p}(A)} \Big);$$

(d) for every $0 < s < +\infty$,

$$\|u_\eta^{\mathrm{sm}} - u_\eta^{\mathrm{op}}\|_{L^p(Q^m_{1+\gamma})} \leq \sup_{v \in \mathbb{B}^m} \|\tau_{\psi_\eta v}(u_\eta^{\mathrm{op}}) - u_\eta^{\mathrm{op}}\|_{L^p(Q^m_{1+\gamma})}.$$

Here,

$$A = \bigcup_{x \in Q^m_{1+\gamma} \cap \operatorname{supp} \mathrm{D}\psi_\eta} B^m_{\psi_\eta(x)}(x).$$

By the triangle inequality, for every $v \in \mathbb{B}^m$, we have

$$\begin{aligned} &\|\tau_{\psi_\eta v}(\mathrm{D}^j u_\eta^{\mathrm{op}}) - \mathrm{D}^j u_\eta^{\mathrm{op}}\|_{L^p(Q^m_{1+\gamma})} \\ &\quad \leq \|\tau_{\psi_\eta v}(\mathrm{D}^j u_\eta^{\mathrm{op}}) - \tau_{\psi_\eta v}(\mathrm{D}^j u)\|_{L^p(Q^m_{1+\gamma})} \\ &\qquad + \|\tau_{\psi_\eta v}(\mathrm{D}^j u) - \mathrm{D}^j u\|_{L^p(Q^m_{1+\gamma})} + \|\mathrm{D}^j u - \mathrm{D}^j u_\eta^{\mathrm{op}}\|_{L^p(Q^m_{1+\gamma})}. \end{aligned}$$

By the change of variable theorem, we find

$$\|\tau_{\psi_\eta v}(\mathrm{D}^j u_\eta^{\mathrm{op}}) - \tau_{\psi_\eta v}(\mathrm{D}^j u)\|_{L^p(Q^m_{1+\gamma})} \lesssim \|\mathrm{D}^j u_\eta^{\mathrm{op}} - \mathrm{D}^j u\|_{L^p(Q^m_{1+2\gamma})}.$$

A similar estimate holds for the Gagliardo seminorm. Furthermore, observing that $\operatorname{supp} \mathrm{D}\psi_\eta \subset \mathscr{U}^m_\eta$ and using that $\psi_\eta \leq \rho\eta$, we have $A \subset \mathscr{U}^m_\eta + Q^m_{\rho\eta}$. Combining this with estimate (iii) in Proposition 2.3.1 applied with $\omega = \mathscr{U}^m_\eta + Q^m_{\rho\eta}$, we deduce that

(a) if $0 < s < 1$, then

$$\eta^s |u^{\mathrm{sm}}_\eta - u^{\mathrm{op}}_\eta|_{W^{s,p}(Q^m_{1+\gamma})} \lesssim \sup_{v \in \mathbb{B}^m} \eta^s |\tau_{\psi_\eta v}(u) - u|_{W^{s,p}(Q^m_{1+\gamma})} + \eta^s |u|_{W^{s,p}(\mathscr{U}^m_\eta + Q^m_{2\rho\eta})} + \|u\|_{L^p(\mathscr{U}^m_\eta + Q^m_{2\rho\eta})};$$

(b) if $s \geq 1$, then for every $j \in \{1, \dots, k\}$,

$$\eta^j \|\mathrm{D}^j u^{\mathrm{sm}}_\eta - \mathrm{D}^j u^{\mathrm{op}}_\eta\|_{L^p(Q^m_{1+\gamma})} \lesssim \sup_{v \in \mathbb{B}^m} \eta^j \|\tau_{\psi_\eta v}(\mathrm{D}^j u) - \mathrm{D}^j u\|_{L^p(Q^m_{1+\gamma})} + \sum_{i=1}^{j} \eta^i \|\mathrm{D}^i u\|_{L^p(\mathscr{U}^m_\eta + Q^m_{2\rho\eta})};$$

(c) if $s \geq 1$ and $\sigma \neq 0$, then for every $j \in \{1, \dots, k\}$,

$$\eta^{j+\sigma} |\mathrm{D}^j u^{\mathrm{sm}}_\eta - \mathrm{D}^j u^{\mathrm{op}}_\eta|_{W^{\sigma,p}(Q^m_{1+\gamma})} \lesssim \sup_{v \in \mathbb{B}^m} \eta^{j+\sigma} |\tau_{\psi_\eta v}(\mathrm{D}^j u) - \mathrm{D}^j u|_{W^{\sigma,p}(Q^m_{1+\gamma})} + \sum_{i=1}^{j} \Big(\eta^i \|\mathrm{D}^i u\|_{L^p(\mathscr{U}^m_\eta + Q^m_{2\rho\eta})} + \eta^{i+\sigma} |\mathrm{D}^i u|_{W^{\sigma,p}(\mathscr{U}^m_\eta + Q^m_{2\rho\eta})} \Big);$$

(d) for every $0 < s < +\infty$,

$$\|u^{\mathrm{sm}}_\eta - u^{\mathrm{op}}_\eta\|_{L^p(Q^m_{1+\gamma})} \lesssim \sup_{v \in \mathbb{B}^m} \|\tau_{\psi_\eta v}(u) - u\|_{L^p(Q^m_{1+\gamma})} + \|u\|_{L^p(\mathscr{U}^m_\eta + Q^m_{2\rho\eta})}.$$

Finally, we apply thickening to the map u^{sm}_η. Choose $0 < \underline{\rho} < \rho$, let $\Phi^{\mathrm{th}}_\eta \colon \mathbb{R}^m \setminus \mathscr{T}^{d^*}_\eta \to \mathbb{R}^m$ be the smooth map given by Proposition 2.5.1 applied with parameter $\underline{\rho}$ and with $\Omega = Q^m_{1+\gamma}$, where we recall that $T^{d^*}_\eta$ is the skeleton dual to U^m_η, and we set

$$u^{\mathrm{th}}_\eta = u^{\mathrm{sm}}_\eta \circ \Phi^{\mathrm{th}}_\eta.$$

This map coincides with u^{sm}_η outside of $\mathscr{U}^m_\eta + Q^m_{\underline{\rho}\eta}$. Since $d + 1 > sp$, Proposition 2.5.1 ensures that $u^{\mathrm{th}}_\eta \in W^{s,p}(Q^m_{1+\gamma}; \mathbb{R}^\nu)$, and moreover, the following estimates hold:

(a) if $0 < s < 1$, then

$$\eta^s |u^{\mathrm{th}}_\eta - u^{\mathrm{sm}}_\eta|_{W^{s,p}(Q^m_{1+\gamma})} \lesssim \eta^s |u^{\mathrm{sm}}_\eta|_{W^{s,p}(\mathscr{U}^m_\eta + Q^m_{\underline{\rho}\eta})} + \|u^{\mathrm{sm}}_\eta\|_{L^p(\mathscr{U}^m_\eta + Q^m_{\underline{\rho}\eta})};$$

(b) if $s \geq 1$, then for every $j \in \{1, \dots, k\}$,

$$\eta^j \|\mathrm{D}^j u_\eta^{\mathrm{th}} - \mathrm{D}^j u_\eta^{\mathrm{sm}}\|_{L^p(Q_{1+\gamma}^m)} \lesssim \sum_{i=1}^{j} \eta^i \|\mathrm{D}^i u_\eta^{\mathrm{sm}}\|_{L^p(\mathscr{U}_\eta^m + Q_{\underline{\rho}\eta}^m)};$$

(c) if $s \geq 1$ and $\sigma \neq 0$, then for every $j \in \{1, \dots, k\}$,

$$\eta^{j+\sigma} |\mathrm{D}^j u_\eta^{\mathrm{th}} - \mathrm{D}^j u_\eta^{\mathrm{sm}}|_{W^{\sigma,p}(Q_{1+\gamma}^m)} \lesssim \sum_{i=1}^{j} \Big(\eta^i \|\mathrm{D}^i u_\eta^{\mathrm{sm}}\|_{L^p(\mathscr{U}_\eta^m + Q_{\underline{\rho}\eta}^m)} + \eta^{i+\sigma} |\mathrm{D}^i u_\eta^{\mathrm{sm}}|_{W^{\sigma,p}(\mathscr{U}_\eta^m + Q_{\underline{\rho}\eta}^m)}\Big);$$

(d) for every $0 < s < +\infty$,

$$\|u_\eta^{\mathrm{th}} - u_\eta^{\mathrm{sm}}\|_{L^p(Q_{1+\gamma}^m)} \lesssim \|u_\eta^{\mathrm{sm}}\|_{L^p(\mathscr{U}_\eta^m + Q_{\underline{\rho}\eta}^m)}.$$

Hence, invoking estimate (i) in Proposition 2.4.1 with $\Omega = \mathscr{U}_\eta^m + Q_{(\rho+\underline{\rho})\eta}^m$ and $\omega = U_\eta^m + Q_{\underline{\rho}\eta}^m$, and then estimate (iii) in Proposition 2.3.1 with $\omega = U_\eta^m + Q_{2\rho\eta}^m$, we obtain

(a) if $0 < s < 1$, then

$$\eta^s |u_\eta^{\mathrm{th}} - u_\eta^{\mathrm{sm}}|_{W^{s,p}(Q_{1+\gamma}^m)} \lesssim \eta^s |u|_{W^{s,p}(\mathscr{U}_\eta^m + Q_{2\rho\eta}^m)} + \|u\|_{L^p(\mathscr{U}_\eta^m + Q_{2\rho\eta}^m)};$$

(b) if $s \geq 1$, then for every $j \in \{1, \dots, k\}$,

$$\eta^j \|\mathrm{D}^j u_\eta^{\mathrm{th}} - \mathrm{D}^j u_\eta^{\mathrm{sm}}\|_{L^p(Q_{1+\gamma}^m)} \lesssim \sum_{i=1}^{j} \eta^i \|\mathrm{D}^i u\|_{L^p(\mathscr{U}_\eta^m + Q_{2\rho\eta}^m)};$$

(c) if $s \geq 1$ and $\sigma \neq 0$, then for every $j \in \{1, \dots, k\}$,

$$\eta^{j+\sigma} |\mathrm{D}^j u_\eta^{\mathrm{th}} - \mathrm{D}^j u_\eta^{\mathrm{sm}}|_{W^{\sigma,p}(Q_{1+\gamma}^m)} \lesssim \sum_{i=1}^{j} \Big(\eta^i \|\mathrm{D}^i u\|_{L^p(\mathscr{U}_\eta^m + Q_{2\rho\eta}^m)} + \eta^{i+\sigma} |\mathrm{D}^i u|_{W^{\sigma,p}(\mathscr{U}_\eta^m + Q_{2\rho\eta}^m)}\Big);$$

(d) for every $0 < s < +\infty$,

$$\|u_\eta^{\mathrm{th}} - u_\eta^{\mathrm{sm}}\|_{L^p(Q_{1+\gamma}^m)} \lesssim \|u\|_{L^p(\mathscr{U}_\eta^m + Q_{2\rho\eta}^m)}.$$

Using the triangle inequality, we conclude that, letting $\mathscr{A}_\mu = \mathscr{U}_\eta^m + Q_{2\rho\eta}^m$,

(a) if $0 < s < 1$, then

$$\eta^s |u_\eta^{\mathrm{th}} - u_\eta|_{W^{s,p}(Q^m_{1+\gamma})} \lesssim \sup_{v \in \mathbb{B}^m} \eta^s |\tau_{\psi_\eta v}(u) - u|_{W^{s,p}(Q^m_{1+\gamma})} + \eta^s |u|_{W^{s,p}(\mathscr{A}_\mu)} + \|u\|_{L^p(\mathscr{A}_\mu)};$$

(b) if $s \geq 1$, then for every $j \in \{1, \dots, k\}$,

$$\eta^j \|\mathrm{D}^j u_\eta^{\mathrm{th}} - \mathrm{D}^j u\|_{L^p(Q^m_{1+\gamma})} \lesssim \sup_{v \in \mathbb{B}^m} \eta^j \|\tau_{\psi_\eta v}(\mathrm{D}^j u) - \mathrm{D}^j u\|_{L^p(Q^m_{1+\gamma})} + \sum_{i=1}^{j} \eta^i \|\mathrm{D}^i u\|_{L^p(\mathscr{A}_\mu)};$$

(c) if $s \geq 1$ and $\sigma \neq 0$, then for every $j \in \{1, \dots, k\}$,

$$\begin{aligned}\eta^{j+\sigma} |\mathrm{D}^j u_\eta^{\mathrm{th}} - \mathrm{D}^j u|_{W^{\sigma,p}(Q^m_{1+\gamma})} &\lesssim \sup_{v \in \mathbb{B}^m} \eta^{j+\sigma} |\tau_{\psi_\eta v}(\mathrm{D}^j u) - \mathrm{D}^j u|_{W^{\sigma,p}(Q^m_{1+\gamma})} \\ &\quad + \sum_{i=1}^{j} \Bigl(\eta^i \|\mathrm{D}^i u\|_{L^p(\mathscr{A}_\mu)} + \eta^{i+\sigma} |\mathrm{D}^i u|_{W^{\sigma,p}(\mathscr{A}_\mu)} \Bigr);\end{aligned}$$

(d) for every $0 < s < +\infty$,

$$\|u_\eta^{\mathrm{th}} - u\|_{L^p(Q^m_{1+\gamma})} \lesssim \sup_{v \in \mathbb{B}^m} \|\tau_{\psi_\eta v}(u) - u\|_{L^p(Q^m_{1+\gamma})} + \|u\|_{L^p(\mathscr{A}_\mu)}.$$

Due to our choice of ψ_η, and since $d \leq sp$ and

$$Q^m \subset K^m_\eta \subset Q^m_{1+\gamma} \subset \{x \in Q^m_{1+2\gamma} \colon \operatorname{dist}(x, \partial Q^m_{1+2\gamma}) \geq \psi_\eta(x)\},$$

according to estimates (2.5.6) and (2.5.7), we have

$$\begin{aligned}\operatorname{Dist}_{\mathscr{N}}(u_\eta^{\mathrm{th}}(\mathscr{K}^m_\eta \setminus \mathscr{T}^{d^*}_\eta)) \leq \max\biggl\{ &\max_{\sigma^m \in K^m_\eta \setminus E^m_\eta} C \frac{1}{\eta^{\frac{m}{sp}-1}} \|\mathrm{D}u\|_{L^{sp}(\sigma^m + Q^m_{2\rho\eta})}, \\ &\sup_{x \in \mathscr{U}^\ell_\eta + Q^m_{\underline{\rho}\eta}} C' \fint_{Q^m_r(x)} \fint_{Q^m_r(x)} |u^{\mathrm{op}}_\eta(y) - u^{\mathrm{op}}_\eta(z)| \,\mathrm{d}y \,\mathrm{d}z \biggr\} \qquad (2.6.3)\end{aligned}$$

if $s \geq 1$, respectively

$$\operatorname{Dist}_{\mathscr{N}}(u_\eta^{\mathrm{th}}(\mathscr{K}_\eta^m \setminus \mathscr{T}_\eta^{d^*})) \leq \max\Big\{\max_{\sigma^m \in K_\eta^m \setminus E_\eta^m} C \frac{1}{\eta^{\frac{m}{p}-s}} |u|_{W^{s,p}(\sigma^m + Q_{2\rho\eta}^m)},$$
$$\sup_{x \in \mathscr{U}_\eta^d + Q_{\underline{\rho}\eta}^m} C' \fint_{Q_r^m(x)} \fint_{Q_r^m(x)} |u_\eta^{\mathrm{op}}(y) - u_\eta^{\mathrm{op}}(z)| \,\mathrm{d}y\,\mathrm{d}z\Big\} \quad (2.6.4)$$

if $0 < s < 1$. In (2.6.3) and (2.6.4), we recall that $r > 0$ is a number used in the definition of ψ_η, and chosen sufficiently small to ensure the validity of (2.4.7) or (2.4.8). We note here that, in defining the bad cubes in equation (2.6.1), respectively (2.6.2), we take the constant $C > 0$ which shows up in estimate (2.6.3), respectively (2.6.4). Doing so, by definition of the set of bad cubes E_η^m, the first term in each max is smaller than the radius ι of a tubular neighborhood of $\mathscr{N}$. Moreover, since $d \leq sp$, Proposition 2.3.8 ensures that we may take $r > 0$ so small that the second term in each max is also smaller than ι. Therefore, we deduce that

$$\operatorname{Dist}_{\mathscr{N}}(u_\eta^{\mathrm{th}}(\mathscr{K}_\eta^m \setminus \mathscr{T}_\eta^{d^*})) \leq \iota.$$

This enables us to define $u_\eta = \Pi \circ u_\eta^{\mathrm{th}}$, which, as we already explained at the end of Section 2.5, is smooth on $\mathscr{K}_\eta^m \setminus \mathscr{T}^{d^*}$, and belongs to $\mathscr{R}_{m-\lfloor sp \rfloor -1}(\mathscr{K}_\eta^m; \mathscr{N})$.

Since $Q^m \subset \mathscr{K}_\eta^m$, to conclude, it only remains to prove that $u_\eta \to u$ in $W^{s,p}(\mathscr{K}_\eta^m)$ as $\eta \to 0$. We claim that it suffices to show that $u_\eta^{\mathrm{th}} \to u$ in $W^{s,p}(\mathscr{K}_\eta^m)$ as $\eta \to 0$. Indeed, the map Π is smooth and has uniformly bounded derivatives, and $\mathscr{N}$ is compact. Hence, the continuity of the composition operator from $W^{s,p} \cap L^\infty$ to $W^{s,p}$ — see for instance [BM21, Chapter 15.3] — ensures that, if $u_\eta^{\mathrm{th}} \to u$ in $W^{s,p}(\mathscr{K}_\eta^m)$, then $u_\eta = \Pi \circ u_\eta^{\mathrm{th}}$ converges in $W^{s,p}(\mathscr{K}_\eta^m)$ to $\Pi \circ u = u$.

We now prove that $u_\eta^{\mathrm{th}} \to u$. We start by noting that the continuity of the translation operator implies that

$$\lim_{\eta \to 0} \sup_{v \in \mathbb{B}^m} \|\tau_{\psi_\eta v}(\mathrm{D}^j u) - \mathrm{D}^j u\|_{L^p(Q_{1+\gamma}^m)} = 0$$

and

$$\lim_{\eta \to 0} \sup_{v \in \mathbb{B}^m} |\tau_{\psi_\eta v}(\mathrm{D}^j u) - \mathrm{D}^j u|_{W^{\sigma,p}(Q_{1+\gamma}^m)} = 0$$

for every $j \in \{0, \ldots, k\}$.

We first deal with the case $s \geq 1$. By the Gagliardo–Nirenberg interpolation inequality — see for instance [BM01, BM18] — for every $i \in \{1, \ldots, k\}$, we have $\mathrm{D}^i u \in L^{\frac{sp}{i}}(Q_{1+2\gamma}^m)$.

Hölder's inequality ensures that

$$\|D^i u\|_{L^p(\mathscr{A}_\mu)} \le |\mathscr{A}_\mu|^{\frac{s-i}{sp}} \|D^i u\|_{L^{sp/i}(\mathscr{A}_\mu)},$$

while Lemma 2.6.2 guarantees that

$$|D^i u|_{W^{\sigma,p}(\mathscr{A}_\mu)} \lesssim |\mathscr{A}_\mu|^{\frac{s-i-\sigma}{sp}} \|D^{i+1} u\|^{\sigma}_{L^{sp/(i+1)}(Q^m_{1+2\gamma})} \|D^i u\|^{1-\sigma}_{L^{sp/i}(\mathscr{A}_\mu)}.$$

Here, we use the fact that $Q^m_{1+2\gamma}$ is convex to justify the use of Lemma 2.6.2.

We now wish to estimate the measure of the set $\mathscr{A}_\mu$. We first note that

$$|\mathscr{A}_\mu| \lesssim \operatorname{card}(U^m_\eta)\eta^m. \tag{2.6.5}$$

Then, for every $\sigma^m \in U^m_\eta$, there exists $\tau^m \in E^m_\eta$ which intersects σ^m, and thus $\tau^m + Q^m_{2\rho\eta} \subset \sigma^m + Q^m_{2(1+\rho)\eta}$. If we write $\sigma^m = Q^m_\eta(a)$, we find $\tau^m + Q^m_{2\rho\eta} \subset Q^m_{\alpha\eta}(a))$ with $\alpha = 3 + 2\rho$. Hence,

$$\tau^m + Q^m_{2\rho\eta} \subset Q^m_{\alpha\eta}(a) \cap Q^m_{1+2\gamma}.$$

We deduce from the definition of E^m_η that

$$\iota < C\frac{1}{\eta^{\frac{m}{sp}-1}}\|Du\|_{L^{sp}(\tau^m+Q^m_{2\rho\eta})} \le C\frac{1}{\eta^{\frac{m}{sp}-1}}\|Du\|_{L^{sp}(Q^m_{\alpha\eta}(a)\cap Q^m_{1+2\gamma})}.$$

Since the number of overlaps between one of the cubes $Q^m_{\alpha\eta}(a)$ and all the other ones is bounded from above by a number depending only on m, summing over all cubes in U^m_η and using the additivity of the integral, we deduce that

$$\begin{aligned}\operatorname{card}(U^m_\eta) &\lesssim \frac{1}{\eta^{m-sp}} \sum_{Q^m_\eta(a)\in U^m_\eta} \int_{Q^m_{\alpha\eta}(a)\cap Q^m_{1+2\gamma}} |Du|^{sp} \\ &\lesssim \frac{1}{\eta^{m-sp}} \int_{(\mathscr{U}^m_\eta + Q^m_{2(1+\rho)\eta})\cap Q^m_{1+2\gamma}} |Du|^{sp}.\end{aligned} \tag{2.6.6}$$

For further use, we already note that, in the case $0 < s < 1$, the exact same reasoning leads to

$$\iota < C\frac{1}{\eta^{\frac{m}{p}-s}}|u|_{W^{s,p}(Q^m_{\alpha\eta}(a)\cap Q^m_{1+2\gamma})}.$$

As for the case $s \ge 1$, replacing the additivity of the integral by the superadditivity of the Gagliardo seminorm, we obtain

$$\operatorname{card}(U^m_\eta) \lesssim \frac{1}{\eta^{m-sp}} \int_{(\mathscr{U}^m_\eta + Q^m_{2(1+\rho)\eta}) \cap Q^m_{1+2\gamma}} \int_{(\mathscr{U}^m_\eta + Q^m_{2(1+\rho)\eta}) \cap Q^m_{1+2\gamma}} \frac{|u(x)-u(y)|^p}{|x-y|^{m+sp}} \,\mathrm{d}x \,\mathrm{d}y. \tag{2.6.7}$$

In both cases $s \geq 1$ and $0 < s < 1$, we conclude that

$$\lim_{\eta \to 0} \frac{|\mathscr{A}_\mu|}{\eta^{sp}} = 0. \tag{2.6.8}$$

Indeed, we first use estimate (2.6.5) along with (2.6.6), respectively (2.6.7), to deduce that

$$\frac{|\mathscr{A}_\mu|}{\eta^{sp}} \lesssim \|\mathrm{D}u\|^p_{L^p(Q^m_{1+2\gamma})}, \quad \text{respectively} \quad \frac{|\mathscr{A}_\mu|}{\eta^{sp}} \lesssim |u|^p_{W^{s,p}(Q^m_{1+2\gamma})}.$$

In particular, $|U^m_\eta + Q^m_{2\rho\eta}| \to 0$. Using this information along with Lebesgue's lemma, we invoke again estimate (2.6.6), respectively (2.6.7), to deduce (2.6.8).

We next proceed as follows. When $s \geq 1$, we find

$$\sum_{i=1}^{j} \eta^{i-j} \|\mathrm{D}^i u\|_{L^p(\mathscr{A}_\mu)} \leq \sum_{i=1}^{j} \eta^{s-j} \biggl(\frac{|\mathscr{A}_\mu|}{\eta^{sp}}\biggr)^{\frac{s-i}{sp}} \|\mathrm{D}^i u\|_{L^{sp/i}(\mathscr{A}_\mu)} \to 0 \quad \text{as } \eta \to 0 \tag{2.6.9}$$

and

$$\begin{aligned}
&\sum_{i=1}^{j} \Bigl(\eta^i \|\mathrm{D}^i u\|_{L^p(\mathscr{A}_\mu)} + \eta^{i+\sigma} |\mathrm{D}^i u|_{W^{\sigma,p}(\mathscr{A}_\mu)}\Bigr) \\
&\quad \lesssim \sum_{i=1}^{j} \eta^{s-j-\sigma} \biggl(\frac{|\mathscr{A}_\mu|}{\eta^{sp}}\biggr)^{\frac{s-i}{sp}} \|\mathrm{D}^i u\|_{L^{sp/i}(\mathscr{A}_\mu)} \\
&\qquad + \sum_{i=1}^{j-1} \eta^{s-j-\sigma} \biggl(\frac{|\mathscr{A}_\mu|}{\eta^{sp}}\biggr)^{\frac{s-i-\sigma}{sp}} \|\mathrm{D}^i u\|^{1-\sigma}_{L^{sp/i}(\mathscr{A}_\mu)} \|\mathrm{D}^{i+1} u\|^{\sigma}_{L^{sp/(i+1)}(Q^m_{1+2\gamma})} \\
&\qquad + |\mathrm{D}^j u|_{W^{\sigma,p}(\mathscr{A}_\mu)} \to 0 \quad \text{as } \eta \to 0.
\end{aligned} \tag{2.6.10}$$

For the last term in (2.6.10), we use (2.6.8) and the Lebesgue lemma. Similarly, we have

$$\|u\|_{L^p(U^m_\eta + Q^m_{2\rho\eta})} \to 0 \quad \text{as } \eta \to 0.$$

This completes the proof that $u^{\mathrm{th}}_\eta \to u$ in $W^{s,p}(Q^m_{1+2\gamma})$ when $s \geq 1$.

The case $0 < s < 1$ is concluded analogously. We note that since $\mathscr{N}$ is compact, we have $u \in L^\infty$. Therefore, we have

$$|u|_{W^{s,p}(\mathscr{A}_\mu)} + \eta^{-s}\|u\|_{L^p(\mathscr{A}_\mu)} \lesssim |u|_{W^{s,p}(\mathscr{A}_\mu)} + \eta^{-s}|\mathscr{A}_\mu|^{\frac{1}{p}}$$
$$\lesssim |u|_{W^{s,p}(\mathscr{A}_\mu)} + \left(\frac{|\mathscr{A}_\mu|}{\eta^{sp}}\right)^{\frac{1}{p}} \to 0 \quad \text{as } \eta \to 0.$$

Combining this with the fact that $\|u\|_{L^p(\mathscr{A}_\mu)} \to 0$ as $\eta \to 0$, we deduce that $u^{\mathrm{th}}_\eta \to u$ in $W^{s,p}(Q^m_{1+\gamma})$ as $\eta \to 0$ when $0 < s < 1$. This completes the proof of Theorem 2.6.1. □

We now explain how to deal with more general domains. The first step is to be able to implement the dilation procedure used at the beginning of the proof. The method we used adapts without any modification to domains that are starshaped with respect to one of their points. However, using a more involved technique, it is possible to work with even more general domains. The reader may consult [BM21, Lemma 15.25] for an implementation of this technique on smooth domains using the normal vector, or [BPVS25] for an argument on continuous bounded domains using local parametrizations. Here we show that the approach even works under the weaker *segment condition*.

We recall that Ω satisfies the *segment condition* whenever, for every $x \in \partial\Omega$, there exists an open set $U_x \subset \mathbb{R}^m$ containing x and a nonzero vector $z_x \in \mathbb{R}^m$ such that, if $y \in U_x \cap \overline{\Omega}$, then $y + tz_x \in \Omega$ for every $0 < t < 1$.

Lemma 2.6.3. *Let $\Omega \subset \mathbb{R}^m$ be a bounded open domain satisfying the segment condition. For every $\gamma > 0$ sufficiently small, there exists a smooth diffeomorphism $\Phi_\gamma \colon \mathbb{R}^m \to \mathbb{R}^m$ such that $\Phi_\gamma(\overline{\Omega}) \subset \Omega$ and*

$$\mathrm{D}^j\Phi_\gamma \to \mathrm{id} \quad \textit{uniformly on } \mathbb{R}^m \textit{ for every } j \in \mathbb{N} \textit{ as } \gamma \to 0.$$

Geometrically, the segment condition means that Ω cannot lie on both sides of $\partial\Omega$. A typical example of a domain Ω not satisfying this assumption is given by two open cubes whose boundaries share a common face. It is known — see for instance [Ada75, 3.17] — that there exists a $W^{1,p}$ map on this domain which cannot be approximated by $C^\infty(\overline{\Omega})$ maps, even in the real valued case.

Proof of Lemma 2.6.3. Let $B_\delta = B^{m-1}_\delta \times (-\delta, \delta)$ be a cylinder of radius and half-height δ. Since $\partial\Omega$ is compact, there exists a finite number of points $x_1, \dots, x_n \in \mathbb{R}^m$ and associated isometries $T_1, \dots, T_n$ of $\mathbb{R}^m$ mapping 0 to x_i such that

$$\partial\Omega \subset \bigcup_{i=1}^{n} T_i(B_{\delta/2}), \tag{2.6.11}$$

and also associated nonzero vectors $z_1, \dots, z_n \in \mathbb{R}^m$ such that, if $y \in T_i(B_\delta) \cap \overline{\Omega}$, then $y + tz_i \in \Omega$ for every $0 < t < 1$. Let $\psi\colon \mathbb{R}^{m-1} \to [0,1]$ be a smooth map such that $\psi(x) = 1$ if $x \in B_{\delta/2}$ and $\psi(x) = 0$ if $x \in \mathbb{R}^{m-1} \setminus B_{3\delta/4}$. For $0 < \gamma < 1$, we define $\Phi_{i,\gamma}\colon \mathbb{R}^m \to \mathbb{R}^m$ by

$$\Phi_{i,\gamma}(x) = x + \gamma\psi(T_i^{-1}(x))z_i.$$

If $\gamma < (\|D\psi\|_{L^\infty}|z_i|)^{-1}$, we observe that $\Phi_{i,\gamma}$ is a smooth diffeomorphism. Moreover, by construction of the vectors z_i, we have $\Phi_{i,\gamma}(\Omega) \subset \Omega$.

We let

$$\Phi_\gamma = \Phi_{n,\gamma} \circ \cdots \circ \Phi_{1,\gamma}.$$

We observe that

$$D^j\Phi_\gamma \to \mathrm{id} \quad \text{uniformly on } \mathbb{R}^m \text{ for every } j \in \mathbb{N} \text{ as } \gamma \to 0.$$

By (2.6.11) and the construction of the maps $\Phi_{i,\gamma}$, for every $x \in \partial\Omega$, there exists $i \in \{1, \dots, n\}$ such that $\Phi_{i,\gamma}(x) \in \Omega$, and this shows that $\Phi_\gamma(\overline{\Omega}) \subset \Omega$. This proves that the family of maps Φ_γ satisfies the conclusions of the lemma. □

Using this construction, we observe that, if $\Omega \subset \mathbb{R}^m$ is a bounded domain satisfying the segment condition and $u \in W^{s,p}(\Omega; \mathscr{N})$, the map $u_\gamma = u \circ \Phi_\gamma$ belongs to $W^{s,p}(\Omega_\gamma; \mathscr{N})$, where $\Omega_\gamma = \Phi_\gamma^{-1}(\Omega)$ is an open subset of $\mathbb{R}^m$ containing $\overline{\Omega}$. Moreover, $u_\gamma \to u$ in $W^{s,p}(\Omega; \mathscr{N})$ as $\gamma \to 0$.

Therefore, we may carry out the same reasoning as in the proof of Theorem 2.6.1 by choosing a cubication K^m_η such that $\Omega \subset \mathscr{K}^m_\eta \subset \Omega_\gamma$.

The other place in the proof of Theorem 2.6.1 where we used a specific assumption on the domain is when we applied Lemma 2.6.2, because we needed convexity to justify the use of this lemma. However, this may be easily bypassed. Indeed, since we work on a dilated domain, by dilating slightly more if necessary, we may assume that $u \in W^{s,p}(\widetilde{\Omega})$ for some open set $\widetilde{\Omega} \subset \mathbb{R}^m$ containing $\overline{\Omega}_\gamma$. It then suffices to apply instead Lemma 2.6.2 to the map $u\psi \in W^{s,p}(\mathbb{R}^m)$, where $\psi\colon \mathbb{R}^m \to [0,1]$ is a smooth map such that $\psi = 1$ on Ω_γ and $\psi = 0$ on $\mathbb{R}^m \setminus \widetilde{\Omega}$.

Taking these modifications into consideration, the proof of Theorem 2.6.1 above can be carried out the exact same way on any bounded domain $\Omega \subset \mathbb{R}^m$ satisfying the segment condition. This leads to the following result.

Theorem 2.6.4. *Let $\Omega \subset \mathbb{R}^m$ be a bounded domain satisfying the segment condition. If $sp < m$, then the class $\mathscr{R}_{m-\lfloor sp \rfloor-1}(\Omega; \mathscr{N})$ is dense in $W^{s,p}(\Omega; \mathscr{N})$.*

We finally move to the general case where the domain can be a manifold. We recall that, in this context, our domain $\mathscr{M}$ is assumed to be a smooth compact Riemannian manifold, isometrically embedded into $\mathbb{R}^N$. In this chapter, we also allow $\mathscr{M}$ to be a manifold *with boundary*. Our next result is the following counterpart of Theorem 2.6.1 when the domain is a smooth manifold.

Theorem 2.6.5. *If $sp < m$, then the class $\mathscr{R}_{m-\lfloor sp\rfloor-1}(\mathscr{M};\mathscr{N})$ is dense in $W^{s,p}(\mathscr{M};\mathscr{N})$.*

We first prove Theorem 2.6.5 when $\mathscr{M}$ has no boundary. This allows us to rely on the nearest point projection onto $\mathscr{M}$. In the end, we shall briefly explain how to deduce the case with boundary from the case without boundary.

Hence, we first assume that $\mathscr{M}$ has no boundary. Then, Theorem 2.6.5 can be deduced from Theorem 2.6.4, by extending the function we want to approximate on a tubular neighborhood of $\mathscr{M}$ and using a slicing argument. The key observation is that, if $\iota > 0$ is the radius of a tubular neighborhood of $\mathscr{M}$, then for every $u \in W^{s,p}(\mathscr{M};\mathscr{N})$, the map $v = u \circ \Pi$ belongs to $W^{s,p}(\mathscr{M} + B^N_{\iota/2};\mathscr{N})$. Indeed, for any summable function $w\colon \mathscr{M} \to [0,+\infty]$, we deduce from the coarea formula that

$$\int_{\mathscr{M}+B^N_{\iota/2}} w\circ\Pi \lesssim \int_{\mathscr{M}}\left(\int_{\Pi^{-1}(x)\cap(\mathscr{M}+B^N_{\iota/2})} w(\Pi(y))\,\mathrm{d}\mathscr{H}^{N-m}(y)\right)\mathrm{d}x \lesssim \iota^{N-m}\int_{\mathscr{M}} w < +\infty.$$

The conclusion then follows from the theory of Fuglede maps presented in Section 2.3 (valid also for maps between manifolds, see [BPVS25]).

To implement this strategy of extension and slicing, we need the following transversality result.

Lemma 2.6.6. *Let $\Sigma \subset \mathbb{R}^N$ be an ℓ-dimensional hyperplane. For almost every $a \in \mathbb{R}^N$, the set $\mathscr{M} \cap (\Sigma + a)$ is a smooth submanifold of $\mathscr{M}$ of dimension $m - N + \ell$ — or the empty set if $\ell < N - m$. Moreover, if $\mathscr{M} \cap (\Sigma + a) \neq \varnothing$, then for every $x \in \mathscr{M}$ and every a as above, we have*

$$\operatorname{dist}(x, \mathscr{M} \cap (\Sigma + a)) \leq C \operatorname{dist}(x, \Sigma + a),$$

for some constant $C > 0$ depending on $\mathscr{M}$, Σ, and a.

The proof of this lemma relies on Sard's theorem. As it is essentially a special case of Lemma 3.2.7 that will be proved in the next chapter, we omit the argument here.

Proof of Theorem 2.6.5 when $\mathscr{M}$ has no boundary. Let $u \in W^{s,p}(\mathscr{M};\mathscr{N})$. Let $\iota > 0$ be the radius of a tubular neighborhood of $\mathscr{M}$, and let $\Pi\colon \mathscr{M} + B^N_\iota \to \mathscr{M}$ be the nearest point projection. We define $\Omega = \mathscr{M} + B^N_{\iota/2}$, which is a smooth bounded open subset of $\mathbb{R}^{\tilde{\nu}}$. As explained above, the map $v = u \circ \Pi$ belongs to $W^{s,p}(\Omega;\mathscr{N})$. Therefore, Theorem 2.6.4 ensures the existence of a sequence $(v_n)_{n\in\mathbb{N}}$ of maps in $\mathscr{R}_{N-\lfloor sp\rfloor-1}(\Omega;\mathscr{N})$ converging to v in $W^{s,p}(\Omega)$ as $n \to +\infty$. Invoking Lemma 2.6.6, we deduce that for almost every

$a \in B^N_{\iota/2}$, the map $u_{n,a} = \tau_a(v_n)_{|\mathscr{M}} \colon \mathscr{M} \to \mathscr{N}$ belongs to the class $\mathscr{R}_{m-\lfloor sp \rfloor -1}(\mathscr{M};\mathscr{N})$. Here, we recall that the translation $\tau_a(v_n)$ is defined by $\tau_a(v_n)(x) = v_n(x+a)$. Indeed, the first part of Lemma 2.6.6 ensures that the singular set of $u_{n,a}$ is as in the definition of the class $\mathscr{R}_{m-\lfloor sp \rfloor -1}$. On the other hand, the distance estimate in Lemma 2.6.6 implies that the estimates on the derivatives of $u_{n,a}$ are satisfied.

Moreover, using a slicing argument, we find that for almost every $a \in B^N_{\iota/2}$, up to extraction of a subsequence, $(u_{n,a})_{n\in\mathbb{N}}$ converges in $W^{s,p}(\mathscr{M})$ to the map $\tau_a(v)_{|\mathscr{M}}$. This can be seen, for instance, using the theory of Fuglede maps presented in Section 2.3. Indeed, consider a summable map $w \colon \Omega \to [0,+\infty]$, and let $i \colon \mathscr{M} \to \Omega$ be the inclusion map. We observe that $\tau_a(v)_{|\mathscr{M}} = v \circ (i+a)$, and we estimate

$$\int_{B^N_{\iota/2}} \int_{\mathscr{M}} w(i(x)+a)\,\mathrm{d}x\,\mathrm{d}a = \int_{\mathscr{M}} \int_{B^N_{\iota/2}(x)} w(a)\,\mathrm{d}a\,\mathrm{d}x \leq |\mathscr{M}| \|w\|_{L^1(\Omega)} < +\infty.$$

Therefore, for almost every $a \in B^N_{\iota/2}$, $w \circ (i+a)$ is summable on $\mathscr{M}$. If we now choose w to be a detector for the $W^{s,p}$ convergence, then, up to a subsequence independent of a, we have

$$u_{n,a} = v_n \circ (i+a) \to v \circ (i+a) = \tau_a(v)_{|\mathscr{M}} \quad \text{in } W^{s,p}(\mathscr{M}) \text{ as } n \to +\infty.$$

On the other hand, by the continuity of translations in $W^{s,p}$, we know that $\tau_a(v)_{|\mathscr{M}} \to v_{|\mathscr{M}} = u$ in $W^{s,p}(\mathscr{M})$ as $a \to 0$ (more precisely, we should rely on an argument in the spirit of [BPVS25, Example 2.5], since there is again a slicing involved here).

We conclude the proof by invoking a diagonal argument: choosing a suitable sequence $(a_n)_{n\in\mathbb{N}}$ in $B^N_{\iota/2}$ such that $a_n \to 0$, the maps $u_n = u_{n,a_n}$ belong to $\mathscr{R}_{m-\lfloor sp \rfloor -1}(\mathscr{M};\mathscr{N})$ and converge to u in $W^{s,p}(\mathscr{M})$ as $n \to +\infty$. □

Proof of Theorem 2.6.5 when $\mathscr{M}$ has non-empty boundary. The key idea is to view $\mathscr{M}$, or more precisely any compact subset of the interior of $\mathscr{M}$, as a subset of a smooth manifold *without* boundary, embedded in $\mathbb{R}^N \times \mathbb{R}$, identifying $\mathbb{R}^N$ with $\mathbb{R}^N \times \{0\}$. For this, we rely on [BPVS17, Lemma 3.4], which is a consequence of the collar neighborhood theorem.

Let K be any compact subset in the relative interior of $\mathscr{M}$. From [BPVS17, Lemma 3.4], we deduce that there exists a smooth compact submanifold $\tilde{\mathscr{M}}$ of $\mathbb{R}^N \times \mathbb{R}$, without boundary, such that

$$K \times \{0\} \subset \tilde{\mathscr{M}} \quad \text{and} \quad \pi(\tilde{\mathscr{M}}) \subset \mathscr{M},$$

where $\pi \colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ is the projection onto the first N variables.

Let $u \in W^{s,p}(\mathscr{M};\mathscr{N})$. The map $v = u \circ \pi$ belongs to $W^{s,p}(\tilde{\mathscr{M}};\mathscr{N})$. Hence, by Theorem 2.6.5 for manifolds without boundary, there exists a sequence $(v_n^K)_{n\in\mathbb{N}}$ of maps in $\mathscr{R}_{m-\lfloor sp\rfloor-1}(\tilde{\mathscr{M}};\mathscr{N})$ such that $v_n^K \to v$ in $W^{s,p}(\tilde{\mathscr{M}})$. In particular, $(v_n^K)_{|K} \to u_{|K}$ in $W^{s,p}(K)$.

Now, we observe that, for every $\varepsilon > 0$ sufficiently small, if we take $K = K_\varepsilon$ such that $\mathscr{M} \setminus K_\varepsilon$ is contained in a uniform neighborhood of radius ε of $\partial\mathscr{M}$, then $u_{|K_\varepsilon}$ may be dilated to a map $u_\varepsilon \in W^{s,p}(\mathscr{M};\mathscr{N})$. Moreover, if we denote by $u_{n,\varepsilon}$ the corresponding dilations of the maps $(v_n^{K_\varepsilon})_{|K_\varepsilon}$, we have both

$$u_\varepsilon \to u \quad \text{as } \varepsilon \to 0 \quad \text{and} \quad u_{n,\varepsilon} \to u_\varepsilon \quad \text{as } n \to +\infty.$$

We conclude using a diagonal argument. □

2.7 Shrinking

This section is dedicated to the shrinking procedure. As we explained in Section 2.2, shrinking is actually a more involved version of a scaling argument, whose purpose is to modify a given map in order to obtain a better map whose energy is controlled. The main result of this section is the following proposition, counterpart of [BPVS15, Proposition 8.1] in the fractional setting, which provides the shrinking construction. We emphasize that, similar to thickening but unlike opening, the map Φ does not depend on the map $u \in W^{s,p}$ it shall be composed with.

Proposition 2.7.1. *Let $d \in \{0,\dots,m-1\}$, $\eta > 0$, $0 < \mu < \frac{1}{2}$, $0 < \tau < \frac{1}{2}$, K^m be a cubication in $\mathbb{R}^m$ of radius η, and T^{d^*} be the dual skeleton of K^d. There exists a smooth map $\Phi\colon \mathbb{R}^m \to \mathbb{R}^m$ such that*

- (i) *Φ is injective;*
- (ii) *for every $\sigma^m \in K^m$, $\Phi(\sigma^m) \subset \sigma^m$;*
- (iii) $\operatorname{Supp}\Phi \subset \mathscr{T}^{d^*} + Q^m_{2\mu\eta}$ *and* $\Phi(\mathscr{T}^{d^*} + Q^m_{\tau\mu\eta}) \supset \mathscr{T}^{d^*} + Q^m_{\mu\eta}$.

If in addition $d+1 > sp$, then for every $u \in W^{s,p}(\mathscr{K}^m;\mathbb{R}^\nu)$ and every $v \in W^{s,p}(\mathscr{K}^m;\mathbb{R}^\nu)$ such that $u = v$ on the complement of $\mathscr{T}^{d^*} + Q^m_{\mu\eta}$, we have $u \circ \Phi \in W^{s,p}(\mathscr{K}^m;\mathbb{R}^\nu)$, and moreover, the following estimates hold:

- (a) *if $0 < s < 1$, then*

$$\begin{aligned}
&(\mu\eta)^s|u\circ\Phi - v|_{W^{s,p}(\mathscr{K}^m)} \\
&\leq C\Big((\mu\eta)^s|u|_{W^{s,p}(\mathscr{K}^m\cap(\mathscr{T}^{d^*}+Q^m_{2\mu\eta})\setminus(\mathscr{T}^{d^*}+Q^m_{\mu\eta}))} + \|u\|_{L^p(\mathscr{K}^m\cap(\mathscr{T}^{d^*}+Q^m_{2\mu\eta})\setminus(\mathscr{T}^{d^*}+Q^m_{\mu\eta}))}\Big) \\
&\quad + C\tau^{\frac{d+1-sp}{p}}\Big((\mu\eta)^s|u|_{W^{s,p}(\mathscr{K}^m\cap(\mathscr{T}^{d^*}+Q^m_{2\mu\eta}))} + \|u\|_{L^p(\mathscr{K}^m\cap(\mathscr{T}^{d^*}+Q^m_{2\mu\eta}))}\Big) \\
&\quad + C(\mu\eta)^s|v|_{W^{s,p}(\mathscr{K}^m\cap(\mathscr{T}^{d^*}+Q^m_{2\mu\eta}))} + C\|v\|_{L^p(\mathscr{K}^m\cap(\mathscr{T}^{d^*}+Q^m_{2\mu\eta}))};
\end{aligned}$$

(b) *if $s \geq 1$, then for every $j \in \{1, \dots, k\}$,*

$$
\begin{aligned}
&(\mu\eta)^j \|\mathrm{D}^j(u \circ \Phi) - \mathrm{D}^j v\|_{L^p(\mathscr{K}^m)} \\
&\quad \leq C \sum_{i=1}^{j} (\mu\eta)^i \|\mathrm{D}^i u\|_{L^p(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}) \setminus (\mathscr{T}^{d^*} + Q^m_{\mu\eta}))} \\
&\qquad + C\tau^{\frac{d+1-sp}{p}} \sum_{i=1}^{j} (\mu\eta)^i \|\mathrm{D}^i u\|_{L^p(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}))} \\
&\qquad + C(\mu\eta)^j \|\mathrm{D}^j v\|_{L^p(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}))};
\end{aligned}
$$

(c) *if $s \geq 1$ and $\sigma \neq 0$, then for every $j \in \{1, \dots, k\}$,*

$$
\begin{aligned}
&(\mu\eta)^{j+\sigma} |\mathrm{D}^j(u \circ \Phi) - \mathrm{D}^j v|_{W^{\sigma,p}(\mathscr{K}^m)} \\
&\quad \leq C \sum_{i=1}^{j} \Big((\mu\eta)^i \|\mathrm{D}^i u\|_{L^p(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}) \setminus (\mathscr{T}^{d^*} + Q^m_{\mu\eta}))} \\
&\qquad\qquad + (\mu\eta)^{i+\sigma} |\mathrm{D}^i u|_{W^{\sigma,p}(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}) \setminus (\mathscr{T}^{d^*} + Q^m_{\mu\eta}))} \Big) \\
&\qquad + C\tau^{\frac{d+1-sp}{p}} \sum_{i=1}^{j} \Big((\mu\eta)^i \|\mathrm{D}^i u\|_{L^p(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}))} \\
&\qquad\qquad + (\mu\eta)^{i+\sigma} |\mathrm{D}^i u|_{W^{\sigma,p}(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}))} \Big) \\
&\qquad + C(\mu\eta)^j \|\mathrm{D}^j v\|_{L^p(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}))} \\
&\qquad\qquad + C(\mu\eta)^{j+\sigma} |\mathrm{D}^j v|_{W^{\sigma,p}(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}))};
\end{aligned}
$$

(d) *for $0 < s < +\infty$,*

$$
\begin{aligned}
\|u \circ \Phi - v\|_{L^p(\mathscr{K}^m)} &\leq C\|u\|_{L^p(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}) \setminus (\mathscr{T}^{d^*} + Q^m_{\mu\eta}))} \\
&\quad + C\tau^{\frac{d+1-sp}{p}} \|u\|_{L^p(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}))} + C\|v\|_{L^p(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}))};
\end{aligned}
$$

for some constant $C > 0$ depending on m, s, and p.

For integer order estimates, we could avoid mentioning the map v in the statement of Proposition 2.7.1 and only establish energy estimates for $u \circ \Phi$ alone on $\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta})$, as in [BPVS15], as the estimates above then follow from the assumption $u = v$ outside of $\mathscr{T}^{d^*} + Q^m_{2\mu\eta}$ using the additivity of the integral. However, for fractional order estimates, we face the usual problem linked to the lack of additivity of the Gagliardo seminorm.

We pause here to explain how Proposition 2.7.1 will be used in the proof of the strong density theorem. Given u and v as above, Proposition 2.7.1 allows us to control, *via* a suitable choice of $\tau > 0$, the energy of $u \circ \Phi$ in terms of the energy of v alone. Indeed, given $\mu > 0$ and $\varepsilon > 0$, if we choose $\tau = \tau_\mu$ sufficiently small — depending on u and v — then, using the fact that $u = v$ outside of $\mathscr{T}^{d^*} + Q^m_{\mu\eta}$, we find

(a) if $0 < s < 1$, then

$$(\mu\eta)^s |u \circ \Phi - v|_{W^{s,p}(\mathscr{K}^m)} \\ \leq C'\Big((\mu\eta)^s |v|_{W^{s,p}(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}))} + \|v\|_{L^p(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}))}\Big) + \varepsilon;$$

(b) if $s \geq 1$, then for every $j \in \{1, \dots, k\}$,

$$(\mu\eta)^j \|\mathrm{D}^j(u \circ \Phi) - \mathrm{D}^j v\|_{L^p(\mathscr{K}^m)} \leq C' \sum_{i=1}^{j} (\mu\eta)^i \|\mathrm{D}^i v\|_{L^p(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}))} + \varepsilon;$$

(c) if $s \geq 1$ and $\sigma \neq 0$, then for every $j \in \{1, \dots, k\}$,

$$(\mu\eta)^{j+\sigma} |\mathrm{D}^j(u \circ \Phi) - \mathrm{D}^j v|_{W^{\sigma,p}(\mathscr{K}^m)} \\ \leq C' \sum_{i=1}^{j} \Big((\mu\eta)^i \|\mathrm{D}^i v\|_{L^p(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}))} + (\mu\eta)^{i+\sigma} |\mathrm{D}^i v|_{W^{\sigma,p}(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}))}\Big) + \varepsilon;$$

(d) for every $0 < s < +\infty$,

$$\|u \circ \Phi - v\|_{L^p(\mathscr{K}^m)} \leq C' \|v\|_{L^p(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}))} + \varepsilon;$$

for some constant $C' > 0$ depending on m, s, and p. The reason for the extra term ε in the right-hand sides of the above estimates is to cover the case where u is identically 0 while v does not vanish on $\mathscr{T}^{d^*} + Q^m_{\mu\eta}$. Estimates (a) to (d) will be used in the proof of the density of smooth maps.

This section is organized as follows. In a first time, we explain the construction of the building blocks for shrinking, and we prove their geometric properties. Then we state the analytic estimates satisfied by the composition of a $W^{s,p}$ map u with those building blocks. Finally, we explain how to suitably combine the building blocks in order to obtain the global shrinking construction, along with the required properties.

We start with the construction of the building blocks for shrinking, which is very similar to thickening. Therefore, in this section, we shall follow an analogous path to the one in Section 2.5. We start by introducing some additional notation, similar to Sections 2.3 and 2.5. Let $0 < \underline{\mu} < \mu < \overline{\mu} < 1$ and $0 < \tau < \underline{\mu}/\mu$ be fixed. Given

$l \in \{1, \ldots, m\}$, we set

$$B_1 = B^l_{\tau\mu\eta} \times Q^{m-l}_{(1-\overline{\mu})\eta}, \quad Q_2 = Q^l_{\underline{\mu}\eta} \times Q^{m-l}_{(1-\overline{\mu})\eta}, \quad Q_3 = Q^l_{\mu\eta} \times Q^{m-l}_{(1-\mu)\eta}.$$

Note that $B_1 \subset Q_2 \subset Q_3$. The rectangle Q_3 contains the geometric support of the building block Φ, that is, $\Phi = \mathrm{id}$ outside of Q_3. The rectangle Q_2 is shrinked into the cylinder B_1: we have $\Phi(B_1) \supset Q_2$. As usual, the region in between serves as a transition region.

As we did for thickening, we split the construction of the building block Φ into two parts. First, we deal with the geometric properties that need to be satisfied by Φ independently of the map u, and then, we move to the Sobolev estimates satisfied by $u \circ \Phi$. We take Φ to be exactly the map given by [BPVS15, Proposition 8.3], and we therefore only recall briefly how this map is built, referring the reader to [BPVS15] for the details. Once again, the main change in our approach is that we establish the Sobolev estimates first for the building blocks, and then we glue them together in order to obtain the estimates given by Proposition 2.7.1.

Analogously to Section 2.5, we define $\zeta\colon \mathbb{R}^m \to \mathbb{R}$ by

$$\zeta(x) = \sqrt{|x'|^2 + (\mu\eta)^2\theta\Big(\frac{x''}{\mu\eta}\Big) + (\mu\eta)^2\varepsilon\tau^2} \tag{2.7.1}$$

for every $x = (x', x'') \in \mathbb{R}^l \times \mathbb{R}^{m-l}$. Here, $\theta\colon \mathbb{R}^{m-l} \to \mathbb{R}$ is defined similarly as in Section 2.5. We choose $1 < q < +\infty$ sufficiently large so that there exist $0 < r_1 < r_2$ satisfying

$$Q^{m-l}_{(1-\overline{\mu})/\mu} \subset \{x'' \in \mathbb{R}^{m-l} \colon |x''|_q < r_1\} \subset \{x'' \in \mathbb{R}^{m-l} \colon |x''|_q < r_2\} \subset Q^{m-l}_{(1-\mu)/\mu}.$$

Then, we pick a nondecreasing smooth map $\tilde{\theta}\colon \mathbb{R}_+ \to [0, 1]$ such that $\tilde{\theta}(r) = 0$ if $0 \leq r \leq r_1$ and $\tilde{\theta}(r) = 1$ if $r \geq r_2$. Finally, we let $\theta(x'') = \tilde{\theta}(|x''|_q)$. With this definition, the map θ is smooth and satisfies $\theta(x'') = 0$ if $x'' \in Q^{m-l}_{(1-\overline{\mu})/\mu}$ and $\theta(x'') = 1$ if $x'' \in \mathbb{R}^{m-d} \setminus Q^{m-l}_{(1-\mu)/\mu}$. The number $\varepsilon > 0$ is to be determined later on, depending only on $\underline{\mu}/\mu$. As we will see in the course of the proof, the extra term involving τ, which was not present in Section 2.5, serves to obtain a desingularized construction.

We are now ready to state the geometric properties of Φ, which are the purpose of Proposition 2.7.2 below.

Proposition 2.7.2. *Let $l \in \{1, \ldots, m\}$, $\eta > 0$, $0 < \underline{\mu} < \mu < \overline{\mu} < 1$, and $0 < \tau < \underline{\mu}/\mu$. There exists a smooth function $\Phi\colon \mathbb{R}^m \to \mathbb{R}^m$ of the form $\Phi(x) = (\lambda(x)x', x'')$, with $\lambda\colon \mathbb{R}^m \to [1, +\infty)$, and such that* (*i*) Φ *is injective;*

(*ii*) $\operatorname{Supp}\Phi \subset Q_3$;

(*iii*) $\Phi(B_1) \supset Q_2$;

(*iv*) *for every* $x \in Q_3$,

$$|D^j\Phi(x)| \le C\frac{\mu\eta}{\zeta^j(x)} \quad \textit{for every } j \in \mathbb{N}_*,$$

and for every $x \in \mathbb{R}^m$,

$$|\mathrm{D}^j\Phi(x)| \le C\frac{(\mu\eta)^{1-j}}{\tau^j} \quad \textit{for every } j \in \mathbb{N}_*,$$

for some constant $C > 0$ *depending on* j, m, $\mu/\overline{\mu}$ *and* $\underline{\mu}/\mu$;

(*v*) *for every* $x \in \mathbb{R}^m$,

$$\mathcal{J}\Phi(x) \ge C'\frac{(\mu\eta)^\beta}{\zeta^\beta(x)} \quad \textit{for every } 0 < \beta < l,$$

and for every $x \in B_1$,

$$\mathcal{J}\Phi(x) \ge C'\frac{1}{\tau^l},$$

for some constant $C' > 0$ *depending on* β, j, m, $\mu/\overline{\mu}$ *and* $\underline{\mu}/\mu$.

Proof. As we announced, we use the same construction as in [BPVS15, Proposition 8.3]. Similar to thickening, we start by constructing an intermediate map $\Psi\colon \mathbb{R}^m \to \mathbb{R}^m$ which satisfies the conclusion of Proposition 2.7.2 with the rectangles Q_i replaced by the cylinders B_i defined as

$$B_2 = B^l_{\underline{\mu}\eta} \times Q^{m-l}_{(1-\overline{\mu})\eta}, \quad B_3 = B^l_{\mu\eta} \times Q^{m-l}_{(1-\mu)\eta}.$$

It will then suffice to compose Ψ with a suitable diffeomorphism $\Theta\colon \mathbb{R}^m \to \mathbb{R}^m$ dilating B_2 to a set containing Q_2 in order to obtain the desired map Φ.

We let $\varphi\colon (0, +\infty) \to [1, +\infty)$ be a smooth function such that

(a) for $0 < r \le \tau\sqrt{1+\varepsilon}$,

$$\varphi(r) = \frac{\underline{\mu}/\mu}{r}\sqrt{1+\varepsilon}\Big(1 + \frac{b}{\ln\frac{1}{r}}\Big);$$

(b) for $r \ge 1$, $\varphi(r) = 1$;

(c) the function $r \in (0, +\infty) \mapsto r\varphi(r)$ is increasing.

This is possible provided that we choose ε such that

$$(\underline{\mu}/\mu)\sqrt{1+\varepsilon} < 1$$

and then $b > 0$ such that

$$(\underline{\mu}/\mu)\sqrt{1+\varepsilon}\Big(1 + \frac{b}{\ln \frac{1}{(\underline{\mu}/\mu)\sqrt{1+\varepsilon}}}\Big) < 1.$$

Then, we define $\lambda\colon \mathbb{R}^m \to [1, +\infty)$ by

$$\lambda(x) = \varphi\Big(\frac{\zeta(x)}{\mu\eta}\Big),$$

and finally

$$\Psi(x', x'') = (\lambda(x', x'')x', x'').$$

The injectivity of Ψ relies on assumption (c) on φ. The fact that $\operatorname{Supp}\Psi \subset B_3$ uses assumption (b) on φ, observing that $\zeta(x) \geq \mu\eta$ whenever $x \in \mathbb{R}^m \setminus B_3$, and hence $\lambda(x) = 1$. To prove (iii), we note that if $x = (x', x'') \in B_1$ and $t \geq 0$, we have

$$\Psi(tx', x'') = \Bigg(t\varphi\Bigg(\sqrt{t^2\Big|\frac{x'}{\mu\eta}\Big|^2 + \varepsilon\tau^2}\Bigg)x', x''\Bigg),$$

where we used the fact that θ vanishes inside of $Q^{m-l}_{(1-\underline{\mu})/\mu}$. For $t = 0$, the factor in front of x' vanishes, while for $t = \tau$, it is larger than $\underline{\mu}\eta/|x'| \geq 1$. We conclude by invoking the intermediate value theorem. The proof of (iv) amounts to estimate $|\mathrm{D}^j\lambda|$ using the Faà di Bruno formula, and then conclude using Leibniz's rule. We obtain the second estimate from the first one by noting that $\zeta \geq (\mu\eta)\sqrt{\varepsilon}\tau$. The proof of (v) again involves explicitly computing $\mathcal{J}\Psi$ as the determinant of a perturbation of a linear map, and then estimating the obtained expression. The second estimate relies on the fact that if $x = (x', x'') \in B_1$, then $|x'| \leq (\mu\eta)\tau$ and $\theta\Big(\frac{x''}{\mu\eta}\Big) = 0$, whence $\zeta(x) \leq (\mu\eta)\sqrt{1+\varepsilon}\tau$. We refer the reader to [BPVS15]*Lemma 8.5 for the details.

We then let $\Theta\colon \mathbb{R}^m \to \mathbb{R}^m$ be a smooth diffeomorphism also of the form $\Theta(x) = (\tilde{\lambda}(x)x', x'')$, with $\tilde{\lambda}\colon \mathbb{R}^m \to [1, +\infty)$, such that Θ is supported in Q_3, maps B_2 on a set containing Q_2, and satisfies the estimates

$$(\mu\eta)^{j-1}|\mathrm{D}^j\Theta| \leq C_1 \quad \text{and} \quad 0 < C_2 \leq \mathcal{J}\Theta \leq C_3 \quad \text{on } \mathbb{R}^m;$$

see [BPVS15, Lemma 8.4]. Using the composition formula for the Jacobian and the Faà di Bruno formula, we conclude, as for thickening, that $\Phi = \Theta \circ \Psi$ is the desired map. □

We now turn to the Sobolev estimates satisfied by $u \circ \Phi$.

Proposition 2.7.3. *Let $l > sp$. Let Φ be as in Proposition 2.7.2. Let $\omega \subset \mathbb{R}^m$ be such that $Q_2 \subset \omega \subset B^m_{c\mu\eta}$ for some $c > 0$, and assume that there exists $c' > 0$ such that*

$$|B^m_\lambda(z) \cap (\omega \setminus Q_2)| \geq c'\lambda^m \quad \text{for every } z \in \omega \setminus Q_2 \text{ and } 0 < \lambda \leq \frac{1}{2}\operatorname{diam}\omega. \tag{2.7.2}$$

For every $u \in W^{s,p}(\Phi^{-1}(\omega);\mathbb{R}^\nu)$, we have $u \circ \Phi \in W^{s,p}(\Phi^{-1}(\omega);\mathbb{R}^\nu)$, and moreover, the following estimates hold:

(a) *if $0 < s < 1$, then*

$$|u \circ \Phi|_{W^{s,p}(\Phi^{-1}(\omega))} \leq C|u|_{W^{s,p}(\omega\setminus Q_2)} + C\tau^{\frac{l-sp}{p}}|u|_{W^{s,p}(\omega)};$$

(b) *if $s \geq 1$, then for every $j \in \{1, \dots, k\}$,*

$$(\mu\eta)^j\|D^j(u\circ\Phi)\|_{L^p(\Phi^{-1}(\omega))} \leq C\sum_{i=1}^{j}(\mu\eta)^i\|D^i u\|_{L^p(\omega\setminus Q_2)} + C\tau^{\frac{l-jp}{p}}\sum_{i=1}^{j}(\mu\eta)^i\|D^i u\|_{L^p(\omega)};$$

(c) *if $s \geq 1$ and $\sigma \neq 0$, then for every $j \in \{1, \dots, k\}$,*

$$\begin{aligned}(\mu\eta)^{j+\sigma}&|D^j(u\circ\Phi)|_{W^{\sigma,p}(\Phi^{-1}(\omega))}\\ &\leq C\sum_{i=1}^{j}\Big((\mu\eta)^i\|D^i u\|_{L^p(\omega\setminus Q_2)} + (\mu\eta)^{i+\sigma}|D^i u|_{W^{\sigma,p}(\omega\setminus Q_2)}\Big)\\ &\quad + C\tau^{\frac{l-(j+\sigma)p}{p}}\sum_{i=1}^{j}\Big((\mu\eta)^i\|D^i u\|_{L^p(\omega)} + (\mu\eta)^{i+\sigma}|D^i u|_{W^{\sigma,p}(\omega)}\Big);\end{aligned}$$

(d) *for every $0 < s < +\infty$,*

$$\|u\circ\Phi\|_{L^p(\Phi^{-1}(\omega))} \leq C\|u\|_{L^p(\omega\setminus Q_2)} + C\tau^{\frac{l}{p}}\|u\|_{L^p(\omega)};$$

for some constant $C > 0$ depending on s, m, p, c, c', $\underline{\mu}/\mu$, and $\mu/\overline{\mu}$.

We encounter again the assumption that balls centered at a point of ω significantly intersect ω. We call the attention of the reader to the fact that, in the proof of Proposition 2.7.1, Proposition 2.7.3 will be applied with ω being a domain more complicated than just a rectangle. This contrasts with the situation encountered in Sections 2.3 and 2.5.

In the proof of Proposition 2.7.3, we need the counterpart of Lemma 2.5.4 for the map ζ used for shrinking. The proof is the same as the proof of Lemma 2.5.4, since both constructions are identical up to an additive constant under the square root, and is therefore omitted.

Lemma 2.7.4. *For every $x, y \in \mathbb{R}^m$, there exists a Lipschitz path $\gamma\colon [0,1] \to \mathbb{R}^m$ from x to y such that*

$$|\gamma|_{C^{0,1}([0,1])} \leq C|x-y|,$$

for some constant $C > 0$ depending only on m, and such that $\zeta \geq \min(\zeta(x), \zeta(y))$ along γ, where ζ is the map defined in (2.7.1).

We are now ready to prove Proposition 2.7.3.

Proof of Proposition 2.7.3. As for thickening, the integer order estimate when $s \geq 1$ is proved exactly as [BPVS15, Corollary 8.2], but is presented here as a prelude for the more involved fractional order case.

By the Faà di Bruno formula, we estimate

$$|\mathrm{D}^j(u \circ \Phi)(x)|^p \lesssim \sum_{i=1}^{j} \sum_{\substack{1 \leq t_1 \leq \cdots \leq t_i \\ t_1 + \cdots + t_i = j}} |\mathrm{D}^i u(\Phi(x))|^p |\mathrm{D}^{t_1}\Phi(x)|^p \cdots |\mathrm{D}^{t_i}\Phi(x)|^p$$

for every $j \in \{1, \ldots, k\}$ and $x \in \Phi^{-1}(\omega)$. Let $0 < \beta < l$. Using the estimates on the derivatives and the Jacobian of Φ, we find $|\mathrm{D}^t\Phi| \lesssim \frac{(\mathcal{J}\Phi)^{\frac{t}{\beta}}}{(\mu\eta)^{t-1}}$, and therefore

$$\begin{aligned}
|\mathrm{D}^j(u \circ \Phi)(x)|^p &\lesssim \sum_{i=1}^{j} \sum_{\substack{1 \leq t_1 \leq \cdots \leq t_i \\ t_1 + \cdots + t_i = j}} |D^i u(\Phi(x))|^p \frac{(\mathcal{J}\Phi(x))^{\frac{t_1 p}{\beta}}}{(\mu\eta)^{(t_1-1)p}} \cdots \frac{(\mathcal{J}\Phi(x))^{\frac{t_i p}{\beta}}}{(\mu\eta)^{(t_i-1)p}} \\
&\lesssim \sum_{i=1}^{j} |\mathrm{D}^i u(\Phi(x))|^p \frac{(\mathcal{J}\Phi(x))^{\frac{jp}{\beta}}}{(\mu\eta)^{(j-i)p}}.
\end{aligned}$$

Since $jp \leq sp < l$, we may choose $\beta = jp$. Hence,

$$|\mathrm{D}^j(u \circ \Phi)(x)|^p \lesssim \sum_{i=1}^{j} |\mathrm{D}^i u(\Phi(x))|^p \frac{\mathcal{J}\Phi(x)}{(\mu\eta)^{(j-i)p}}.$$

Since Φ is injective, the change of variable theorem ensures that

$$\begin{aligned}\int_{\Phi^{-1}(\omega\setminus Q_2)} (\mu\eta)^{jp}|\mathrm{D}^j(u\circ\Phi)|^p &\lesssim \int_{\Phi^{-1}(\omega\setminus Q_2)} \sum_{i=1}^{j} (\mu\eta)^{ip}|\mathrm{D}^i u(\Phi(x))|^p \mathcal{J}\Phi(x)\,\mathrm{d}x\\ &\lesssim \sum_{i=1}^{j}\int_{\omega\setminus Q_2} (\mu\eta)^{ip}|\mathrm{D}^i u|^p.\end{aligned}$$

Combining inclusion (iii) and estimates (iv) and (v) in Proposition 2.7.2, we find

$$|\mathrm{D}^j(u\circ\Phi)(x)|^p \lesssim \tau^{l-jp}\sum_{i=1}^{j}|\mathrm{D}^i u(\Phi(x))|^p \frac{\mathcal{J}\Phi(x)}{(\mu\eta)^{(j-i)p}}$$

for every $x\in\Phi^{-1}(Q_2)\subset B_1$. Using again the change of variable theorem, we deduce that

$$\begin{aligned}\int_{\Phi^{-1}(Q_2)} (\mu\eta)^{jp}|\mathrm{D}^j(u\circ\Phi)|^p &\lesssim \int_{\Phi^{-1}(Q_2)} \tau^{l-jp}\sum_{i=1}^{j} (\mu\eta)^{ip}|\mathrm{D}^i u(\Phi(x))|^p \mathcal{J}\Phi(x)\,\mathrm{d}x\\ &\lesssim \tau^{l-jp}\sum_{i=1}^{j}\int_{Q_2} (\mu\eta)^{ip}|\mathrm{D}^i u|^p.\end{aligned}$$

We conclude by additivity of the integral, combining the estimates on $\Phi^{-1}(\omega\setminus Q_2)$ and on $\Phi^{-1}(Q_2)$.

The proof of the estimate at order 0 relies on the same decomposition and change of variable, noting that in particular $\mathcal{J}\Phi\geq C_1$ to handle the region $\Phi^{-1}(\omega\setminus Q_2)$.

We now move to the fractional estimate when $0<s<1$. We observe that, as in (2.5.4), we have

$$\frac{|\Phi(x)-\Phi(y)|}{|x-y|}\lesssim\frac{\mu\eta}{\zeta(y)}\quad\text{for every } x, y\in\omega. \tag{2.7.3}$$

We start by splitting, in the spirit of the proof of Proposition 2.5.3,

$$\iint\limits_{\substack{\Phi^{-1}(\omega)\times\Phi^{-1}(\omega)\\ \zeta(x)\leq\zeta(y)}} \frac{|u\circ\Phi(x)-u\circ\Phi(y)|^p}{|x-y|^{m+sp}}\,\mathrm{d}x\,\mathrm{d}y = I_1+I_2+I_3+I_4, \tag{2.7.4}$$

where we have set

$$I_1 = \iint\limits_{\substack{\Phi^{-1}(\omega\setminus Q_2)\times\Phi^{-1}(\omega\setminus Q_2)\\ \zeta(x)\le\zeta(y)}} \frac{|u\circ\Phi(x)-u\circ\Phi(y)|^p}{|x-y|^{m+sp}}\,\mathrm{d}x\,\mathrm{d}y, \quad I_2 = \iint\limits_{\substack{\Phi^{-1}(Q_2)\times\Phi^{-1}(Q_2)\\ \zeta(x)\le\zeta(y)}} \frac{|u\circ\Phi(x)-u\circ\Phi(y)|^p}{|x-y|^{m+sp}}\,\mathrm{d}x\,\mathrm{d}y,$$

$$I_3 = \iint\limits_{\substack{\Phi^{-1}(\omega\setminus Q_2)\times\Phi^{-1}(Q_2)\\ \zeta(x)\le\zeta(y)}} \frac{|u\circ\Phi(x)-u\circ\Phi(y)|^p}{|x-y|^{m+sp}}\,\mathrm{d}x\,\mathrm{d}y, \quad I_4 = \iint\limits_{\substack{\Phi^{-1}(Q_2)\times\Phi^{-1}(\omega\setminus Q_2)\\ \zeta(x)\le\zeta(y)}} \frac{|u\circ\Phi(x)-u\circ\Phi(y)|^p}{|x-y|^{m+sp}}\,\mathrm{d}x\,\mathrm{d}y.$$

Estimating the right-hand side of (2.7.4) is similar to Step 2 in the case $0 < s < 1$ of Proposition 2.5.3. The novelty here is that we need to be more careful with the domains on which the estimates are performed. Indeed, in order to obtain (a), we need to estimate the right-hand side of (2.7.4) by a sum of terms that are either preceded by a suitable power of τ, or involve only the energy of u on $\omega \setminus Q_2$.

We begin with I_2. We define

$$\mathscr{B}_{x,y} = B^m_{|\Phi(x)-\Phi(y)|}\left(\frac{\Phi(x)+\Phi(y)}{2}\right) \cap Q_2,$$

so that

$$I_2 \lesssim \int_{\Phi^{-1}(Q_2)} \int_{\Phi^{-1}(Q_2)} \fint_{\mathscr{B}_{x,y}} \frac{|u\circ\Phi(x) - u(z)|^p}{|x-y|^{m+sp}}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$$

We observe that $|\mathscr{B}_{x,y}| \gtrsim |\Phi(x)-\Phi(y)|^m$ due to the fact that Q_2 is a rectangle with comparable sidelengths. Moreover, $|\Phi(x) - z| \le \frac{3}{2}|\Phi(x)-\Phi(y)|$. Hence, using Tonelli's theorem, we find

$$\begin{aligned}\int_{\Phi^{-1}(Q_2)} \int_{\Phi^{-1}(Q_2)} \fint_{\mathscr{B}_{x,y}} &\frac{|u\circ\Phi(x) - u(z)|^p}{|x-y|^{m+sp}}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\ &\lesssim \int_{\Phi^{-1}(Q_2)} \int_{Q_2} \int_{\mathscr{Y}_{x,z}} \frac{|u\circ\Phi(x)-u(z)|^p}{|x-y|^{m+sp}|\Phi(x)-z|^m}\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x,\end{aligned}$$

where

$$\mathscr{Y}_{x,z} = \{y \in \Phi^{-1}(Q_2) \colon z \in \mathscr{B}_{x,y}\} \subset \{y \in \mathbb{R}^m \colon |\Phi(x)-z| \lesssim \frac{\mu\eta}{\zeta(x)}|x-y|\}.$$

Therefore,

$$\int_{\Phi^{-1}(Q_2)} \int_{Q_2} \int_{\mathscr{Y}_{x,z}} \frac{|u \circ \Phi(x) - u(z)|^p}{|x-y|^{m+sp}|\Phi(x) - z|^m} \,\mathrm{d}y \,\mathrm{d}z \,\mathrm{d}x \lesssim \int_{\Phi^{-1}(Q_2)} \int_{Q_2} \frac{|u \circ \Phi(x) - u(z)|^p}{|\Phi(x) - z|^{m+sp}} \frac{(\mu\eta)^{sp}}{\zeta(x)^{sp}} \,\mathrm{d}z \,\mathrm{d}x. \quad (2.7.5)$$

Now we use: (i) the fact that $\zeta(x) \gtrsim \mu\eta\tau$; (ii) the second estimate on $\mathcal{J}\Phi$ — valid on $\Phi^{-1}(Q_2) \subset B_1$ — and (iii) the change of variable theorem to get

$$\int_{\Phi^{-1}(Q_2)} \int_{Q_2} \frac{|u \circ \Phi(x) - u(z)|^p}{|\Phi(x) - z|^{m+sp}} \frac{(\mu\eta)^{sp}}{\zeta(x)^{sp}} \,\mathrm{d}z \,\mathrm{d}x \lesssim \tau^{dl-sp} \int_{Q_2} \int_{Q_2} \frac{|u(x) - u(y)|^p}{|x-y|^{m+sp}} \,\mathrm{d}x \,\mathrm{d}y.$$

The three other terms are handled similarly, so we only point out the required changes. We define instead

$$\mathscr{B}_{x,y} = B^m_{|\Phi(x)-\Phi(y)|}\left(\frac{\Phi(x) + \Phi(y)}{2}\right) \cap (\omega \setminus Q_2).$$

For I_3, we split

$$I_3 \lesssim \int_{\Phi^{-1}(\omega\setminus Q_2)} \int_{\Phi^{-1}(Q_2)} \fint_{\mathscr{B}_{x,y}} \frac{|u \circ \Phi(x) - u(z)|^p}{|x-y|^{m+sp}} \,\mathrm{d}z \,\mathrm{d}y \,\mathrm{d}x + \int_{\Phi^{-1}(\omega\setminus Q_2)} \int_{\Phi^{-1}(Q_2)} \fint_{\mathscr{B}_{x,y}} \frac{|u \circ \Phi(y) - u(z)|^p}{|x-y|^{m+sp}} \,\mathrm{d}z \,\mathrm{d}y \,\mathrm{d}x.$$

Note that we still have $|\mathscr{B}_{x,y}| \gtrsim |\Phi(x) - \Phi(y)|^m$, using this time the assumption on the volume of balls centered in $\omega \setminus Q_2$. We then pursue as for the second term in the right-hand side of (2.7.4): we use Tonelli's theorem, and after that, we integrate with respect to y. Similar to (2.7.5), we deduce that

$$I_3 \lesssim \int_{\Phi^{-1}(\omega\setminus Q_2)} \int_{\omega\setminus Q_2} \frac{|u \circ \Phi(x) - u(z)|^p}{|\Phi(x) - z|^{m+sp}} \frac{(\mu\eta)^{sp}}{\zeta(x)^{sp}} \,\mathrm{d}z \,\mathrm{d}x + \int_{\Phi^{-1}(Q_2)} \int_{\omega\setminus Q_2} \frac{|u \circ \Phi(x) - u(z)|^p}{|\Phi(x) - z|^{m+sp}} \frac{(\mu\eta)^{sp}}{\zeta(x)^{sp}} \,\mathrm{d}z \,\mathrm{d}x.$$

Invoking the change of variable theorem, using the first estimate on $\mathcal{J}\Phi$ with $\beta = sp$ for the first term and the second estimate on $\mathcal{J}\Phi$ for the second term, we conclude that

$$I_3 \leq \left(\int_{\omega\setminus Q_2} \int_{\omega\setminus Q_2} \frac{|u(x) - u(y)|^p}{|x-y|^{m+sp}} \,\mathrm{d}x \,\mathrm{d}y + \tau^{l-sp} \int_{\omega} \int_{\omega} \frac{|u(x) - u(y)|^p}{|x-y|^{m+sp}} \,\mathrm{d}x \,\mathrm{d}y\right).$$

By the exact same reasoning,

$$I_4 \leq \left(\int_{\omega \setminus Q_2} \int_{\omega \setminus Q_2} \frac{|u(x) - u(y)|^p}{|x-y|^{m+sp}} \, \mathrm{d}x \, \mathrm{d}y + \tau^{d-sp} \int_{\omega} \int_{\omega} \frac{|u(x) - u(y)|^p}{|x-y|^{m+sp}} \, \mathrm{d}x \, \mathrm{d}y \right),$$

while

$$I_1 \lesssim \int_{\omega \setminus Q_2} \int_{\omega \setminus Q_2} \frac{|u(x) - u(y)|^p}{|x-y|^{m+sp}} \, \mathrm{d}x \, \mathrm{d}y.$$

Collecting the estimates for the right-hand side of (2.7.4), we arrive at estimate (a) of Proposition 2.7.3.

We finish with the estimate for the Gagliardo seminorm in the case $s \geq 1$. We consider $x, y \in \Phi^{-1}(\omega)$ such that, without loss of generality, $\zeta(x) \leq \zeta(y)$. As usual, using the Faà di Bruno formula, the multilinearity of the differential, and the estimates on the derivatives of Φ, we write

$$\begin{aligned} &|\mathrm{D}^j(u \circ \Phi)(x) - \mathrm{D}^j(u \circ \Phi)(y)| \\ &\quad \lesssim \sum_{i=1}^{j} \bigg(|\mathrm{D}^i u \circ \Phi(x) - \mathrm{D}^i u \circ \Phi(y)| \frac{(\mu\eta)^i}{\zeta(y)^j} \\ &\qquad + \sum_{t=1}^{j} |\mathrm{D}^i u \circ \Phi(x)| |\mathrm{D}^t \Phi(x) - \mathrm{D}^t \Phi(y)| \frac{(\mu\eta)^{i-1}}{\zeta(x)^{j-t}} \bigg). \end{aligned} \tag{2.7.6}$$

For the second term in (2.7.6), we proceed once again by splitting the integral over $B^m_r(x)$ and $\mathbb{R}^m \setminus B^m_r(x)$ with $r = \zeta(x)$ to arrive at

$$\int_{\substack{\Phi^{-1}(\omega) \\ \zeta(x) \leq \zeta(y)}} \frac{|\mathrm{D}^t \Phi(x) - \mathrm{D}^t \Phi(y)|^p}{|x-y|^{m+\sigma p}} \, \mathrm{d}y \lesssim \frac{(\mu\eta)^p}{\zeta(x)^{(t+\sigma)p}}.$$

Hence,

$$\begin{aligned} \iint_{\substack{\Phi^{-1}(\omega) \times \Phi^{-1}(\omega) \\ \zeta(x) \leq \zeta(y)}} &\frac{|\mathrm{D}^i u \circ \Phi(x)|^p |\mathrm{D}^t \Phi(x) - \mathrm{D}^t \Phi(y)|^p}{|x-y|^{m+\sigma p}} \frac{(\mu\eta)^{(i-1)p}}{\zeta(x)^{(j-t)p}} \, \mathrm{d}x \, \mathrm{d}y \\ &\lesssim \int_{\Phi^{-1}(\omega)} |\mathrm{D}^i u \circ \Phi(x)|^p \frac{(\mu\eta)^{ip}}{\zeta(x)^{(j+\sigma)p}} \, \mathrm{d}x. \end{aligned} \tag{2.7.7}$$

We then argue as for the integer order term. We split the integral in the right-hand side of (2.7.7) over the regions $\omega \setminus Q_2$ and Q_2. Owing to the change of variable theorem, using the first estimate on $\mathcal{J}\Phi$ with $\beta = (j+\sigma)p$ over $\omega \setminus Q_2$ and the second estimate on $\mathcal{J}\Phi$ over Q_2, we obtain

$$
\begin{aligned}
&\int_{\Phi^{-1}(\omega)} |\mathrm{D}^i u \circ \Phi(x)|^p \frac{1}{\zeta(x)^{(j+\sigma)p}} \,\mathrm{d}x \\
&\qquad \lesssim (\mu\eta)^{ip-(j+\sigma)p} \int_{\omega\setminus Q_2} |\mathrm{D}^i u|^p + \tau^{l-(j+\sigma)p} (\mu\eta)^{ip-(j+\sigma)p} \int_{Q_2} |\mathrm{D}^i u|^p.
\end{aligned}
$$

As for thickening, the first term in (2.7.6) is handled exactly as in the case $0 < s < 1$, taking into account the presence of the factor $\frac{(\mu\eta)^{ip}}{\zeta(y)^{jp}}$: we use the same splitting as in (2.7.4), and then the usual averaging argument. Doing so, we deduce that the first term in (2.7.6) is bounded from above by a constant multiple of

$$
\begin{aligned}
&\int_{\Phi^{-1}(\omega\setminus Q_2)} \int_{\omega\setminus Q_2} \frac{|\mathrm{D}^i u \circ \Phi(x) - \mathrm{D}^i u(z)|^p}{|\Phi(x) - z|^{m+\sigma p}} \frac{(\mu\eta)^{(i+\sigma)p}}{\zeta(x)^{(j+\sigma)p}} \,\mathrm{d}z \,\mathrm{d}x \\
&\qquad + \int_{\Phi^{-1}(Q_2)} \int_{\omega} \frac{|\mathrm{D}^i u \circ \Phi(x) - \mathrm{D}^i u(z)|^p}{|\Phi(x) - z|^{m+\sigma p}} \frac{(\mu\eta)^{(i+\sigma)p}}{\zeta(x)^{(j+\sigma)p}} \,\mathrm{d}z \,\mathrm{d}x. \qquad (2.7.8)
\end{aligned}
$$

An additional use of the change of variable theorem shows that (2.7.8) is estimated, up to a constant factor, by

$$
\begin{aligned}
&(\mu\eta)^{(i-j)p} \int_{\omega\setminus Q_2} \int_{\omega\setminus Q_2} \frac{|\mathrm{D}^i u(x) - \mathrm{D}^i u(y)|^p}{|x-y|^{m+\sigma p}} \,\mathrm{d}x \,\mathrm{d}y \\
&\qquad + \tau^{l-(j+\sigma)p} (\mu\eta)^{(i-j)p} \int_{\omega} \int_{\omega} \frac{|\mathrm{D}^i u(x) - \mathrm{D}^i u(y)|^p}{|x-y|^{m+\sigma p}} \,\mathrm{d}z \,\mathrm{d}x.
\end{aligned}
$$

Gathering the estimates for both terms in (2.7.6), we obtain the desired conclusion, hence finishing the proof of Proposition 2.7.3. □

Now that we have at our disposal the building blocks for the shrinking procedure, we are ready to prove Proposition 2.7.1. As usual, for the convenience of the reader, we start with an informal presentation of the construction.

We first apply shrinking around the vertices of the dual skeleton $\mathscr{T}^{d^*}$, with parameters $0 < \mu_{m-1} < \nu_m < \mu_m$ and $\frac{\tau\mu}{\nu_m}$, where $\mu_{m-1} \geq \mu$ and $\mu_m \leq 2\mu$. This shrinks a neighborhood of size $\mu_{m-1}\eta$ of these vertices into a neighborhood of size $\tau\mu\eta$. We then apply shrinking around the edges of $\mathscr{T}^{d^*}$ with parameters $0 < \mu_{m-2} < \nu_{m-1} < \mu_{m-1}$ and $\frac{\tau\mu}{\nu_{m-1}}$, where $\mu_{m-2} \geq \mu$. This shrinks the part of a neighborhood of size $\mu_{m-2}\eta$ of the edges of $\mathscr{T}^{d^*}$ lying at distance at most $\mu_{m-1}\eta$ of the $(m-1)$-faces of $\mathscr{K}^m$ into a neighborhood of size $\tau\mu\eta$ of those edges. But since the part of the neighborhood of size $\mu_{m-2}\eta$ lying at distance more than $\mu_{m-1}\eta$ of the $(m-1)$-faces of K^m has already been shrinked during the previous step, we conclude that the whole neighborhood of size $\mu_{m-2}\eta$ of $\mathscr{T}^1$ is shrinked into a neighborhood of size $\tau\mu\eta$. We continue this procedure by downward induction until we reach the dimension d^*, which produces the desired map Φ.

We illustrate this induction procedure in Figures 2.10, 2.11, and 2.12. Here, we take $m = 2$ and $d = 0$. In Figure 2.10, which corresponds to the first step of the induction, the values in the gray region around the center of the cube in the left part of the figure are shrinked into the much smaller gray region on the right. During the next step, depicted in Figure 2.11, the values in gray around the edges of the cube on the left are shrinked into the much smaller gray region around the dual skeleton on the right. The combination of both steps is shown in Figure 2.12. The values in the region in gray on the left are shrinked into the small neighborhood of the dual skeleton in gray on the right.

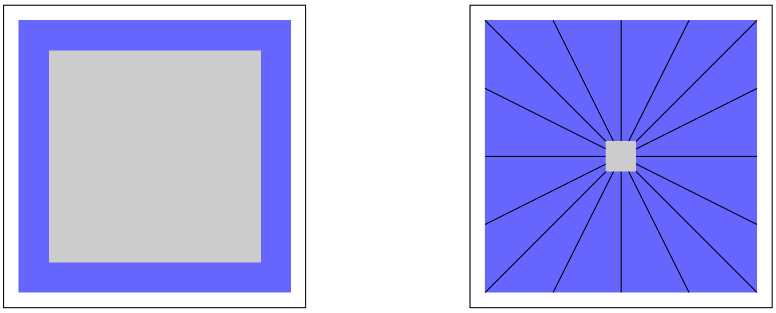

Figure 2.10 – Shrinking around vertices

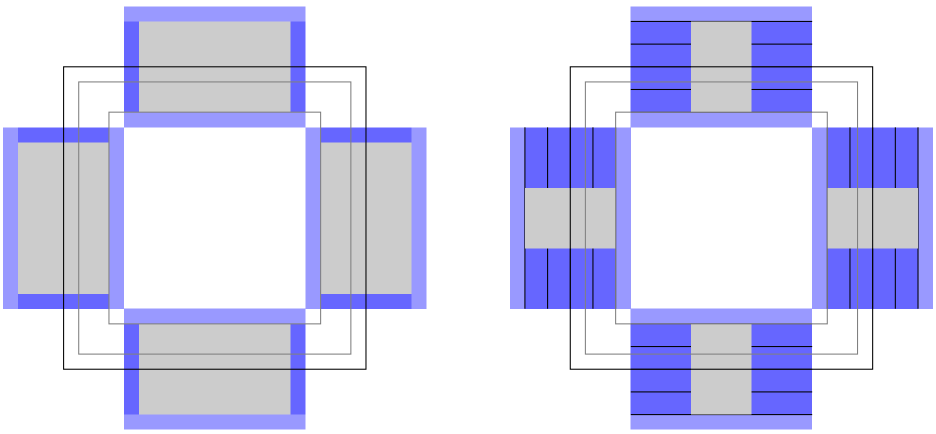

Figure 2.11 – Shrinking around edges

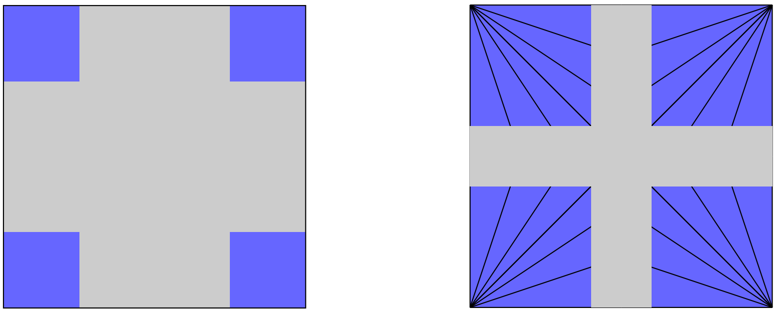

Figure 2.12 – Final shrinking at order 1

As we will see, the induction procedure is more involved than in the case of thickening, and relies on Proposition 2.7.3 applied with domains more general than rectangles.

Proof of Proposition 2.7.1. The map Φ is constructed by downward induction. We consider finite sequences $(\mu_i)_{d\leq i\leq m}$ and $(\nu_i)_{d\leq i\leq m}$ such that

$$0 < \mu = \mu_d < \nu_{d+1} < \mu_{d+1} < \cdots < \mu_{m-1} < \nu_m < \mu_m \leq 2\mu.$$

We first define $\Phi^m = \mathrm{id}$. Then, assuming that Φ^l has been defined for some $l \in \{d+1,\dots,m\}$, we identify any $\sigma^l \in K^l$ with $Q^l_\eta \times \{0\}^{m-l}$, and we let Φ_{σ^l} be the map given by Proposition 2.7.2 applied around σ^l with parameters $\underline{\mu} = \mu_{l-1}$, $\mu = \nu_l$, $\overline{\mu} = \mu_l$, and $\tau\mu/\nu_l$. We define $\Psi^l \colon \mathbb{R}^m \to \mathbb{R}^m$ by

$$\Psi^l(x) = \begin{cases} \Phi_{\sigma^l}(x) & \text{if } x \in T_{\sigma^l}(Q_3) \text{ for some } \sigma^l \in \mathscr{K}^l, \\ x & \text{otherwise,} \end{cases}$$

where T_{σ^l} is an isometry of $\mathbb{R}^m$ mapping $Q^l_\eta \times \{0\}^{m-l}$ to σ^l. We then let $\Phi^{l-1} = \Psi^l \circ \Phi^l$. The required map is given by $\Phi = \Phi^d$.

Properties (i) to (iii) are already contained in [BPVS15, Proposition 8.1], so it only remains to prove the Sobolev estimates. The argument is similar to the one used in the proof of Proposition 2.5.1. We proceed by induction. One of the issues is how to remove inductively neighborhoods of dual skeletons. We let $Q_4 = Q^l_{2\mu\eta} \times Q^{m-l}_{(1-\mu)\eta}$, so that $Q_3 \subset Q_4$ for every $l \in \{d+1,\dots,m\}$. First, we note that invoking Proposition 2.7.3 with $\omega = Q_4 \setminus T^{-1}_{\sigma^l}(\mathscr{T}^{m-l-1} + Q^m_{\mu_l\eta})$ ensures that

(a) if $0 < s < 1$, then

$$|u \circ \Phi_{\sigma^l}|_{W^{s,p}(T_{\sigma^l}(Q_4)\setminus(\mathscr{T}^{m-l-1}+Q^m_{\mu_l\eta}))} \lesssim |u|_{W^{s,p}(T_{\sigma^l}(Q_4)\setminus(\mathscr{T}^{m-l}+Q^m_{\mu_{l-1}\eta}))} + \tau^{\frac{l-sp}{p}} |u|_{W^{s,p}(T_{\sigma^l}(Q_4))};$$

(b) if $s \ge 1$, then for every $j \in \{1, \dots, k\}$,

$$(\mu\eta)^j \|D^j(u \circ \Phi_{\sigma^l})\|_{L^p(T_{\sigma^l}(Q_4)\setminus(\mathcal{T}^{m-l-1}+Q^m_{\mu_l\eta}))} \lesssim \sum_{i=1}^{j}(\mu\eta)^i\|D^i u\|_{L^p(T_{\sigma^l}(Q_4)\setminus(\mathcal{T}^{m-l}+Q^m_{\mu_{l-1}\eta}))} + \tau^{\frac{l-jp}{p}}\sum_{i=1}^{j}(\mu\eta)^i\|D^i u\|_{L^p(T_{\sigma^l}(Q_4))};$$

(c) if $s \ge 1$ and $\sigma \ne 0$, then for every $j \in \{1, \dots, k\}$,

$$\begin{aligned}(\mu\eta)^{j+\sigma}|D^j(u \circ \Phi_{\sigma^l})|_{W^{\sigma,p}(T_{\sigma^l}(Q_4)\setminus(\mathcal{T}^{m-l-1}+Q^m_{\mu_l\eta}))} \\ \lesssim \sum_{i=1}^{j}\Big((\mu\eta)^i\|D^i u\|_{L^p(T_{\sigma^l}(Q_4)\setminus(\mathcal{T}^{m-l}+Q^m_{\mu_{l-1}\eta}))}+(\mu\eta)^{i+\sigma}|D^i u|_{W^{\sigma,p}(T_{\sigma^l}(Q_4)\setminus(\mathcal{T}^{m-l}+Q^m_{\mu_{l-1}\eta}))}\Big) \\ + \tau^{\frac{l-(j+\sigma)p}{p}}\sum_{i=1}^{j}\Big((\mu\eta)^i\|D^i u\|_{L^p(T_{\sigma^l}(Q_4))} + (\mu\eta)^{i+\sigma}|D^i u|_{W^{\sigma,p}(T_{\sigma^l}(Q_4))}\Big);\end{aligned}$$

(d) for every $0 < s < +\infty$,

$$\|u \circ \Phi_{\sigma^l}\|_{L^p(T_{\sigma^l}(Q_4)\setminus(\mathcal{T}^{m-l-1}+Q^m_{\mu_l\eta}))} \lesssim \|u\|_{L^p(T_{\sigma^l}(Q_4)\setminus(\mathcal{T}^{m-l}+Q^m_{\mu_{l-1}\eta}))} + \tau^l\|u\|_{L^p(T_{\sigma^l}(Q_4))}.$$

Indeed, we have: (i) $(T_{\sigma^l}(Q_4) \setminus (\mathcal{T}^{m-l-1} + Q^m_{\mu_l\eta})) \setminus T_{\sigma^l}(Q_2) \subset T_{\sigma^l}(Q_4) \setminus (\mathcal{T}^{m-l} + Q^m_{\mu_{l-1}\eta})$; (ii) $Q_4 \setminus T^{-1}_{\sigma^l}(\mathcal{T}^{m-l-1} + Q^m_{\mu_l\eta}) \subset \Phi^{-1}(\omega)$; and (iii) ω satisfies the condition on the volume of balls required to apply Proposition 2.7.3. Affirmation (ii) is a consequence of the fact that Φ has the specific form $\Phi(x) = (\lambda(x)x', x'')$ with $\lambda\colon \mathbb{R}^m \to [1, +\infty)$. Affirmation(iii) follows from the fact that $\omega \setminus Q_2$ is actually a rectangle to which other rectangles have been removed. We note that, for convenience of notation, we let $T^{-1} = \varnothing$.

Using the additivity of the integral or Lemma 2.2.1 combined with the usual finite number of overlaps argument, we deduce that

(a) if $0 < s < 1$, then

$$\begin{aligned}(\mu\eta)^s|u \circ \Psi^l|_{W^{s,p}(\mathcal{K}^m\cap(\mathcal{T}^{l^*}+Q^m_{2\mu\eta})\setminus(\mathcal{T}^{m-l-1}+Q^m_{\mu_l\eta}))} \\ \lesssim (\mu\eta)^s|u|_{W^{s,p}(\mathcal{K}^m\cap(\mathcal{T}^{l^*}+Q^m_{2\mu\eta})\setminus(\mathcal{T}^{m-l}+Q^m_{\mu_{l-1}\eta}))} + \|u\|_{L^p(\mathcal{K}^m\cap(\mathcal{T}^{l^*}+Q^m_{2\mu\eta})\setminus(\mathcal{T}^{m-l}+Q^m_{\mu_{l-1}\eta}))} \\ + \tau^{\frac{d+1-sp}{p}}\Big((\mu\eta)^s|u|_{W^{s,p}(\mathcal{K}^m\cap(\mathcal{T}^{l^*}+Q^m_{2\mu\eta}))} + \|u\|_{L^p(\mathcal{K}^m\cap(\mathcal{T}^{l^*}+Q^m_{2\mu\eta}))}\Big);\end{aligned}$$

(b) if $s \geq 1$, then for every $j \in \{1, \dots, k\}$,

$$
\begin{aligned}
&(\mu\eta)^j \|\mathrm{D}^j (u \circ \Psi^l)\|_{L^p(\mathscr{K}^m \cap (\mathscr{T}^{l^*} + Q^m_{2\mu\eta}) \setminus (\mathscr{T}^{m-l-1} + Q^m_{\mu_l \eta}))} \\
&\quad \lesssim \sum_{i=1}^{j} (\mu\eta)^i \|\mathrm{D}^i u\|_{L^p(\mathscr{K}^m \cap (\mathscr{T}^{l^*} + Q^m_{2\mu\eta}) \setminus (\mathscr{T}^{m-l} + Q^m_{\mu_{l-1}\eta}))} \\
&\qquad + \tau^{\frac{d+1-sp}{p}} \sum_{i=1}^{j} (\mu\eta)^i \|\mathrm{D}^i u\|_{L^p(\mathscr{K}^m \cap (\mathscr{T}^{l^*} + Q^m_{2\mu\eta}))};
\end{aligned}
$$

(c) if $s \geq 1$ and $\sigma \neq 0$, then for every $j \in \{1, \dots, k\}$,

$$
\begin{aligned}
&(\mu\eta)^{j+\sigma} |\mathrm{D}^j (u \circ \Psi^l)|_{W^{\sigma,p}(\mathscr{K}^m \cap (\mathscr{T}^{l^*} + Q^m_{2\mu\eta}) \setminus (\mathscr{T}^{m-l-1} + Q^m_{\mu_l \eta}))} \\
&\quad \lesssim \sum_{i=1}^{j} \Big((\mu\eta)^i \|\mathrm{D}^i u\|_{L^p(\mathscr{K}^m \cap (\mathscr{T}^{l^*} + Q^m_{2\mu\eta}) \setminus (\mathscr{T}^{m-l} + Q^m_{\mu_{l-1}\eta}))} \\
&\qquad + (\mu\eta)^{i+\sigma} |\mathrm{D}^i u|_{W^{\sigma,p}(\mathscr{K}^m \cap (\mathscr{T}^{l^*} + Q^m_{2\mu\eta}) \setminus (\mathscr{T}^{m-l} + Q^m_{\mu_{l-1}\eta}))} \Big) \\
&\qquad + \tau^{\frac{d+1-sp}{p}} \sum_{i=1}^{j} \Big((\mu\eta)^i \|\mathrm{D}^i u\|_{L^p(\mathscr{K}^m \cap (\mathscr{T}^{l^*} + Q^m_{2\mu\eta}))} \\
&\qquad\quad + (\mu\eta)^{i+\sigma} |\mathrm{D}^i u|_{W^{\sigma,p}(\mathscr{K}^m \cap (\mathscr{T}^{l^*} + Q^m_{2\mu\eta}))} \Big);
\end{aligned}
$$

(d) for every $0 < s < +\infty$,

$$
\begin{aligned}
&\|u \circ \Psi^l\|_{L^p(\mathscr{K}^m \cap (\mathscr{T}^{l^*} + Q^m_{2\mu\eta}) \setminus (\mathscr{T}^{m-l-1} + Q^m_{\mu_l \eta}))} \\
&\qquad \lesssim \|u\|_{L^p(\mathscr{K}^m \cap (\mathscr{T}^{l^*} + Q^m_{2\mu\eta}) \setminus (\mathscr{T}^{m-l} + Q^m_{\mu_{l-1}\eta}))} + \tau^{\frac{d+1-sp}{p}} \|u\|_{L^p(\mathscr{K}^m \cap (\mathscr{T}^{l^*} + Q^m_{2\mu\eta}))}.
\end{aligned}
$$

In particular, since $\tau < 1$, another application of Proposition 2.7.3 yields the following simpler estimates:

(a) if $0 < s < 1$, then

$$
\begin{aligned}
&(\mu\eta)^s |u \circ \Psi^l|_{W^{s,p}(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}))} \\
&\qquad \lesssim (\mu\eta)^s |u|_{W^{s,p}(K^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}))} + \|u\|_{L^p(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}))};
\end{aligned}
$$

(b) if $s \geq 1$, then for every $j \in \{1, \ldots, k\}$,

$$(\mu\eta)^j \|D^j(u \circ \Psi^l)\|_{L^p(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}))} \lesssim \sum_{i=1}^{j} (\mu\eta)^i \|D^i u\|_{L^p(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}))};$$

(c) if $s \geq 1$ and $\sigma \neq 0$, then for every $j \in \{1, \ldots, k\}$,

$$\begin{aligned}
&(\mu\eta)^{j+\sigma} |D^j(u \circ \Psi^l)|_{W^{\sigma,p}(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}))} \\
&\quad \lesssim \sum_{i=1}^{j} \Big((\mu\eta)^i \|D^i u\|_{L^p(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}))} + (\mu\eta)^{i+\sigma} |D^i u|_{W^{\sigma,p}(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}))} \Big);
\end{aligned}$$

(d) for every $0 < s < +\infty$,

$$\|u \circ \Psi^l\|_{L^p(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}))} \lesssim \|u\|_{L^p(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}))}.$$

Combining both these sets of estimates through a downward induction procedure on l, we arrive at the following estimates:

(a) if $0 < s < 1$, then

$$\begin{aligned}
&(\mu\eta)^s |u \circ \Phi|_{W^{s,p}(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}))} \\
&\quad \lesssim (\mu\eta)^s |u|_{W^{s,p}(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}) \setminus (\mathscr{T}^{d^*} + Q^m_{\mu\eta}))} + \|u\|_{L^p(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}) \setminus (\mathscr{T}^{d^*} + Q^m_{\mu\eta}))} \\
&\qquad + \tau^{\frac{d+1-sp}{p}} \Big((\mu\eta)^s |u|_{W^{s,p}(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}))} + \|u\|_{L^p(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}))} \Big);
\end{aligned}$$

(b) if $s \geq 1$, then for every $j \in \{1, \ldots, k\}$,

$$\begin{aligned}
(\mu\eta)^j \|D^j(u \circ \Phi)\|_{L^p(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}))} &\lesssim \sum_{i=1}^{j} (\mu\eta)^i \|D^i u\|_{L^p(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}) \setminus (\mathscr{T}^{d^*} + Q^m_{\mu\eta}))} \\
&\quad + \tau^{\frac{d+1-sp}{p}} \sum_{i=1}^{j} (\mu\eta)^i \|D^i u\|_{L^p(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}))};
\end{aligned}$$

(c) if $s \geq 1$ and $\sigma \neq 0$, then for every $j \in \{1, \dots, k\}$,

$$
\begin{aligned}
&(\mu\eta)^{j+\sigma}|\mathrm{D}^j(u \circ \Phi)|_{W^{\sigma,p}(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}))} \\
&\qquad \lesssim \sum_{i=1}^{j} \Big((\mu\eta)^i \|\mathrm{D}^i u\|_{L^p(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}) \setminus (\mathscr{T}^{d^*} + Q^m_{\mu\eta}))} \\
&\qquad\qquad + (\mu\eta)^{i+\sigma}|\mathrm{D}^i u|_{W^{\sigma,p}(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}) \setminus (\mathscr{T}^{d^*} + Q^m_{\mu\eta}))} \Big) \\
&\qquad\quad + \tau^{\frac{d+1-sp}{p}} \sum_{i=1}^{j} \Big((\mu\eta)^i \|\mathrm{D}^i u\|_{L^p(K^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}))} \\
&\qquad\qquad + (\mu\eta)^{i+\sigma}|\mathrm{D}^i u|_{W^{\sigma,p}(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}))} \Big);
\end{aligned}
$$

(d) for every $0 < s < +\infty$,

$$
\begin{aligned}
&\|u \circ \Phi\|_{L^p(\mathscr{K}^m \cap (T^{d^*} + Q^m_{2\mu\eta}))} \\
&\qquad \lesssim \|u\|_{L^p(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}) \setminus (\mathscr{T}^{d^*} + Q^m_{\mu\eta}))} + \tau^{\frac{d+1-sp}{p}} \|u\|_{L^p(\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta}))}.
\end{aligned}
$$

The conclusion follows from the fact that $u = v$ outside of $\mathscr{T}^{d^*} + Q^m_{\mu\eta}$, by noting that actually $\operatorname{Supp}\Phi \subset \mathscr{T}^{d^*} + Q^m_{\nu_m\eta}$, and using once again the additivity of the integral or Lemma 2.2.1. □

2.8 Density of smooth maps

In view of the density of the class $\mathscr{R}$, in order to prove the density of smooth maps, it suffices to show that maps of the class $\mathscr{R}$ may be approximated by smooth maps with values into $\mathscr{N}$. As we already announced, the basic idea to do so is to remove the singularities of maps in the class $\mathscr{R}$ by filling them with a smooth map. The key tool in this direction is the following lemma, which relies on the fact that $\mathscr{K}^d$ is a homotopy retract of the complement $\mathscr{K}^m \setminus \mathscr{T}^{d^*}$ of the dual skeleton $\mathscr{T}^{d^*}$. The statement we present is from [BPVS15, Proposition 7.1], but similar ideas were already used, e.g., in [Whi86, Section 1], [Haj94, Section 2], or [HL03a, Section 6].

Lemma 2.8.1. *Let K^m be a cubication in $\mathbb{R}^m$ of radius $\eta > 0$, $d \in \{0, \dots, m-1\}$, T^{d^*} the dual skeleton of K^d, and $u \in C^\infty(\mathscr{K}^m \setminus \mathscr{T}^{d^*}; \mathscr{N})$. If there exists $f \in C^0(\mathscr{K}^m; \mathscr{N})$ such that $f_{|\mathscr{K}^d} = u_{|\mathscr{K}^d}$, then for every $0 < \mu < 1$, there exists $v \in C^\infty(\mathscr{K}^m; \mathscr{N})$ such that $v = u$ on $\mathscr{K}^m \setminus (\mathscr{T}^{d^*} + Q^m_{\mu\eta})$.*

In order to apply Lemma 2.8.1, it is useful to know when a continuous map from $\mathscr{K}^d$

to $\mathscr{N}$ may be extended to a continuous map from $\mathscr{K}^m$ to $\mathscr{N}$. Following Hang F. and Lin F. [HL03a], we introduce the notion of *extension property*.

Definition 2.8.2. *Let K^m be a cubication in $\mathbb{R}^m$ and $d \in \{0, \dots, m-1\}$. We say that K^m has the d-extension property with respect to $\mathscr{N}$ whenever, every continuous map $f\colon \mathscr{K}^d \to \mathscr{N}$ has an extension $g \in C^0(\mathscr{K}^m; \mathscr{N})$.*

This definition is slightly different from the one given by Hang F. and Lin F. [HL03a], and more in the spirit of the recent work by P. Bousquet, A. Ponce, and J. Van Schaftingen [BPVS25]. In [HL03a], Hang F. and Lin F. required that

$$\begin{aligned}&\text{for every continuous map } f\colon \mathscr{K}^{d+1} \to \mathscr{N},\\ &f_{|\mathscr{K}^d} \text{ has an extension } g \in C^0(\mathscr{K}^m; \mathscr{N}).\end{aligned} \tag{2.8.1}$$

To draw the parallel between both settings, let us mention that the d-extension property as stated in Definition 2.8.2 is equivalent to the assumption that (2.8.1) holds along with $\pi_d(\mathscr{N}) = \{0\}$.

The identification of the key role played by the extension property in the strong density problem was one of the major contributions of [HL03a]. In this respect, we start with the following proposition, which provides an approximation result for maps in the class $\mathscr{R}$ as the ones used in the proof of Theorem 2.6.1. All the other results in this section will be deduced from this proposition.

Proposition 2.8.3. *Let K^m be a cubication in $\mathbb{R}^m$. Let $d \in \{0, \dots, m-1\}$ be such that $d = \lfloor sp \rfloor$, and T^{d^*} the dual skeleton of K^d. If K^m has the d-extension property with respect to $\mathscr{N}$, then $C^\infty(\mathscr{K}^m; \mathscr{N})$ is dense in $C^\infty(\mathscr{K}^m \setminus \mathscr{T}^{d^*}; \mathscr{N}) \cap W^{s,p}(\mathscr{K}^m; \mathscr{N})$ with respect to the $W^{s,p}$ distance.*

Proof. Let $u \in C^\infty(\mathscr{K}^m \setminus \mathscr{T}^{d^*}; \mathscr{N}) \cap W^{s,p}(\mathscr{K}^m; \mathscr{N})$. We denote by η the radius of the cubication K^m. The d-extension property of K^m with respect to $\mathscr{N}$ then ensures that $u_{|\mathscr{K}^d}$ extends to a continuous map from $\mathscr{K}^m$ to $\mathscr{N}$. Therefore, Lemma 2.8.1 implies that, for every $0 < \mu < 1$, there exists a map $u^{\mathrm{ex}}_\mu \in C^\infty(\mathscr{K}^m; \mathscr{N})$ such that $u^{\mathrm{ex}}_\mu = u$ on $\mathscr{K}^m \setminus (\mathscr{T}^{d^*} + Q^m_{\mu\eta})$.

We now apply shrinking to this map u^{ex}_μ. More precisely, we assume that $\mu < \frac{1}{2}$, we take $0 < \tau < \frac{1}{2}$ and we define $u^{\mathrm{sh}}_{\tau,\mu} = u^{\mathrm{ex}}_\mu \circ \Phi^{\mathrm{sh}}_{\tau,\mu}$, where $\Phi^{\mathrm{sh}}_{\tau,\mu}$ is provided by Proposition 2.7.1. By Proposition 2.7.1 and the remark that follows that statement, choosing $\tau = \tau_\mu$ sufficiently small and $\varepsilon = \mu$, we deduce that, with $\mathscr{A}_\mu = \mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta})$,

(a) if $0 < s < 1$, then

$$(\mu\eta)^s |u^{\mathrm{sh}}_{\tau_\mu,\mu} - u|_{W^{s,p}(\mathscr{K}^m)} \lesssim (\mu\eta)^s |u|_{W^{s,p}(\mathscr{A}_\mu)} + \|u\|_{L^p(\mathscr{A}_\mu)} + \mu;$$

(b) if $s \geq 1$, then for every $j \in \{1, \dots, k\}$,

$$(\mu\eta)^j \|\mathrm{D}^j u^{\mathrm{sh}}_{\tau_\mu,\mu} - \mathrm{D}^j u\|_{L^p(\mathscr{K}^m)} \lesssim \sum_{i=1}^{j} (\mu\eta)^i \|\mathrm{D}^i u\|_{L^p(\mathscr{A}_\mu)} + \mu;$$

(c) if $s \geq 1$ and $\sigma \neq 0$, then for every $j \in \{1, \dots, k\}$,

$$(\mu\eta)^{j+\sigma} |\mathrm{D}^j u^{\mathrm{sh}}_{\tau_\mu,\mu} - \mathrm{D}^j u|_{W^{\sigma,p}(\mathscr{K}^m)} \lesssim \sum_{i=1}^{j} \Big((\mu\eta)^i \|\mathrm{D}^i u\|_{L^p(\mathscr{A}_\mu)} + (\mu\eta)^{i+\sigma} |\mathrm{D}^i u|_{W^{\sigma,p}(\mathscr{A}_\mu)} \Big) + \mu;$$

(d) for every $0 < s < +\infty$,

$$\|u^{\mathrm{sh}}_{\tau_\mu,\mu} - u\|_{L^p(\mathscr{K}^m)} \lesssim \|u\|_{L^p(\mathscr{A}_\mu)} + \mu.$$

Using the compactness of $\mathscr{N}$ and the fact that $u \in W^{s,p}(\mathscr{K}^m)$, we deduce from the Gagliardo–Nirenberg inequality that $D^i u \in L^{\frac{sp}{i}}(\mathscr{K}^m)$. Therefore, by Hölder's inequality,

$$\|\mathrm{D}^i u\|_{L^p(\mathscr{A}_\mu)} \leq |\mathscr{K}^m \cap (\mathscr{T}^{d^*} + Q^m_{2\mu\eta})|^{\frac{s-i}{sp}} \|\mathrm{D}^i u\|_{L^{sp/i}(\mathscr{A}_\mu)}.$$

Similarly, using Lemma 2.6.2, for every $i \in \{1, \dots, k-1\}$, we find that

$$|\mathrm{D}^i u|_{W^{\sigma,p}(\mathscr{A}_\mu)} \lesssim |\mathscr{A}_\mu|^{\frac{s-i-\sigma}{sp}} \|\mathrm{D}^i u\|^{1-\sigma}_{L^{sp/i}(\mathscr{A}_\mu)} \|\mathrm{D}^{i+1} u\|^{\sigma}_{L^{sp/(i+1)}(\mathscr{A}_\mu)}.$$

(Strictly speaking, Lemma 2.6.2 requires to work with two open sets $\omega \Subset \Omega$, with Ω being convex. However, we already saw that this assumption may be easily bypassed in the situation we are facing here. Indeed, this can be done, for instance, relying on the existence of a continuous extension operator from $W^{s,p}(\mathscr{K}^m; \mathbb{R}^\nu)$ to $W^{s,p}(\mathbb{R}^m; \mathbb{R}^\nu)$.) Moreover, using the fact that $u \in L^\infty(\mathscr{K}^m)$ since $\mathscr{N}$ is compact, we have

$$\|u\|_{L^p(\mathscr{A}_\mu)} \lesssim |\mathscr{A}_\mu|^{\frac{1}{p}}.$$

On the other hand, we observe that $|\mathscr{A}_\mu| \lesssim (\mu\eta)^{d+1}$. Therefore,

(a) if $0 < s < 1$, then

$$|u^{\mathrm{sh}}_{\tau_\mu,\mu} - u|_{W^{s,p}(\mathscr{K}^m)} \lesssim |u|_{W^{s,p}(\mathscr{A}_\mu)} + (\mu\eta)^{\frac{d+1-sp}{p}} + \mu;$$

(b) if $s \geq 1$, then for every $j \in \{1, \ldots, k\}$,

$$\|D^j u^{\mathrm{sh}}_{\tau_\mu,\mu} - D^j u\|_{L^p(\mathscr{K}^m)} \lesssim \sum_{i=1}^{j} (\mu\eta)^{i-j+\frac{s-i}{sp}(d+1)} \|D^i u\|_{L^{sp/i}(\mathscr{A}_\mu)} + \mu;$$

(c) if $s \geq 1$ and $\sigma \neq 0$, then for every $j \in \{1, \ldots, k\}$,

$$\begin{aligned} |D^j u^{\mathrm{sh}}_{\tau_\mu,\mu} - D^j u|_{W^{\sigma,p}(\mathscr{K}^m)} &\lesssim \sum_{i=1}^{j} (\mu\eta)^{i-j-\sigma+\frac{s-i}{sp}(d+1)} \|D^i u\|_{L^{sp/i}(\mathscr{A}_\mu)} \\ &+ \sum_{i=1}^{k-1} (\mu\eta)^{i-j+\frac{s-i-\sigma}{sp}(d+1)} \|D^i u\|^{1-\sigma}_{L^{sp/i}(\mathscr{A}_\mu)} \|D^{i+1} u\|^{\sigma}_{L^{sp/(i+1)}(\mathscr{A}_\mu)} + |D^j u|_{W^{\sigma,p}(\mathscr{A}_\mu)} + \mu; \end{aligned}$$

(d) for every $0 < s < +\infty$,

$$\|u^{\mathrm{sh}}_{\tau_\mu,\mu} - u\|_{L^p(\mathscr{K}^m)} \lesssim (\mu\eta)^{\frac{d+1}{p}} + \mu.$$

Since $sp < d+1$, we observe that all the powers on $\mu\eta$ above are positive. Moreover, since $|\mathscr{A}_\mu| \to 0$ as $\mu \to 0$, we deduce from Lebesgue's lemma that all Lebesgue norms and Gagliardo seminorms above tend to 0 when $\mu \to 0$.

This shows that $u^{\mathrm{sh}}_{\tau_\mu,\mu} \to u$ in $W^{s,p}(\mathscr{K}^m)$, and since $u^{\mathrm{sh}}_{\tau_\mu,\mu} \in C^\infty(\mathscr{K}^m; \mathscr{N})$, the proof is complete. □

Theorem 1.4.5 in the introduction follows from Proposition 2.8.3 by using the fact that a ball satisfies (2.8.1) for any target manifold $\mathscr{N}$. This was already present in [HL03a]. For a proof, the reader may also consult [BPVS15, Proposition 7.3]. (Both these references deal with a cube instead of a ball, but the extension property is preserved by homeomorphism.)

More generally, in the case where the domain is an open subset of $\mathbb{R}^m$, we have the following result.

Theorem 2.8.4. *If $\Omega \subset \mathbb{R}^m$ is a bounded open set which satisfies the segment condition, and if we may find a sequence $(\eta_n)_{n\in\mathbb{N}}$ of positive real numbers such that $\eta_n \to 0$ and such that for every $n \in \mathbb{N}$, the cubication $K^m_{\eta_n}$ used in the proof of Theorem 2.6.4 satisfies the $\lfloor sp \rfloor$-extension property with respect to $\mathscr{N}$, then $H^{s,p}_S(\Omega; \mathscr{N}) = W^{s,p}(\Omega; \mathscr{N})$.*

Under an assumption as weak as the segment condition, it is not clear how to link the topology of the cubications $\mathscr{K}^m_{\eta_n}$ containing Ω to the topology of Ω itself. However, in the case where Ω is a smooth domain, the topological assumption above can be clarified. Indeed, in this case, using a retraction along the normal vector to $\partial\Omega$, one may show that, if K^m_η is a cubication of radius $\eta > 0$ in $\mathbb{R}^m$ for $\eta > 0$ sufficiently small such that K^m_η is made only of cubes that intersect $\overline{\Omega}$, then $\mathscr{K}^m_\eta$ is homotopic to Ω. This implies that, if we endow Ω with a structure of CW-complex, then the d-extension property of K^m_d is equivalent to the d-extension property of Ω, and this does not depends on the choice of CW-complex structure on Ω; see e.g. [HL03a, Section 2]. Here, analogously to the definition on a cubication, we say that Ω has the d-extension property with respect to $\mathscr{N}$ whenever any map $f \in C^0(\Omega^d; \mathscr{N})$ has an extension $g \in C^0(\Omega; \mathscr{N})$, where Ω^d denotes the d-skeleton of the CW-complex structure on Ω.

This leads to the following theorem.

Theorem 2.8.5. *Let $\Omega \subset \mathbb{R}^m$ be a smooth bounded open domain. If $sp < m$ and if Ω has the $\lfloor sp \rfloor$-extension property with respect to $\mathscr{N}$, then $H^{s,p}_S(\overline{\Omega}; \mathscr{N}) = W^{s,p}(\Omega; \mathscr{N})$.*

We finally consider the general case where the domain is a smooth compact, connected Riemannian manifold $\mathscr{M}$ of dimension m, isometrically embedded in $\mathbb{R}^N$. As we did for Theorem 2.6.5, we may restrict to the case where $\mathscr{M}$ has empty boundary, since the general case reduces to this special case by embedding into a larger manifold without boundary.

In this setting, a tubular neighborhood of $\mathscr{M}$ is homotopic to $\mathscr{M}$ through the nearest point projection, and therefore has the d-extension property if and only if $\mathscr{M}$ has the d-extension property. We may thus proceed as for Theorem 2.6.5 to deduce the following result.

Theorem 2.8.6. *If $sp < m$ and if $\mathscr{M}$ has the $\lfloor sp \rfloor$-extension property with respect to $\mathscr{N}$, then $H^{s,p}_S(\mathscr{M}; \mathscr{N}) = W^{s,p}(\mathscr{M}; \mathscr{N})$.*

Proof. We first assume that $\mathscr{M}$ has an empty boundary. Let $\iota > 0$ be the radius of a tubular neighborhood of $\mathscr{M}$, let Π denote the nearest point projection onto $\mathscr{M}$, and let $\Omega = \mathscr{M} + B^N_{\iota/2}$. Given $u \in W^{s,p}(\mathscr{M}; \mathscr{N})$, as explained before the proof of Theorem 2.6.5, the map $v = u \circ \Pi$ belongs to $W^{s,p}(\Omega; \mathscr{N})$. By the observation above, Ω has the $\lfloor sp \rfloor$-extension property. Therefore, by Theorem 2.8.5, there exists a sequence $(v_n)_{n\in\mathbb{N}}$ in $C^\infty(\overline{\Omega}; \mathscr{N})$ which converges to v in $W^{s,p}(\Omega)$. We conclude as in the proof of Theorem 2.6.5, using a slicing argument to find a sequence $(a_n)_{n\in\mathbb{N}}$ in $B^N_{\iota/2}$ such that $a_n \to 0$ as $n \to +\infty$ satisfying $u_n = \tau_{a_n}(v_n)_{|\mathscr{M}} \to u$ in $W^{s,p}(\mathscr{M})$.

The case where $\mathscr{M}$ is allowed to have non-empty boundary is deduced from the empty boundary case exactly as for the proof of Theorem 2.6.5, and we therefore omit the proof. □

Chapter 3

An improved dense class

Résumé

Dans ce chapitre, on pousse plus avant l'étude de la question (Q2) portant sur la densité des fonctions presque lisses. Dans le Chapitre 2, nous avons travaillé avec une classe d'applications qui sont lisses en dehors d'une union finie de sous-variétés de dimension $(m - \lfloor sp \rfloor - 1)$. Dans ce chapitre, nous montrons qu'il est possible de se restreindre à la classe plus réduite des applications qui sont lisses en dehors d'*une seule* sous-variété de dimension $(m - \lfloor sp \rfloor - 1)$, ce qui interdit à l'ensemble singulier d'exhiber des croisements. Ceci répond à une question posée par H. Brezis et P. Mironescu.

Dans le cas particulier où la topologie de $\mathscr{N}$ est suffisamment simple et pour certaines valeurs de s et p, ce résultat était déjà connu comme une conséquence de la *méthode de la projection*, dont les origines remontent aux travaux de H. Federer et W. Fleming. Comme premier résultat, nous mettons en œuvre cette méthode dans la gamme complète des s et p pour lesquels il était attendu que la méthode s'applique. Dans le cas d'une cible arbitraire, nous mettons au point un argument topologique permettant de supprimer les croisements dans l'ensemble singulier des applications presque lisses obtenues *via* la technique de F. Bethuel et que nous avons étudiée dans le Chapitre 2.

Abstract

In this chapter, we push further the study of question (Q2) concerning the density of almost smooth maps. In Chapter 2, the class we worked with is a set of maps which are smooth outside of a finite union of $(m - \lfloor sp \rfloor - 1)$-dimensional submanifolds. In this chapter, we show that one can restrict to the smaller class of maps that are smooth outside of *one* $(m - \lfloor sp \rfloor - 1)$-dimensional submanifold, hence ruling out the possibility for the singular set to exhibit crossings. This answers a question raised by H. Brezis and P. Mironescu.

In the special case where $\mathscr{N}$ has a sufficiently simple topology and for some values of s and p, this result was known to follow from the *method of projection*, which takes its roots in the work of H. Federer and W. Fleming. As a first result, we implement

this method in the full range of s and p in which it was expected to be applicable. In the case of a general target manifold, we devise a topological argument that allows to remove the self-intersections in the singular set of the maps obtained via Bethuel's technique and that we investigated in Chapter 2.

3.1 Introduction

In this chapter, we study the density of an improved version of the class $\mathscr{R}$. For the sake of convenience, in this chapter only, we work with the following slightly more stringent definition of the class $\mathscr{R}$ for maps defined on bounded open sets of $\mathbb{R}^m$. We let $\mathscr{R}_i(\Omega;\mathscr{N})$ be the set of all maps u such that there exists a set $\mathscr{S} = \mathscr{S}_u \subset \mathbb{R}^m$ which is a finite union of closedly embedded i-dimensional submanifolds of $\mathbb{R}^m$ and such that $u \in C^\infty(\overline{\Omega} \setminus \mathscr{S};\mathscr{N})$ and

$$|\mathrm{D}^j u(x)| \leq C\frac{1}{\operatorname{dist}(x,\mathscr{S})^j} \quad \text{for every } x \in \Omega \text{ and } j \in \mathbb{N}_*,$$

where $C > 0$ is a constant depending on u and j.

We recall importantly that the singular set $\mathscr{S}$ *depends on the map* u. Moreover, when we write $u \in C^\infty(\overline{\Omega} \setminus \mathscr{S};\mathscr{N})$, this means that there exists an open set $U \subset \mathbb{R}^m$ such that $\Omega \Subset U$ and a map $v \in C^\infty(U \setminus \mathscr{S};\mathscr{N})$ such that $u = v$ on $\Omega \setminus \mathscr{S}$. By *closedly embedded*, we mean that the manifold should be a closed subset of $\mathbb{R}^m$, which should not be confused with a *closed manifold*, which is a compact manifold without boundary. The difference with the definition we work with in the other chapters is that $\mathscr{S}$ is a submanifold of the whole $\mathbb{R}^m$ instead of merely Ω. As a consequence, the estimate on the derivatives of u may depend on parts of $\mathscr{S}$ that lie outside of Ω — although we may always restrict to the part of $\mathscr{S}$ lying in a neighborhood of $\overline{\Omega}$, enlarging the constant C if necessary. This technical detail will be of crucial importance for us later on, when we require stability properties of the class $\mathscr{R}$ under composition with local diffeomorphisms for instance.

We now introduce the following subclass of the class $\mathscr{R}$ on open domains.

Definition 3.1.1. *The class $\mathscr{R}_i^{\mathrm{uncr}}(\Omega;\mathscr{N})$ is the set of all $u \in \mathscr{R}_i(\Omega;\mathscr{N})$ such that the singular set $\mathscr{S}$ is a closedly embedded i-dimensional submanifold of $\mathbb{R}^m$.*

Our main result in this chapter reads as follows.

Theorem 3.1.2. *Assume that $sp < m$. The class $\mathscr{R}^{\mathrm{uncr}}_{m-\lfloor sp\rfloor-1}(Q^m;\mathscr{N})$ is dense in $W^{s,p}(Q^m;\mathscr{N})$.*

We recall that $Q^m = (-1,1)^m$ is the open unit cube in $\mathbb{R}^m$. The key feature of Theorem 3.1.2 above is to assert that one may *avoid the crossings* in the singular sets of the almost smooth maps that are dense in $W^{s,p}(Q^m;\mathscr{N})$. Indeed, since the singular set of a map in $\mathscr{R}_{m-\lfloor sp\rfloor-1}(Q^m;\mathscr{N})$ is a *union* of submanifolds, it may exhibit crossings at the points where those manifolds intersect. In fact, the singular sets of the maps constructed in the existing proofs of the density of almost smooth maps for arbitrary target manifolds, such that the one we presented in Chapter 2, *do* exhibit crossings, as they arise as dual skeletons of decompositions of the domain into cubes. In Chapter 2, the crossings come from the application of the thickening procedure, explained in Section 2.5. On the other hand, $\mathscr{R}^{\mathrm{uncr}}$ corresponds to all the maps in $\mathscr{R}$ such that their singular set is *uncrossed*, that is, does not have crossings. This explains our choice of notation for the class $\mathscr{R}^{\mathrm{uncr}}$.

We emphasize that, in full generality, Theorem 3.1.2 is new *even in the case* $s = 1$. This answers in particular a question raised by H. Brezis and P. Mironescu; see e.g. the discussion in [BM21, Chapter 10].

Up to now, the only approach to prove strong density results that was able to provide the density of the class $\mathscr{R}^{\mathrm{uncr}}$ instead of merely the class $\mathscr{R}$ was based on the *method of projection*, famously devised by H. Federer and W. Fleming [FF60] in their study of normal and integral currents. We refer the reader to Section 1.4.4 in the introduction for the history of the method as well as a sketch of its application for sphere-valued maps. In the first part of this chapter, in Section 3.2, we show that the method of projection indeed works in its full expected applicability range, that is, for any $0 < s < +\infty$ and any $(\lfloor sp\rfloor - 1)$-connected target manifold $\mathscr{N}$. This answers a question raised by P. Bousquet, A. Ponce, and J. Van Schaftingen; see [BPVS14, Section 2]. Although not allowing to prove Theorem 3.1.2 in its full generality, this result is interesting per se: (i) it gives the full range of applicability of the method of singular projection; (ii) it provides a much simpler proof of Theorem 2.6.4 in the particular case of an $(\lfloor sp\rfloor - 1)$-connected target; and (iii) it has the advantage of applying to a general domain Ω, unlike our proof of Theorem 3.1.2.

We briefly explain the additional difficulties arising when implementing the method of projection in the full range $0 < s < +\infty$ of fractional Sobolev spaces. The main novel difficulty in our setting is that we need to establish fractional estimates for a general singular projection. Indeed, up to now, these estimates either were obtained by relying on the specific form of the projection for a particular target, as in [Bou07] for $\mathscr{N} = \mathbb{S}^1$, or were deduced from the integer order estimates through the Gagliardo–Nirenberg interpolation inequality when $0 < s < 1$, as in [BPVS14]. However, for $s > 1$, this approach would force us to exclude some relevant values of the parameters s and p.

To illustrate the need for direct estimates, let us see what can be obtained by interpolation. Assume for instance that one wants to prove the density of the class $\mathscr{R}_0(\mathbb{B}^2;\mathbb{S}^1)$ in $W^{s,p}(\mathbb{B}^2;\mathbb{S}^1)$ in the case $1 \leq sp < 2$ — which is the only relevant one. One typically wants to interpolate $W^{s,p}$ between L^r and $W^{k,q}$, with $k \in \mathbb{N}$ satisfying $k > s$. For this to hold, one is lead to choose r and q satisfying the relation

$$\frac{1}{p} = \frac{1-\theta}{r} + \frac{\theta}{q},$$

where $\theta \in (0,1)$ satisfies

$$s = 0 \cdot (1-\theta) + k\theta,$$

that is, $\theta = s/k$. The key assumption to implement successfully Federer and Fleming's averaging argument over a, which essentially requires that the $W^{k,q}$-norm of $x \mapsto \frac{x}{|x|}$ over $\mathbb{B}^2$ should be finite, is therefore that $kq < 2$; see the model computation in (1.4.4) in the introduction for the case $k = 1$. If $0 < s < 1$, then we may take $k = 1$, and hence the condition is $q < 2$. But if $1 < s < 2$, then $k \geq 2$, and this implies that q should be chosen less that 1, which is not possible. Therefore, one sees that some ranges of values of s and p that are relevant in the problem of strong density of the class $\mathscr{R}$ cannot be handled by interpolation when $s > 1$ is not an integer. For the record, we note that the above model case is exactly the one treated by P. Bousquet [Bou07], using direct fractional estimates relying on the specific form of the singular projection when the target is a circle. Similarly, for maps $\mathbb{B}^3 \to \mathbb{S}^2$ with $2 < sp < 3$, one cannot handle the case $2 < s < 3$ by means of interpolation. To the best of our knowledge, no direct estimates are available in the existing literature for this case, and hence the method of projection could not be implemented in this setting up to now.

Nevertheless, even knowing the full range of validity of the method of projection is not sufficient to prove Theorem 3.1.2 in full generality. Indeed, as will be proved in Lemma 3.2.2, a singular projection is only available when the target is $(\lfloor sp \rfloor - 1)$-connected. Moreover, as will be discussed in Section 3.2.3, there is little hope that even all $(\lfloor sp \rfloor - 1)$-connected manifolds admit a singular projection whose singular set is a submanifold, which is required to deduce the density of the uncrossed class $\mathscr{R}^{\mathrm{uncr}}$ using the method of projection. Therefore, proving Theorem 3.1.2 in the general case requires to find a different approach. This is the purpose of the second part of this chapter, in Section 3.3.

We introduce a more rigid version of the class $\mathscr{R}$, where the singular set is required to coincide with the dual skeleton of some cubication of $\mathbb{R}^m$.

Definition 3.1.3. *The class $\mathscr{R}_i^{\mathrm{rig}}(\Omega;\mathscr{N})$ is the set of all $u \in \mathscr{R}_i(\Omega;\mathscr{N})$ such that the singular set $\mathscr{S}$ is the i-dimensional dual skeleton of some cubication of $\mathbb{R}^m$.*

The fact that the class $\mathscr{R}^{\mathrm{rig}}$ — recall that we omit the subscript when we want to speak about the class $\mathscr{R}$ or one of its variants without specifying the dimension of the singular set — is a subclass of the class $\mathscr{R}$ is a consequence of the fact that the singular set of maps in $\mathscr{R}^{\mathrm{rig}}$ may be taken to be a finite union of i-dimensional hyperplanes. Indeed, the dual skeleton of a cubication of $\mathbb{R}^m$ is a union of affine spaces, and one may keep only a finite number of them since Ω is bounded.

It is a consequence of the proof of Theorem 2.6.4 — but not of Theorem 2.6.4 itself — that the more rigid class $\mathscr{R}^{\mathrm{rig}}_{m-\lfloor sp\rfloor-1}(\Omega;\mathscr{N})$ is dense in $W^{s,p}(\Omega;\mathscr{N})$. Indeed, the maps in $\mathscr{R}_{m-\lfloor sp\rfloor-1}$ that are constructed in the proof of Theorem 2.6.4 to approximate a given map in $W^{s,p}$ actually belong to $\mathscr{R}^{\mathrm{rig}}_{m-\lfloor sp\rfloor-1}$, using the fact that the thickening procedure produces a singular set corresponding to the dual skeleton of the cubication used in the proof.

To prove the density of the class $\mathscr{R}^{\mathrm{uncr}}$, it therefore suffices to be able to approximate any map in $\mathscr{R}^{\mathrm{rig}}$. This is the main goal of Section 3.3. Our proof is in two main steps. First, we devise a topological procedure that removes the crossings between the orthogonal hyperplanes constituting the singular set of a general map in $\mathscr{R}^{\mathrm{rig}}$. This procedure, which itself consists of several steps, only requires to modify the initial map on a small set, but comes without any estimate. In order to obtain, from the previous construction, a better map with suitable estimates, we rely in a second step on the shrinking procedure from [BPVS15] that we presented in Section 2.7.

The new ingredient is therefore the topological procedure to uncross the singularities. This procedure is explained in Section 3.3.1 in particular cases that allow for more simple notation and illustrative figures, before the general case, presented in Section 3.3.2. At the core of the argument lies the following model problem. It is well-known that the 1-skeleton $\mathscr{K}^1$ of the unit cube $\overline{Q^3}$ is a retract of $\overline{Q^3}\setminus\mathscr{T}^1$, where T^1 is the dual skeleton of K^1. Is it possible to write instead $\mathscr{K}^1$ as a retract of $\overline{Q^3}\setminus\mathscr{S}$, where $\mathscr{S}$ is a 1-dimensional *submanifold* of $\mathbb{R}^3$, that is, without crossing? Although it may come as very surprising, the answer to this question is actually *yes*. Elaborating on the construction allowing to obtain such a retraction is the cornerstone of the topological step of our proof in Section 3.3.

As a concluding remark, we comment on the dimension of the singular set in the class $\mathscr{R}$ and its variants. Indeed, as we explained, the content of Theorem 3.1.2 is to provide the strong density of an improved version of the class $\mathscr{R}$. Another natural idea to improve the density result given by Theorem 1.4.5 would be to try reducing the dimension of the singular set, that is, to prove the density of the class $\mathscr{R}_i$ for some $i < m-\lfloor sp\rfloor-1$. However, and as we already explained in the introduction, it turns out that, in presence of the topological obstruction ruling out the density of $C^\infty(\Omega;\mathscr{N})$ in $W^{s,p}(\Omega;\mathscr{N})$, the only value of i for which $\mathscr{R}_i$ is dense in $W^{s,p}$ is $i = m-\lfloor sp\rfloor-1$. For smaller i, the same topological obstruction also rules out the density of the class $\mathscr{R}_i$, while for larger i, $\mathscr{R}_i$ is not even a subset of $W^{s,p}$. See [BPVS15, Section 6] for a detailed proof in the case where $s \in \mathbb{N}_*$. The argument may be carried out similarly for fractional order spaces.

3.2 The method of singular projection

3.2.1 Singular projections: definitions and main result

This section is devoted to the study of the full range of applicability of the method of singular projection for density problems. We start by defining the precise notion of singular projection that we consider.

Definition 3.2.1. *Let $\ell \in \{2, \dots, \nu\}$. An ℓ-singular projection onto $\mathscr{N}$ is a smooth map $P\colon \mathbb{R}^\nu \setminus \Sigma \to \mathscr{N}$ such that $P_{|\mathscr{N}} = \mathrm{id}_{\mathscr{N}}$ and*

$$|\mathrm{D}^j P(x)| \leq C \frac{1}{\operatorname{dist}(x, \Sigma)^j} \quad \text{for every } x \in \mathbb{R}^\nu \setminus \Sigma \text{ and } j \in \mathbb{N}_*$$

for some constant $C > 0$ depending on j and $\mathscr{N}$, where $\Sigma \subset \mathbb{R}^\nu \setminus \mathscr{N}$ is either the underlying set of a finite $(\nu - \ell)$-subskeleton in $\mathbb{R}^\nu$ or a closedly embedded $(\nu - \ell)$-submanifold of $\mathbb{R}^\nu$.

At this stage, the reader may wonder why we split the form of allowed singular sets for singular projections into two types, instead of considering more generally maps that are singular outside of a finite union of closedly embedded $(\nu - \ell)$-submanifolds of $\mathbb{R}^\nu$, which would include both cases in Definition 3.2.1. The answer is given by the combination of the two following lemmas.

Lemma 3.2.2. *If there exists a continuous map $P\colon \mathbb{R}^\nu \setminus \Sigma \to \mathscr{N}$ such that $P_{|\mathscr{N}} = \mathrm{id}_{\mathscr{N}}$, where Σ is a finite union of closedly embedded $(\nu - \ell)$-submanifolds of $\mathbb{R}^\nu$, then $\mathscr{N}$ is $(\ell - 2)$-connected.*

Lemma 3.2.3. *If $\mathscr{N}$ is $(\ell - 2)$-connected, then it admits an ℓ-singular projection, whose singular set is the underlying set of a finite $(\nu - \ell)$-subskeleton in $\mathbb{R}^\nu$.*

We first comment Lemma 3.2.2 and 3.2.3 before giving their proof. The combination of both these lemmas shows at the same time that the $(\ell - 2)$-connectedness of $\mathscr{N}$ is a *necessary and sufficient* condition for the existence of an ℓ-singular projection onto $\mathscr{N}$, and that allowing the singular set to be a finite union of closedly embedded $(\nu - \ell)$-submanifolds of $\mathbb{R}^\nu$ would not have broadened the range of target manifolds admitting a singular projection. Meanwhile, assuming that Σ is the underlying set of a finite $(\nu - \ell)$-subskeleton in $\mathbb{R}^\nu$ will allow for technical simplifications in the sequel, in addition to being the natural form of singular set arising when performing the proof of Lemma 3.2.3.

We also have allowed for singular sets that are given by *one* submanifold of $\mathbb{R}^\nu$, that is, that do not exhibit crossings, because we are precisely interested in studying the density of the class $\mathscr{R}_i^{\mathrm{uncr}}$, whose maps have a singular set without crossings.

Proof of Lemma 3.2.2. The key observation is the fact that $\mathbb{R}^\nu \setminus \Sigma$ is $(\ell - 2)$-connected. Taking this for granted, the conclusion follows directly from the fact that a retract of an $(\ell - 2)$-connected space is itself $(\ell - 2)$-connected. Namely, for every $i \in \{0, \dots, \ell - 2\}$, we have the commutative diagram

$$\begin{tikzcd}
 & \{0\} = \pi_i(\mathbb{R}^\nu \setminus \Sigma) \arrow[dr, "P_*"] & \\
\pi_i(\mathscr{N}) \arrow[ur, hook] \arrow[rr, "\mathrm{id}_{\mathscr{N}}"] & & \pi_i(\mathscr{N})
\end{tikzcd}$$

for the maps induced between the homotopy groups. This implies that the identity map on the group $\pi_i(\mathscr{N})$ is the zero map, whence $\pi_i(\mathscr{N}) = \{0\}$ for every $i \in \{0, \dots, \ell - 2\}$.

The fact that $\mathbb{R}^\nu \setminus \Sigma$ is $(\ell - 2)$-connected is presumably well-known, but it seems difficult to find a proof of it in the general case, so we provide one for the convenience of the reader. One may consult [MVS21a, Lemma 3.8] for a proof in the case where Σ is an affine space. Our argument relies on the same idea. We show that, if U is an $(\ell - 2)$-connected open subset of $\mathbb{R}^\nu$ and $\mathscr{O}$ a closedly embedded $(\nu - \ell)$-submanifold of U, then $U \setminus \mathscr{O}$ is $(\ell - 2)$-connected. The conclusion then follows by removing inductively each manifold constituting Σ, and using this claim at each step to show that the resulting set remains $(\ell - 2)$-connected.

To prove the claim, let $i \in \{0, \dots, \ell - 2\}$ and $f \colon \mathbb{S}^i \to U \setminus \mathscr{O}$ be a continuous map. Since U is $(\ell - 2)$-connected, there exists a continuous map $g \colon \overline{\mathbb{B}^{i+1}} \to U$ such that $g = f$ on $\mathbb{S}^i$. Moreover, by a standard regularization process, we may assume that g is smooth on $\mathbb{B}^{i+1}$. Since $f(\mathbb{S}^i)$ is closed and $U \setminus \mathscr{O}$ is open, there exists $\delta > 0$ sufficiently small such that, for any $a \in B_\delta^\nu$, $(f(\mathbb{S}^i) - a) \cap (U \setminus \mathscr{O}) = \varnothing$. Now, we invoke the following particular case of Lemma 3.2.7 that will be proved below using Sard's theorem: since $i + 1 \leq \ell - 1 < \ell$, for almost every $a \in B_\delta$, we have $g^{-1}(\mathscr{O} + a) = \varnothing$. This implies that, for any such $a \in B_\delta^\nu$, $g - a$ is a continuous extension to $\overline{\mathbb{B}^{i+1}}$ of $f - a$, whence $f - a$ is nullhomotopic. But by our choice of δ, $f - a$ and f are homotopic, which implies that f itself is nullhomotopic. This concludes the proof of the lemma. □

Proof of Lemma 3.2.3. We follow the approach in [VS19, Proposition 4.4]. Let $\iota > 0$ be sufficiently small so that the there exists a smooth retraction Π onto $\mathscr{N}$ defined on $\mathscr{N} + B_\iota^\nu$. Let K^ν be a cubication of $\mathbb{R}^\nu$ of radius $r > 0$. Choosing $r > 0$ sufficiently small, we may find a subskeleton U^ν of K^ν such that $\mathscr{N} \subset \mathscr{U}^\nu \subset \mathscr{N} + B_{\iota/2}$. We let V^ν be the subskeleton of K^ν consisting in all cubes in K^ν that do not intersect some cube in U^ν, and $W^\nu = K^\nu \setminus (U^\nu \cup V^\nu)$.

We define P on $\mathscr{U}^\nu$ by $P = \Pi$, and on $\mathscr{V}^\nu$, we set $P = b$ for some arbitrary $b \in \mathscr{N}$. Hence, it remains to define P on $\mathscr{W}^\nu$. We proceed by induction. For any $\sigma^0 \in W^0$ such that P is not yet defined on σ^0, we let $P = b$ at σ^0. Then, for any $\sigma^1 \in W^1$ on which P is not yet defined, we use the assumption $\pi_0(\mathscr{N}) = \{0\}$ to extend P on σ^1 from its values on W^0. Repeating this process up to dimension $\ell - 1$, we define P on the whole $\mathscr{W}^{\ell-1}$. Finally, we use successive homogeneous extensions to extend P on $\mathscr{W}^\nu \setminus \mathscr{T}^{(\ell-1)^*}$, where $T^{(\ell-1)^*}$ is the dual skeleton to $W^{\ell-1}$. Here, we recall that the homogeneous extension to $\overline{Q^i} \setminus \{0\}$ of a map f defined on ∂Q^i is given by $x \mapsto f(x/|x|_\infty)$. Hence, a first step of homogeneous extension allows us to define P on $\mathscr{W}^\ell$, with one singularity at the center of each ℓ-cell. A second step extends P on $\mathscr{W}^{\ell+1}$, with a singular set given by a finite union of segments, whose endpoints are located at the centers of the $(\ell+1)$-cells and at the centers of the ℓ-cells from the previous step. We pursue this construction up to dimension ν, each step increasing the dimension of the singular set by 1. By the properties of homogeneous extension, the map P that we constructed is indeed a singular projection, with singular set given by $\Sigma = \mathscr{T}^{(\ell-1)^*}$, which concludes the proof.

We observe however that the above argument produces only a Lipschitz map. To obtain a smooth map, one should slightly modify the first step, which relies on topological extension, to define smoothly P on $\mathscr{W}^{\ell-1} + B^\nu_{r/2}$ instead of merely $\mathscr{W}^{\ell-1}$. Then, one should use the thickening procedure instead of homogeneous extension, in order to get a smooth map outside of $\mathscr{T}^{(\ell-1)^*}$ with the required estimates for all derivatives. We also refer the reader to the work of Gastel [Gas16, Proposition 1] for a more detailed but slightly different proof. □

Now that we have defined a precise notion of singular projection, we may state the main result of this section.

Theorem 3.2.4. *Assume that $\Omega \subset \mathbb{R}^m$ is a bounded open set satisfying the segment condition, and that there exists an ℓ-singular projection $P\colon \mathbb{R}^\nu \setminus \Sigma \to \mathscr{N}$ with $sp < \ell$. The class $\mathscr{R}_{m-\ell}(\Omega;\mathscr{N})$ is dense in $W^{s,p}(\Omega;\mathscr{N})$. If in addition Σ is a $(\nu-\ell)$-submanifold of $\mathbb{R}^\nu$, then $\mathscr{R}^{\mathrm{uncr}}_{m-\ell}(\Omega;\mathscr{N})$ is dense in $W^{s,p}(\Omega;\mathscr{N})$.*

As explained in the introduction, in the particular case where the target manifold admits an ℓ-singular projection with $sp < \ell$, this result provides at the same time a simpler proof of the density of the class $\mathscr{R}$ with crossings, which corresponds to Bethuel's theorem and its counterpart for arbitrary $0 < s < +\infty$ (see Chapter 2), and of our main result concerning the density of the uncrossed class $\mathscr{R}^{\mathrm{uncr}}$ provided that the singular set of the target manifold has no crossing.

We note the following important particular case.

Corollary 3.2.5. *Let $sp < d+1$. The class $\mathscr{R}^{\mathrm{uncr}}_{m-d-1}(\Omega; \mathbb{S}^d)$ is dense in $W^{s,p}(\Omega; \mathbb{S}^d)$.*

The case $d = 1$ was already known [Bou07, Theorem 2], but the other cases are presumably new in the general case $0 < s < +\infty$. We note that, as mentioned in the introduction, the case $s = 1$ is already contained in [BZ88] and the case $0 < s < 1$ was proved in [BBM05], see also [BPVS14].

This corollary is also a good opportunity to emphasize that the method of singular projection *does not always provide the good singular set*. Indeed, Corollary 3.2.5 above gives the same size of singular set for every s and p such that $sp < d+1$. However, if $sp < d$, since then $\pi_{\lfloor sp \rfloor}(\mathscr{N}) = \{0\}$, we actually have density of *smooth maps* in $W^{s,p}(\Omega; \mathbb{S}^d)$, while Corollary 3.2.5 only provides the density of the class $\mathscr{R}^{\mathrm{uncr}}_{m-d-1}(\Omega; \mathbb{S}^d)$.

Proof. We note that $P\colon \mathbb{R}^{d+1} \setminus \{0\} \to \mathbb{S}^d$ defined by $P(x) = \frac{x}{|x|}$ is a $(d+1)$-singular projection, and invoke Theorem 3.2.4. □

A similar result holds for the torus $\mathbb{T}^2$, for which a singular 2-projection whose singular set is the union of a circle inside the torus and a line passing through the hole of the torus may be constructed by hand.

Corollary 3.2.6. *Let $sp < 2$. The class $\mathscr{R}^{\mathrm{uncr}}_{m-2}(\Omega; \mathbb{T}^2)$ is dense in $W^{s,p}(\Omega; \mathbb{T}^2)$.*

Consider now the two-holed torus $\mathbb{T}^2 \# \mathbb{T}^2$. Theorem 3.2.4 also applies to this target, but since the singular set constructed by hand — or using Lemma 3.2.3 and the fact that $\mathbb{T}^2 \# \mathbb{T}^2$ is connected — exhibits crossings, we only obtain the density of the class $\mathscr{R}_{m-2}$. One may wonder whether or not it is possible to construct a better singular projection onto $\mathbb{T}^2 \# \mathbb{T}^2$ whose singular set would be a submanifold, to deduce the density of the class $\mathscr{R}^{\mathrm{uncr}}_{m-2}$. We are not able to answer this question, but the discussion in Section 3.2.3 suggests that there is little hope that the answer is *yes*.

3.2.2 Approximation by singular projection

We now turn to the proof of Theorem 3.2.4. The strategy is to rely on classical approximation by convolution, and then project back the approximating maps to the target manifold using the singular projection. Therefore, a first key step is to control the regularity of the singular set which is obtained through this process. In addition, we need a control on the derivatives of the projected map near the singular set. This is the purpose of the following lemma, based on Sard's theorem and the submersion theorem.

Lemma 3.2.7. *Let $v \in C^\infty(\Omega; \mathbb{R}^\nu)$ and let $\Sigma \subset \mathbb{R}^\nu$ be a finite union of $(\nu - \ell)$-dimensional submanifolds of $\mathbb{R}^\nu$. For almost every $a \in \mathbb{R}^\nu$,* (*i*) *the set* $v^{-1}(\Sigma + a)$ *is a finite union of* $(m-\ell)$*-dimensional submanifolds of* Ω*, one for each manifold constituting* Σ *— or the empty set if* $\ell > m$*;*

(*ii*) *if* $\ell \leq m$*, for every compact* $K \subset \Omega$*, there exists a constant* $C > 0$ *depending on* Σ*,* v*,* ℓ*,* K*, and* a *such that, for every* $x \in K$*,*

$$\operatorname{dist}(x, v^{-1}(\Sigma + a)) \leq C \operatorname{dist}(v(x), \Sigma + a).$$

Lemma 3.2.7 is a slight generalization of [BPVS14, Lemma 2.3] to the case where Σ is an arbitrary union of submanifolds, not necessarily affine spaces. The proof of the first part follows the argument given in [BPVS14], but for the second part, we give a different proof, by contradiction.

Proof. For (i), it suffices to consider the case where Σ is made of only one submanifold, as the general case then follows by taking the union over all manifolds constituting Σ. Consider the map $\Psi \colon \Omega \times \Sigma \to \mathbb{R}^\nu$ defined by

$$\Psi(x, z) = v(x) - z.$$

Since Ψ is a smooth map between smooth manifolds, Sard's theorem — see e.g. [Bre93, Chapter II.6] — ensures that, for almost every $a \in \mathbb{R}^\nu$, the linear map $D\Psi(x, z) \colon \mathbb{R}^m \times T_z\Sigma \to \mathbb{R}^\nu$ is surjective for every $(x, z) \in \Psi^{-1}(\{a\})$. If $\ell > m$, this already implies the conclusion, since the domain of this linear map has dimension $m + (\nu - \ell) < \nu$. Therefore, $D\Psi(x, z)$ is never surjective, which forces $\Psi^{-1}(\{a\}) = \varnothing$ for almost every $a \in \mathbb{R}^\nu$. We note that this corresponds to the *easy* case of Sard's theorem, which is nothing else but the fact that the image of a smooth map is a null set when the dimension of the codomain is strictly higher than the dimension of the domain.

If $\ell \leq m$, we pursue by observing that for any $a \in \mathbb{R}^\nu$ such that $D\Psi(x, z)$ is surjective,

$$\mathbb{R}^\nu = D\Psi(x, z)[\mathbb{R}^m \times T_z\Sigma] = Dv(x)[\mathbb{R}^m] + T_z\Sigma \quad \text{for every } (x, z) \in \Psi^{-1}(\{a\}).$$

Furthermore, by definition, we have $(x, z) \in \Psi^{-1}(\{a\})$ if and only if $v(x) = z + a \in \Sigma + a$. Hence, we conclude that

$$\mathbb{R}^\nu = Dv(x)[\mathbb{R}^m] + T_{v(x)}(\Sigma + a) \quad \text{for every } x \in v^{-1}(\Sigma + a).$$

Otherwise stated, for almost every $a \in \mathbb{R}^\nu$, the map v is transversal to $\Sigma + a$. This implies that — see for instance [War83, Theorem 1.39] — for almost every $a \in \mathbb{R}^\nu$, $v^{-1}(\Sigma + a)$ is a smooth submanifold of $\mathbb{R}^m$ of dimension $m - \ell$.

We now turn to the proof of (ii). Once again, it suffices to prove the assertion when

Σ is made of one manifold, since the distance to a union of sets is the minimum of the distances to all those sets. We assume without loss of generality that $a = 0$. Assume by contradiction that there exists a compact set $K \subset \Omega$ and a sequence $(x_n)_{n\in\mathbb{N}}$ in K such that

$$\operatorname{dist}(x_n, v^{-1}(\Sigma)) > n \operatorname{dist}(v(x_n), \Sigma).$$

We note that $x_n \notin v^{-1}(\Sigma)$, otherwise we would have $0 > 0$. As K is compact, up to extraction, we may assume that $x_n \to x \in K$ as $n \to +\infty$. We observe that $\operatorname{dist}(v(x), \Sigma) = 0$, which implies that $v(x) \in \Sigma$, and hence $x \in v^{-1}(\Sigma)$.

For $n \in \mathbb{N}$ sufficiently large, let y_n be the nearest point projection of x_n onto $v^{-1}(\Sigma)$. Since $x_n \notin v^{-1}(\Sigma)$, we have $x_n \neq y_n$. Moreover, by construction of the nearest point projection, we know that $x_n - y_n \in T_{y_n} v^{-1}(\Sigma)^\perp$ for every $n \in \mathbb{N}$, and also $|x_n - y_n| = \operatorname{dist}(x_n, v^{-1}(\Sigma))$. In particular, $y_n \to x$. Up to a further extraction, we may assume that

$$\frac{x_n - y_n}{|x_n - y_n|} \to \xi \in T_x v^{-1}(\Sigma)^\perp \quad \text{as } n \to +\infty.$$

Since v is continuously differentiable, we deduce that

$$\frac{v(x_n) - v(y_n)}{\operatorname{dist}(x_n, v^{-1}(\Sigma))} = \frac{v(x_n) - v(y_n)}{|x_n - y_n|} \to \mathrm{D}v(x)[\xi] \quad \text{as } n \to +\infty.$$

Let us note that, since we are in the situation where

$$\mathbb{R}^\nu = \mathrm{D}v(x)[\mathbb{R}^m] + T_{v(x)}\Sigma,$$

we have

$$\mathbb{R}^\nu = \mathrm{D}v(x)[T_x v^{-1}(\Sigma)^\perp] \oplus T_{v(x)}\Sigma.$$

Indeed, this follows from the fact that $\mathrm{D}v(x)[\zeta] \in T_{v(x)}\Sigma$ for every $\zeta \in T_x v^{-1}(\Sigma)$ and a dimension argument. Therefore, up to replacing the usual scalar product on $\mathbb{R}^\nu$ with a new one, we may assume that the two subspaces involved in the above direct sum are actually orthogonal. This only modifies the distances by a multiplicative constant. Let Π_Σ denote the nearest point projection onto Σ relative to the metric induced by this new scalar product.

By the triangle inequality, we write

$$\frac{|v(x_n) - v(y_n)|}{\operatorname{dist}(x_n, v^{-1}(\Sigma))} \leq \frac{|v(x_n) - \Pi_\Sigma(v(x_n))|}{\operatorname{dist}(x_n, v^{-1}(\Sigma))} + \frac{|\Pi_\Sigma(v(x_n)) - \Pi_\Sigma(v(y_n))|}{\operatorname{dist}(x_n, v^{-1}(\Sigma))},$$

where we made use of the fact that $\Pi_{\Sigma}(v(y_n)) = v(y_n)$ since $v(y_n) \in \Sigma$. We note that $\Pi_{\Sigma}(v(x_n))$ is well-defined for n sufficiently large, as $v(x_n)$ is then close to Σ. The first term in the right-hand side converges to 0 as $n \to +\infty$ by the assumption over $(x_n)_{n\in\mathbb{N}}$, as $|v(x_n) - \Pi_{\Sigma}(v(x_n))| = \operatorname{dist}(v(x_n), \Sigma)$. Concerning the other term, since $\Pi_{\Sigma} \circ v$ is continuously differentiable in a neighborhood of x, we have

$$\frac{\Pi_{\Sigma}(v(x_n)) - \Pi_{\Sigma}(v(y_n))}{\operatorname{dist}(x_n, v^{-1}(\Sigma))} \to \mathrm{D}\Pi_{\Sigma}(v(x))[\mathrm{D}v(x)[\xi]] \quad \text{as } n \to +\infty.$$

Since $\mathrm{D}\Pi_{\Sigma}(v(x))$ is, by construction of the nearest point projection, the orthogonal projection onto $T_{v(x)}\Sigma$, we have $\mathrm{D}\Pi_{\Sigma}(v(x))[\mathrm{D}v(x)[\xi]] = 0$ as a consequence of our choice of scalar product and the fact that $\xi \in T_x v^{-1}(\Sigma)^{\perp}$.

Hence, we conclude that

$$\frac{|v(x_n) - v(y_n)|}{\operatorname{dist}(x_n, v^{-1}(\Sigma))} \to 0 \quad \text{as } n \to +\infty.$$

This implies that $\mathrm{D}v(x)[\xi] = 0$. But, since $\xi \in T_x v^{-1}(\Sigma)^{\perp}$ is a nonzero vector, this contradicts the fact that

$$\mathbb{R}^{\nu} = \mathrm{D}v(x)[T_x v^{-1}(\Sigma)^{\perp}] \oplus T_{v(x)}\Sigma,$$

and concludes the proof. □

The next lemma provides a mean value-type estimate for the derivatives of a singular projection.

Lemma 3.2.8. *Let $\omega \subset \mathbb{R}^{\nu}$ be a bounded set and $P\colon \mathbb{R}^{\nu} \setminus \Sigma \to \mathscr{N}$ be a singular projection. For every $x, y \in \omega \setminus \Sigma$ such that* $\operatorname{dist}(x, \Sigma) \leq \operatorname{dist}(y, \Sigma)$ *and for every $j \in \mathbb{N}_*$,*

$$|\mathrm{D}^j P(x) - \mathrm{D}^j P(y)| \leq C \frac{|x - y|}{\operatorname{dist}(x, \Sigma)^{j+1}}$$

for some constant $C > 0$ depending on Σ and the diameter of ω.

Proof. We claim that there exists $\delta > 0$ depending only on Σ such that, whenever $|x - y| \leq \delta$ and $\operatorname{dist}(x, \Sigma) < \delta$, there exists a Lipschitz path $\gamma\colon [0, 1] \to \mathbb{R}^{\nu} \setminus \Sigma$ satisfying $\gamma(0) = x$, $\gamma(1) = y$, $|\gamma'| \lesssim |x - y|$, and $\operatorname{dist}(\gamma(t), \Sigma) \geq \operatorname{dist}(x, \Sigma)$ for every $t \in [0, 1]$. The conclusion of the lemma follows directly from this claim. Indeed, if $|x - y| \leq \delta$ and $\operatorname{dist}(x, \Sigma) < \delta$, it suffices to apply the mean value theorem along the path γ and to use the estimates on the derivatives of P. If instead $|x - y| \geq \delta$, since $\operatorname{dist}(x, \Sigma)$ is bounded from above on ω, we have

$$|\mathrm{D}^j P(x) - \mathrm{D}^j P(y)| \lesssim \frac{1}{\operatorname{dist}(x, \Sigma)^j} \lesssim \frac{|x-y|}{\operatorname{dist}(x, \Sigma)^{j+1}}.$$

In the case where $|x - y| \leq \delta$ but $\operatorname{dist}(x, \Sigma) \geq \delta$, we only have to invoke the mean value theorem along the straight line between x and y.

We turn to the proof of the claim. We first assume that Σ is a closedly embedded submanifold of $\mathbb{R}^\nu$. We take $R > 0$ so large that $\omega \subset B^\nu_R$, and by a compactness argument, we choose $0 < \delta < R$ sufficiently small so that there exist finitely many open sets U_1, $\dots, U_j \subset \mathbb{R}^\nu$ such that for any $z \in \Sigma \cap B^\nu_{2R}$, there exists $i \in \{1, \dots, j\}$ with $B^\nu_{2\delta}(z) \subset U_i$, and there exist diffeomorphisms $\Phi_i \colon U_i \to B^{\nu-\ell}_{r_i} \times B^\ell_{s_i}$ for some r_i, $s_i > 0$, satisfying $\Phi_i(\Sigma \cap U_i) = B^{\nu-\ell}_{r_i} \times \{0\}$ and such that for every $a \in U_i$, $\operatorname{dist}(a, \Sigma)$ is given by the norm of the second component of $\Phi_i(a)$. Choose $z \in \Sigma \cap B^\nu_{2R}$ such that $x \in B^\nu_\delta(z)$, so that $y \in B^\nu_{2\delta}(z)$. Let $i \in \{1, \dots, j\}$ with $B^\nu_{2\delta}(z) \subset U_i$. We observe that we may connect $\Phi(z)$ and $\Phi(y)$ in $B^{\nu-\ell}_{r_i} \times B^\ell_{s_i}$ by a Lipschitz path $\tilde{\gamma} \colon [0, 1] \to B^{\nu-\ell}_{r_i} \times B^\ell_{s_i}$ with $|\gamma'| \lesssim |\Phi(x) - \Phi(y)|$ and such that the norm of the second component of $\tilde{\gamma}$ is always larger than $\operatorname{dist}(x, \Sigma)$. The conclusion follows by defining $\gamma = \Phi^{-1} \circ \tilde{\gamma}$.

In the case where Σ is a subskeleton, we observe that one may obtain a suitable γ as a succession of line segments and arcs of circle. □

The proof of Theorem 3.2.4 relies on approximation by convolution. It will be instrumental for us to establish estimates for the distance between the convoluted map and the original one, and also estimates on the derivatives of the convoluted map. To state the required estimates in the fractional setting, we introduce the *fractional derivative* as

$$\mathrm{D}^{\sigma,p} v(x) = \left(\int_\Omega \frac{|v(x) - v(y)|^p}{|x-y|^{m+\sigma p}} \,\mathrm{d}y \right)^{\frac{1}{p}}.$$

Let also $\varphi \in C^\infty_c(\mathbb{B}^m)$ be a fixed mollifier, that is,

$$\varphi \geq 0 \quad \text{on } \mathbb{B}^m \quad \text{and} \quad \int_{\mathbb{B}^m} \varphi = 1.$$

Given $\eta > 0$, we define

$$\varphi_\eta(x) = \frac{1}{\eta^m} \varphi\left(\frac{x}{\eta}\right) \quad \text{for every } x \in \mathbb{R}^m.$$

Lemma 3.2.9 corresponds to [BPVS14, Lemma 2.4]. We present the proof for the sake of completeness.

Lemma 3.2.9. *Assume that* $0 < \sigma < 1$ *and let* $v \in W^{\sigma,p}(\Omega;\mathbb{R}^\nu)$*. For every* $\eta > 0$ *and for every* $x \in \Omega$ *such that* $\eta < \operatorname{dist}(x, \partial\Omega)$*,*

(*i*) $|\varphi_\eta * v(x) - v(x)| \leq C\eta^\sigma D^{\sigma,p} v(x)$;

(*ii*) $|D(\varphi_\eta * v)(x)| \leq C'\eta^{\sigma-1} D^{\sigma,p} v(x)$;

for some constants $C > 0$ *depending on* φ *and* $C' > 0$ *depending on* $D\varphi$*.*

Proof. Jensen's inequality ensures that

$$\begin{aligned}|\varphi_\eta * v(x) - v(x)| &\leq \int_{B^m_\eta} \varphi_\eta(h)|v(x-h) - v(x)|\,\mathrm{d}h \\ &\leq \Bigg(\int_{B^m_\eta} \varphi_\eta(h)\eta^{m+\sigma p}\frac{|v(x-h)-v(x)|^p}{|h|^{m+\sigma p}}\,\mathrm{d}h\Bigg)^{\frac{1}{p}}.\end{aligned}$$

Since $\varphi_\eta \lesssim \eta^{-m}$, we conclude that

$$|\varphi_\eta * v(x) - v(x)| \leq C\eta^\sigma D^{\sigma,p} v(x).$$

This proves the first part of the conclusion.

For the second part, by differentiating under the integral, we find

$$D(\varphi_\eta * v)(x) = \int_{B^m_\eta} D\varphi_\eta(h) v(x-h)\,\mathrm{d}h.$$

As $\int_{B^m_\eta} D\varphi_\eta = 0$, we may write

$$|D(\varphi_\eta * v)(x)| \leq \int_{B^m_\eta} |D\varphi_\eta(h)||v(x-h) - v(x)|\,\mathrm{d}h.$$

Since $\int_{B^m_\eta} |D\varphi_\eta| \lesssim \eta^{-1}$, Jensen's inequality ensures that

$$|D(\varphi_\eta * v)(x)| \lesssim \frac{1}{\eta^{\frac{p-1}{p}}}\Bigg(\int_{B^m_\eta} |D\varphi_\eta(h)|\eta^{m+\sigma p}\frac{|v(x-h)-v(x)|^p}{|h|^{m+\sigma p}}\,\mathrm{d}h\Bigg)^{\frac{1}{p}}.$$

We conclude as above by using the fact that $|D\varphi_\eta| \leq \eta^{-m-1}$. □

We are now ready to prove Theorem 3.2.4. As explained in the introduction, we follow the approach in [BPVS14]. However, as we already mentioned, the range where $s \geq 1$ is not an integer is more difficult, as we cannot rely on interpolation, and we need to establish directly estimates on the Gagliardo seminorm.

Proof of Theorem 3.2.4. Let $u \in W^{s,p}(\Omega;\mathscr{N})$. By a standard dilation procedure, we may assume that $u \in W^{s,p}(\omega;\mathscr{N})$ for some open set $\omega \subset \mathbb{R}^m$ such that $\overline{\Omega} \subset \omega$. In particular, there exists $\gamma > 0$ such that $\operatorname{dist}(\Omega;\partial\omega) > 2\gamma$. We note that this is the only point in the proof where we use the regularity of Ω, and that assuming merely the segment condition is sufficient to implement a dilation argument, as we did in Lemma 2.6.3 in Chapter 2. Therefore, for any $0 < \eta \leq \gamma$, the map $u_\eta = \varphi_\eta * u$ is well-defined and smooth on $\Omega_\gamma = \Omega + B^m_\gamma$. After an extension procedure, using e.g. a cutoff function, we may assume that u_η is actually a smooth (non necessarily $\mathscr{N}$-valued) map on the whole $\mathbb{R}^m$, that coincides with $\varphi_\eta * u$ on Ω_γ. Hence, for any $a \in \mathbb{R}^\nu$, the map $v_{\eta,a} = P \circ (u_\eta - a)$ satisfies $v_{\eta,a} \in C^\infty(\mathbb{R}^m \setminus S_{\eta,a};\mathscr{N})$, where $S_{\eta,a} = u_\eta^{-1}(\Sigma + a)$. We recall that $\Sigma \subset \mathbb{R}^\nu$ is the singular set of the singular projection P onto $\mathscr{N}$. Moreover, in the case where Σ is a subskeleton, by adding extra cells if necessary, we may assume that it is a finite union of $(\nu - \ell)$-hyperplanes. By Lemma 3.2.7, we deduce that $S_{\eta,a}$ is a finite union of closed submanifolds of $\mathbb{R}^m$ for almost every $a \in \mathbb{R}^\nu$, and actually a closed submanifold of $\mathbb{R}^m$ when Σ is a submanifold. Additionally, the required estimates on the derivatives of the maps $v_{\eta,a}$ allowing to deduce that they belong to the class $\mathscr{R}_{m-\ell}$ follow from the Faà di Bruno formula as in (3.2.2) below, combined with point (ii) of Lemma 3.2.7 and the fact that u_η has bounded derivatives on Ω. We are going to show that, for every $0 < \eta \leq \gamma$, we may choose such an $a_\eta \in \mathbb{R}^\nu$ so that $a_\eta \to 0$ as $\eta \to 0$ and $v_{\eta,a_\eta} \to u$ in $W^{s,p}(\Omega)$, and this will conclude the proof of the theorem.

For this purpose, we let

$$\alpha = \frac{1}{4}\operatorname{dist}(\Sigma,\mathscr{N})$$

and we choose $\psi \in C^\infty(\mathbb{R}^\nu)$ such that

(a) $\psi(x) = 0$ if $\operatorname{dist}(x,\Sigma) \leq \alpha$;

(b) $\psi(x) = 1$ if $\operatorname{dist}(x,\Sigma) \geq 2\alpha$.

We write

$$v_{\eta,a} = w_{\eta,a} + y_{\eta,a},$$

where

$$w_{\eta,a} = \psi(u_\eta - a)v_{\eta,a} = (\psi P) \circ (u_\eta - a)$$

and

$$y_{\eta,a} = (1 - \psi(u_\eta - a))v_{\eta,a} = ((1-\psi)P) \circ (u_\eta - a).$$

Since the map ψP is smooth with bounded derivatives and since $u_\eta - a_\eta \to u$ in $W^{s,p}(\Omega)$ whenever $a_\eta \to 0$, using the compactness of $\mathscr{N}$ to get a uniform L^∞ bound, we deduce from the continuity of the composition operator — see for instance [BM21, Chapter 15.3] — that $w_{\eta,a_\eta} \to u$ in $W^{s,p}(\Omega)$ provided that we choose $a_\eta \to 0$. It therefore remains to prove that we can choose the a_η so that $y_{\eta,a_\eta} \to 0$ in $W^{s,p}(\Omega)$ in order to conclude.

For this purpose, we are going to show the average estimate

$$\int_{B^\nu_\alpha} \|y_{\eta,a}\|^p_{W^{s,p}(\Omega)}\,\mathrm{d}a \to 0 \quad \text{as } \eta \to 0. \tag{3.2.1}$$

Taking (3.2.1) for granted, we conclude the proof as follows. Markov's inequality ensures that

$$\left|\left\{a \in B^\nu_\alpha\colon \|y_{\eta,a}\|^p_{W^{s,p}(\Omega)} \geq \left(\int_{B^\nu_\alpha} \|y_{\eta,a}\|^p_{W^{s,p}(\Omega)}\,\mathrm{d}a\right)^{\frac{1}{2}}\right\}\right| \leq \left(\int_{B^\nu_\alpha} \|y_{\eta,a}\|^p_{W^{s,p}(\Omega)}\,\mathrm{d}a\right)^{\frac{1}{2}} \to 0.$$

Hence, for every $0 < \eta \leq \gamma$, we may choose $a_\eta \in B^\nu_\alpha$ such that $a_\eta \to 0$ and

$$\|y_{\eta,a}\|_{W^{s,p}(\Omega)} \leq \left(\int_{B^\nu_\alpha} \|y_{\eta,a}\|^p_{W^{s,p}(\Omega)}\,\mathrm{d}a\right)^{\frac{1}{2p}} \to 0,$$

which proves the theorem. It therefore only remains to prove estimate (3.2.1).

We start with the case where $\sigma = 0$, and thus $s = k \in \mathbb{N}_*$. For every $j \in \{1, \dots, k\}$, the Faà di Bruno formula ensures that

$$|\mathrm{D}^j y_{\eta,a}(x)| \lesssim \sum_{i=1}^{j} \sum_{\substack{1 \leq t_1 \leq \cdots \leq t_i \\ t_1 + \cdots + t_i = j}} |\mathrm{D}^i((1-\psi)P)(u_\eta(x) - a)||\mathrm{D}^{t_1} u_\eta(x)| \cdots |\mathrm{D}^{t_i} u_\eta(x)|.$$

Since ψ has bounded derivatives, we obtain

$$|\mathrm{D}^j y_{\eta,a}(x)| \lesssim \sum_{i=1}^{j} \sum_{\substack{1 \leq t_1 \leq \cdots \leq t_i \\ t_1 + \cdots + t_i = j}} \frac{1}{\operatorname{dist}(u_\eta(x) - a, \Sigma)^i} |\mathrm{D}^{t_1} u_\eta(x)| \cdots |\mathrm{D}^{t_i} u_\eta(x)|. \tag{3.2.2}$$

As $\mathscr{N}$ is compact, we also know that

$$|y_{\eta,a}(x)| \text{ is uniformly bounded with respect to } x,\ \eta, \text{ and } a. \tag{3.2.3}$$

Moreover, by definition of ψ, the map $y_{\eta,a}$ is supported on $\{\operatorname{dist}(u_\eta - a, \Sigma) \leq 2\alpha\}$. We observe that, using the fact that $u \in \mathscr{N}$ and the definition of α,

$$\{\mathrm{dist}(u_\eta - a, \Sigma) \le 2\alpha\} \subset \{\mathrm{dist}(u_\eta, \Sigma) \le 3\alpha\} \subset \{|u_\eta - u| \ge \alpha\}.$$

Since $ip \le sp < \ell$, we have that

$$\int_{B^\nu_\alpha} \frac{1}{\mathrm{dist}(u_\eta(x) - a, \Sigma)^{ip}} \,\mathrm{d}a = \int_{B^\nu_\alpha + u_\eta(x)} \frac{1}{\mathrm{dist}(a, \Sigma)^{ip}} \,\mathrm{d}a \le \int_{B^\nu_R} \frac{1}{\mathrm{dist}(a, \Sigma)^{ip}} \,\mathrm{d}a < +\infty,$$

where $R > 0$ is chosen sufficiently large so that $B^\nu_\alpha + u_\eta(x) \subset B^\nu_R$ for every $x \in \Omega$ and $0 < \eta \le \gamma$. Integrating (3.2.2) and (3.2.3), and using Tonelli's theorem and the information on the support of $y_{\eta,a}$, we deduce that

$$\int_{B^\nu_\alpha} \|y_{\eta,a}\|^p_{W^{s,p}(\Omega)} \,\mathrm{d}a \lesssim \int_{\{|u_\eta - u| \ge \alpha\}} 1 + \sum_{j=1}^{k} \sum_{i=1}^{j} \sum_{\substack{1 \le t_1 \le \cdots \le t_i \\ t_1 + \cdots + t_i = j}} |\mathrm{D}^{t_1} u_\eta|^p \cdots |\mathrm{D}^{t_i} u_\eta|^p.$$

Since $u_\eta \to u$ in $L^p(\Omega)$, in particular $u_\eta \to u$ in measure, and therefore $|\{|u_\eta - u| \ge \alpha\}| \to 0$ as $\eta \to 0$. Hölder's inequality ensures that, for $t_1 + \cdots + t_i = j$,

$$\begin{aligned} &\int_{\{|u_\eta - u| \ge \alpha\}} |\mathrm{D}^{t_1} u_\eta|^p \cdots |\mathrm{D}^{t_i} u_\eta|^p \\ &\qquad \le \left(\int_{\{|u_\eta - u| \ge \alpha\}} |\mathrm{D}^{t_1} u_\eta|^{jp/t_1}\right)^{t_1/j} \cdots \left(\int_{\{|u_\eta - u| \ge \alpha\}} |\mathrm{D}^{t_i} u_\eta|^{jp/t_i}\right)^{t_i/j}. \end{aligned}$$

But as $u \in L^\infty(\Omega) \cap W^{k,p}(\Omega)$, we infer from the classical Gagliardo–Nirenberg inequality — see [Gag59] and [Nir59, Lecture 2] — that

$$u \in W^{t_\beta, kp/t_\beta}(\Omega) \subset W^{t_\beta, jp/t_\beta}(\Omega) \quad \text{whenever } 1 \le t_\beta \le k.$$

Invoking Lebesgue's lemma, we conclude that

$$\int_{B_\alpha} \|y_{\eta,a}\|^p_{W^{s,p}(\Omega)} \,\mathrm{d}a \to 0 \quad \text{as } \eta \to 0,$$

which establishes estimate (3.2.1).

We now turn to the case $0 < \sigma < 1$, and we assume that $k \ge 1$. Using the integer order case, we already have

$$\int_{B_\alpha} \|y_{\eta,a}\|^p_{W^{k,p}(\Omega)} \,\mathrm{d}a \to 0 \quad \text{as } \eta \to 0,$$

so that it only remains to prove that

$$\int_{B_\alpha} |D^k y_{\eta,a}|^p_{W^{\sigma,p}(\Omega)} \, da \to 0 \quad \text{as } \eta \to 0.$$

Since $y_{\eta,a}$ is supported on $\{\mathrm{dist}(u_\eta - a, \Sigma) \leq 2\alpha\} \subset \{|u_\eta - u| \geq \alpha\}$, we may write

$$\begin{aligned}
&|D^k y_{\eta,a}|^p_{W^{\sigma,p}(\Omega)} \\
&\leq 2 \int_{\{\mathrm{dist}(u_\eta(x)-a,\Sigma)\leq 2\alpha\}} \int_{\{\mathrm{dist}(u_\eta(x)-a,\Sigma)\leq \mathrm{dist}(u_\eta(y)-a,\Sigma)\}} \frac{|D^k y_{\eta,a}(x) - D^k y_{\eta,a}(y)|^p}{|x-y|^{m+\sigma p}} \, dy \, dx \\
&\leq 2 \int_{\{|u_\eta(x)-u(x)|\geq\alpha\}} \int_{\{\mathrm{dist}(u_\eta(x)-a,\Sigma)\leq \mathrm{dist}(u_\eta(y)-a,\Sigma)\}} \frac{|D^k y_{\eta,a}(x) - D^k y_{\eta,a}(y)|^p}{|x-y|^{m+\sigma p}} \, dy \, dx.
\end{aligned}$$

Given $x, y \in \Omega$, using the Faà di Bruno formula and the multilinearity of the derivative, we obtain

$$|D^k y_{\eta,a}(x) - D^k y_{\eta,a}(y)| \lesssim \sum_{j=1}^{k} \sum_{\substack{1\leq t_1\leq\cdots\leq t_j \\ t_1+\cdots+t_j=k}} \Big(A_{j,t_1,\ldots,t_j} + \sum_{i=1}^{j} B_{i,j,t_1,\ldots,t_j}\Big), \tag{3.2.4}$$

where

$$\begin{aligned}
&A_{j,t_1,\ldots,t_j} \\
&\quad = |D^j((1-\psi)P)(u_\eta(x)-a) - D^j((1-\psi)P)(u_\eta(y)-a)||D^{t_1}u_\eta(x)|\cdots|D^{t_j}u_\eta(x)|
\end{aligned}$$

and

$$\begin{aligned}
&B_{i,j,t_1,\ldots,t_j} \\
&\quad = |D^j((1-\psi)P)(u_\eta(y)-a)|\Big(\prod_{1\leq\beta<i} |D^{t_\beta}u_\eta(x)|\Big)|D^{t_i}u_\eta(x) - D^{t_i}u_\eta(y)|\Big(\prod_{i<\beta\leq j} |D^{t_\beta}u_\eta(y)|\Big).
\end{aligned}$$

To bear in mind more readable terms, the reader may think of the case $j = 1$, where one has

$$A_1 = |D((1-\psi)P)(u_\eta(x)-a) - D((1-\psi)P)(u_\eta(y)-a)||Du_\eta(x)|$$

and

$$B_1 = |D((1-\psi)P)(u_\eta(y)-a)||Du_\eta(x) - Du_\eta(y)|.$$

We observe that $|\mathrm{D}^t u_\eta| \lesssim \eta^{-t}$ for every $t \in \mathbb{N}$. Therefore, (3.2.4) yields

$$|\mathrm{D}^k y_{\eta,a}(x) - \mathrm{D}^k y_{\eta,a}(y)|^p \lesssim \sum_{j=1}^{k} \Big(C_j + \sum_{t=1}^{k} D_{j,t} \Big), \tag{3.2.5}$$

where

$$C_j = \eta^{-kp} |\mathrm{D}^j((1-\psi)P)(u_\eta(x) - a) - \mathrm{D}^j((1-\psi)P)(u_\eta(y) - a)|^p$$

and

$$D_{j,t} = \eta^{-(k-t)p} |\mathrm{D}^j((1-\psi)P)(u_\eta(y) - a)|^p |\mathrm{D}^t u_\eta(x) - \mathrm{D}^t u_\eta(y)|^p.$$

As for the integer case, since $jp \leq kp < \ell$, we have

$$\int_{B^\nu_\alpha} |\mathrm{D}^j((1-\psi)P)(u_\eta(y) - a)|^p \,\mathrm{d}a \lesssim \int_{B^\nu_\alpha} \frac{1}{\operatorname{dist}(u_\eta(y) - a, \Sigma)^{jp}} \,\mathrm{d}a < +\infty.$$

We now integrate with respect to x and y, split the integral in y into two parts, and use again the estimate $|\mathrm{D}^t u_\eta| \lesssim \eta^{-t}$. This yields, for any $r > 0$,

$$\begin{aligned}
&\int_{B^\nu_\alpha} \int_{\{|u_\eta(x) - u(x)| \geq \alpha\}} \int_{\Omega} \frac{D_{j,t}}{|x-y|^{m+\sigma p}} \,\mathrm{d}y \,\mathrm{d}x \,\mathrm{d}a \\
&\qquad \lesssim \eta^{-(k+1)p} \int_{\{|u_\eta(x) - u(x)| \geq \alpha\}} \int_{B^m_r(x)} \frac{1}{|x-y|^{m+(\sigma-1)p}} \,\mathrm{d}y \,\mathrm{d}x \\
&\qquad\quad + \eta^{-kp} \int_{\{|u_\eta(x) - u(x)| \geq \alpha\}} \int_{\mathbb{R}^m \setminus B^m_r(x)} \frac{1}{|x-y|^{m+\sigma p}} \,\mathrm{d}y \,\mathrm{d}x \\
&\qquad \lesssim (\eta^{-(k+1)p} r^{(1-\sigma)p} + \eta^{-kp} r^{-\sigma p}) |\{|u_\eta - u| \geq \alpha\}|.
\end{aligned}$$

Inserting $r = \eta$, we obtain

$$\int_{B^\nu_\alpha} \int_{\{|u_\eta(x) - u(x)| \geq \alpha\}} \int_{\Omega} \frac{D_{j,t}}{|x-y|^{m+\sigma p}} \,\mathrm{d}y \,\mathrm{d}x \,\mathrm{d}a \lesssim \eta^{-sp} |\{|u_\eta - u| \geq \alpha\}|.$$

By the fractional Gagliardo–Nirenberg inequality — see e.g. [BM01, Corollary 3.2] and [BM18, Theorem 1] — we have $u \in W^{\sigma, sp/\sigma}(\Omega)$. Invoking the Markov inequality and Lemma 3.2.9, we find

$$
\begin{aligned}
|\{|u_\eta - u| \geq \alpha\}| &\leq \frac{1}{\alpha^{sp/\sigma}} \int_{\{|u_\eta - u| \geq \alpha\}} |u_\eta - u|^{sp/\sigma} \\
&\lesssim \eta^{sp} \int_{\{|u_\eta - u| \geq \alpha\}} (\mathrm{D}^{\sigma, sp/\sigma} u)^{sp/\sigma}.
\end{aligned}
\tag{3.2.6}
$$

Hence, using Lebesgue's lemma, we conclude that

$$
\eta^{-sp} |\{|u_\eta - u| \geq \alpha\}| \lesssim \int_{\{|u_\eta - u| \geq \alpha\}} (\mathrm{D}^{\sigma, sp/\sigma} u)^{sp/\sigma} \to 0 \quad \text{as } \eta \to 0.
$$

This achieves to estimate the second term in (3.2.5).

For the first term, we also split the integral with respect to y into two parts, but this time we use Lemma 3.2.8 to estimate $|\mathrm{D}^j((1-\psi)P)(u_\eta(x) - a) - \mathrm{D}^j((1-\psi)P)(u_\eta(y) - a)|$ in the ball $B^m_r(x)$. This yields, for every $r > 0$,

$$
\begin{aligned}
&\int_{B^\nu_\alpha} \int_{\{|u_\eta(x) - u(x)| \geq \alpha\}} \int_{\{\mathrm{dist}(u_\eta(x) - a, \Sigma) \leq \mathrm{dist}(u_\eta(y) - a, \Sigma)\}} \frac{C_j}{|x-y|^{m+\sigma p}} \,\mathrm{d}y \,\mathrm{d}x \,\mathrm{d}a \\
&\quad \lesssim \eta^{-kp} \int_{\{|u_\eta(x) - u(x)| \geq \alpha\}} \int_{B^\nu_\alpha} \biggl(\int_{B^m_r(x)} \frac{1}{\mathrm{dist}(u_\eta(x) - a, \Sigma)^{(j+1)p}} \frac{|u_\eta(x) - u_\eta(y)|^p}{|x-y|^{m+\sigma p}} \,\mathrm{d}y \\
&\qquad\qquad + \int_{\mathbb{R}^m \setminus B^m_r(x)} \frac{1}{\mathrm{dist}(u_\eta(x) - a, \Sigma)^{jp}} \frac{1}{|x-y|^{m+\sigma p}} \,\mathrm{d}y \biggr) \,\mathrm{d}a \,\mathrm{d}x.
\end{aligned}
$$

We estimate

$$
\int_{B^m_r(x)} \frac{1}{\mathrm{dist}(u_\eta(x) - a, \Sigma)^{(j+1)p}} \frac{|u_\eta(x) - u_\eta(y)|^p}{|x-y|^{m+\sigma p}} \,\mathrm{d}y \lesssim r^{(1-\sigma)p} \eta^{-p} \frac{1}{\mathrm{dist}(u_\eta(x) - a, \Sigma)^{(j+1)p}}
$$

and

$$
\int_{\mathbb{R}^m \setminus B^m_r(x)} \frac{1}{\mathrm{dist}(u_\eta(x) - a, \Sigma)^{jp}} \frac{1}{|x-y|^{m+\sigma p}} \,\mathrm{d}y \lesssim r^{-\sigma p} \frac{1}{\mathrm{dist}(u_\eta(x) - a, \Sigma)^{jp}}.
$$

Inserting $r = \eta \,\mathrm{dist}(u_\eta(x) - a, \Sigma)$, we obtain

$$
\begin{aligned}
&\int_{B^\nu_\alpha} \int_{\{|u_\eta(x) - u(x)| \geq \alpha\}} \int_{\{\mathrm{dist}(u_\eta(x) - a, \Sigma) \leq \mathrm{dist}(u_\eta(y) - a, \Sigma)\}} \frac{C_j}{|x-y|^{m+\sigma p}} \,\mathrm{d}y \,\mathrm{d}x \,\mathrm{d}a \\
&\quad \lesssim \eta^{-sp} \int_{\{|u_\eta(x) - u(x)| \geq \alpha\}} \int_{B^\nu_\alpha} \frac{1}{\mathrm{dist}(u_\eta(x) - a, \Sigma)^{(j+\sigma)p}} \,\mathrm{d}a \,\mathrm{d}x \lesssim \eta^{-sp} |\{|u_\eta - u| \geq \alpha\}|,
\end{aligned}
$$

where, in the last inequality, we once more made use of the fact that $sp < \ell$. We observe interestingly that our choice of r is not so common. Indeed, in such an optimization argument, one usually takes r to be some suitable power of η. Here, not only our choice is more complex, but it also depends on x and a, the outer variables of integration. Using estimate (3.2.6), we conclude that the above quantity goes to 0 as $\eta \to 0$, which finishes to estimate the first term in (3.2.5). Both terms being controlled, this establishes average estimate (3.2.1), therefore concluding the proof of the theorem when $k \geq 1$ and $0 < \sigma < 1$.

The case $k = 0$ and $0 < \sigma < 1$ is similar, and actually simpler. Indeed, as no derivatives are involved, we have to estimate the difference $((1 - \psi)P)(u_\eta(x) - a) - ((1 - \psi)P)(u_\eta(y) - a)$, which is directly performed with the same technique as for the C_j term in the previous case. Moreover, this range of parameters was already treated in [BPVS14] with a different technique, interpolating with the first order term using the Gagliardo–Nirenberg inequality. We therefore omit the details of the argument. □

3.2.3 Concluding thoughts: What singular set can we hope for?

We conclude this section by considering the question of existence of a singular projection whose singular set is a closed submanifold of $\mathbb{R}^\nu$. We have seen in Lemmas 3.2.2 and 3.2.3 that the $(\ell - 2)$-connectedness of the target manifold $\mathscr{N}$ is a necessary and sufficient condition for the existence of a singular projection, and that the proof produces a singular projection whose singular set is a subskeleton, and therefore exhibits crossings. Since projections whose singular set do not have crossings allow to obtain the density of the class $\mathscr{R}^{\mathrm{uncr}}$ instead of the class $\mathscr{R}$, it is natural to ask whether or not it is always possible to improve singular projections so that their singular set is a submanifold. That is: Does every $(\ell - 2)$-connected manifold admit a singular projection whose singular set is a submanifold?

Although we are not able to answer this question, we give in this section a family of examples suggesting that there is little hope that the answer is *yes*. For every $\ell \in \mathbb{N}_*$, we let $\mathscr{N}_\ell$ denote a connected sum of ℓ copies of the 2-dimensional torus, embedded into $\mathbb{R}^3$. Since $\mathscr{N}_\ell$ is connected, it admits a 2-singular projection. Actually, this projection may even be taken to be the nearest point projection. For $\mathscr{N}_1 = \mathbb{T}^2$, its singular set is the circle forming the core of the torus and a line passing through the hole of the torus. For $\mathscr{N}_2 = \mathbb{T}^2 \# \mathbb{T}^2$, the two-holed torus, its singular set is the eight-figure forming the core of the torus and two lines, each one passing through one of the holes of $\mathscr{N}_2$. One may notice that, in those two examples, the singular set of the natural singular projection onto $\mathbb{T}^2$ is a 1-dimensional submanifold of $\mathbb{R}^3$, while the singular set of the natural singular projection onto $\mathbb{T}^2 \# \mathbb{T}^2$ is only a finite union of 1-dimensional submanifolds of

$\mathbb{R}^3$. It is therefore natural to ask whether or not this can be improved to have a singular projection onto $\mathbb{T}^2\#\mathbb{T}^2$ whose singular set would be a 1-dimensional submanifold of $\mathbb{R}^3$. The same question arises for $\mathscr{N}_\ell$ for every $\ell \geq 2$.

Proposition 3.2.10. *If $\ell \geq 2$, then there is no homotopy retract $P\colon \mathbb{S}^3 \setminus \mathscr{S} \to \mathscr{N}_\ell$ such that $\mathscr{S}$ is a 1-dimensional submanifold of $\mathbb{S}^3$.*

We have stated Proposition 3.2.10 with $\mathbb{S}^3$ instead of $\mathbb{R}^3$, but this is equivalent up to compactification in the case of maps that are constant at infinity — or if the singular set passes through the point at infinity, as it is the case for the $\mathscr{N}_\ell$ above. In Definition 3.2.1, singular projections were required to be continuous retracts of $\mathbb{R}^\nu \setminus \Sigma$ into $\mathscr{N}$, that is, $P \circ i\colon \mathscr{N} \to \mathscr{N} = \mathrm{id}_{\mathscr{N}}$, where i is the inclusion of $\mathscr{N}$ into $\mathbb{R}^\nu \setminus \Sigma$. In Proposition 3.2.10, we consider instead homotopy retracts, that is, P should in addition satisfy that $i \circ P\colon \mathbb{R}^\nu \setminus \Sigma \to \mathbb{R}^\nu \setminus \Sigma$ is homotopic to the identity map. In particular, P induces a homotopy equivalence between $\mathbb{R}^\nu \setminus \Sigma$ and $\mathscr{N}$. This is a stronger requirement than asking merely that P is a continuous retract. Nevertheless, one may check that the usual constructions for a singular projection, like in Lemma 3.2.3, produce a homotopy retract that is constant at infinity, so Proposition 3.2.10 leaves little hope to find a singular projection into $\mathscr{N}_\ell$ whose singular set would be a submanifold when $\ell \geq 2$.

Proof. Assume by contradiction that there exists a homotopy retract $P\colon \mathbb{S}^3 \setminus \mathscr{S} \to \mathscr{N}_\ell$, where $\mathscr{S}$ is a submanifold of $\mathbb{S}^3$. We start by computing the homology groups of $\mathbb{S}^3$, $\mathbb{S}^3 \setminus \mathscr{S}$, and the relative homology groups of $\mathbb{S}^3$ relatively to $\mathbb{S}^3 \setminus \mathscr{S}$. The first homology groups of the sphere are given by

$$H_0(\mathbb{S}^3) = \mathbb{Z}, \quad H_1(\mathbb{S}^3) = \{0\}, \quad H_2(\mathbb{S}^3) = \{0\}, \quad H_3(\mathbb{S}^3) = \mathbb{Z}.$$

We note that we always implicitly consider homology with integer coefficients. On the other hand, since we assumed the existence of the homotopy retract P, it follows that $\mathbb{S}^3 \setminus \mathscr{S}$ and $\mathscr{N}_\ell$ share the same homology groups: $H_j(\mathbb{S}^3 \setminus \mathscr{S}) = H_j(\mathscr{N}_\ell)$ for every $j \in \mathbb{N}$. Therefore, we obtain

$$H_0(\mathbb{S}^3 \setminus \mathscr{S}) = \mathbb{Z}, \quad H_1(\mathbb{S}^3 \setminus \mathscr{S}) = \mathbb{Z}^{2\ell}, \quad H_2(\mathbb{S}^3 \setminus \mathscr{S}) = \mathbb{Z}, \quad H_3(\mathbb{S}^3 \setminus \mathscr{S}) = \{0\}.$$

To obtain the homology groups $H_j(\mathbb{S}^3, \mathbb{S}^3 \setminus \mathscr{S})$, we use the long exact sequence of relative homology groups

$$\cdots \longrightarrow H_j(\mathbb{S}^3) \longrightarrow H_j(\mathbb{S}^3, \mathbb{S}^3 \setminus \mathscr{S}) \longrightarrow H_{j-1}(\mathbb{S}^3 \setminus \mathscr{S}) \longrightarrow H_{j-1}(\mathbb{S}^3) \longrightarrow \cdots.$$

The portion of this sequence for $j = 2$ yields

$$\{0\} \longrightarrow H_2(\mathbb{S}^3, \mathbb{S}^3 \setminus \mathscr{S}) \longrightarrow \mathbb{Z}^{2\ell} \longrightarrow \{0\},$$

which implies that $H_2(\mathbb{S}^3, \mathbb{S}^3 \setminus \mathscr{S}) = \mathbb{Z}^{2\ell}$. We now examine the portion of the sequence with $j = 3$, which translates into

$$\{0\} \longrightarrow \mathbb{Z} \longrightarrow H_3(\mathbb{S}^3, \mathbb{S}^3 \setminus \mathscr{S}) \longrightarrow \mathbb{Z} \longrightarrow \{0\}.$$

As $\mathbb{Z}$ is a free $\mathbb{Z}$-module, the above short exact sequence of abelian groups splits, which implies that necessarily $H_3(\mathbb{S}^3, \mathbb{S}^3 \setminus \mathscr{S}) = \mathbb{Z} \oplus \mathbb{Z}$.

We now recall two important duality principles concerning homology groups. The first one is *Poincaré duality*: If $\mathscr{M}$ is a closed orientable m-dimensional manifold, then the homology group $H_{m-j}(\mathscr{M})$ is isomorphic to the cohomology group $H^j(\mathscr{M})$ for every $j \in \{0, \dots, m\}$; see e.g. [Hat02, Theorem 3.30]. The second one is the *Poincaré–Lefschetz duality*: If K is a compact locally contractible subspace of a closed orientable m-dimensional manifold $\mathscr{M}$, then $H_j(\mathscr{M}, \mathscr{M} \setminus K) \cong H^{m-j}(K)$ for every $j \in \{0, \dots, m\}$; see e.g. [Hat02, Theorem 3.44]. Applied to $\mathscr{M} = \mathbb{S}^3$ and $K = \mathscr{S}$, the Poincaré–Lefschetz duality yields

$$H^0(\mathscr{S}) \cong H_3(\mathbb{S}^3, \mathbb{S}^3 \setminus \mathscr{S}) = \mathbb{Z} \oplus \mathbb{Z} \quad \text{and} \quad H^1(\mathscr{S}) \cong H_2(\mathbb{S}^3, \mathbb{S}^3 \setminus \mathscr{S}) = \mathbb{Z}^{2\ell}.$$

On the other hand, since $\mathscr{S}$ is assumed to be a 1-dimensional submanifold of $\mathbb{S}^3$, the Poincaré duality implies that

$$H_0(\mathscr{S}) \cong H^1(\mathscr{S}) \cong \mathbb{Z}^{2\ell} \quad \text{and} \quad H_1(\mathscr{S}) \cong H^0(\mathscr{S}) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

However, for a 1-dimensional manifold, the groups H_0 and H_1 both coincide with a direct sum of the same number of copies of $\mathbb{Z}$, one for each connected component. Therefore, the above situation is only possible for $\ell = 1$, which concludes the proof. □

We note along the way that, when $\ell = 1$, the above proof shows that the singular set of a homotopy retract to $\mathbb{T}^2$ must have exactly two connected components. This is coherent with what we obtain with the natural construction described above, and shows that the singular set obtained there cannot be improved to be made of only one connected component.

A similar reasoning could be carried out in other situations, provided one is able to compute the required homology groups. For instance, one could examine the situation for non orientable surfaces, relying on homology with coefficients in $\mathbb{Z}/2\mathbb{Z}$ so that

Poincaré duality is also available.

3.3 The general case: the crossings removal procedure

3.3.1 The idea of the method

In this section, we consider the case of a general target manifold $\mathscr{N}$, non necessarily $(\lfloor sp \rfloor - 1)$-connected. In this context where the method of projection cannot be applied, all currently available proofs of the density of the class $\mathscr{R}$ and its variants rely on modifying the map $u \in W^{s,p}(\Omega; \mathscr{N})$ to be approximated on its domain — in contrast with the method of projection, which consists in working on the codomain. In the most general case, as we already explained, there are essentially two ideas of proof. The first one is the method of good and bad cubes, introduced by F. Bethuel [Bet91] to handle the case $W^{1,p}$, and later pursued in the general case $W^{s,p}$ with $0 < s < +\infty$; see [BPVS15] and Chapter 2. The second one is the averaging argument devised by H. Brezis and P. Mironescu [BM15], suited for the case $0 < s < 1$.

Both these ideas require to decompose the domain Q^m into a small grid, and rely crucially on homogeneous extension. In Bethuel's approach, this procedure is used to approximate u on the bad cubes of the grid, while in Brezis and Mironescu's approach, it is used on all the cubes of the grid. By the very definition of homogeneous extension, it is clear that this technique *necessarily* produces maps whose singular set exhibits crossings — except in the case of point singularities.

The key ingredient in the homogeneous extension procedure is the standard retraction $\overline{Q^m} \setminus \mathscr{T}^{d^*} \to \mathscr{K}^d$, where we recall that K^d is the d-skeleton of the unit cube, and T^{d^*} its dual skeleton. In order to perform approximation with maps having a singular set *without* crossings, a natural question would be whether or not there exists another retraction $g \colon \overline{Q^m} \setminus \mathscr{S} \to \mathscr{K}^d$, where here $\mathscr{S}$ would be a d^*-submanifold of $\mathbb{R}^m$, that is, without crossings. This would correspond to a modified version of the usual retraction $\overline{Q^m} \setminus \mathscr{T}^{d^*} \to \mathscr{K}^d$, where the singular set has been uncrossed.

It turns out that such a retraction *does* exist, and is actually quite simple to construct. This may come as very surprising, in view of Proposition 3.2.10. We note importantly that this is not due to the fact that Proposition 3.2.10 requires a homotopy retract, since the map that we are going to construct is actually a homotopy retract. The possibility to obtain such a retraction is instead due to the fact that here, we only require it to be a retraction on a 1-dimensional set, while in Proposition 3.2.10, we imposed a 2-dimensional constraint. This allows for more freedom in our construction.

The procedure to build this retraction g is explained below, with $m = 3$ to allow for illustration. The starting point is the zero-homogeneous map $x \mapsto x/|x|_\infty$, which retracts $\overline{Q^3} \setminus \{0\}$ onto ∂Q^3. Choosing the center of projection to be a point *above* Q^3 instead of *inside* Q^3 yields a continuous map h defined on the whole $\overline{Q^3}$, that retracts $\overline{Q^3}$ onto its four lateral faces and its lower face. We then postcompose the map h with the usual retraction of these five faces minus their centers onto their boundary, which is exactly $\mathscr{K}^1$. This produces the expected continuous retraction $g\colon \overline{Q^3} \setminus \mathcal{S} \to \mathscr{K}^1$, where $\mathcal{S}$ is the inverse image of the centers of the five aforementioned faces under h, which consists of five line segments that emanate from those centers and end up on the top face of Q^3. Those lines do not cross inside Q^3, but they would intersect at the center of projection above Q^3 if they were extended up to there. The situation is depicted on Figure 3.1, where the singularities of g are represented in red, and extended up to the projection point to help visualization.

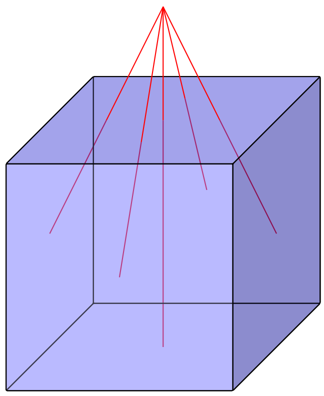

Figure 3.1 – The retraction g and its singular set

Another way of looking at this construction is the following. Viewed from the projection point lying slightly above Q^3, the set of all faces except the top one looks like on Figure 3.2, with the centers of the faces represented in red. The retraction $g\colon \overline{Q^3} \setminus \mathcal{S} \to \mathscr{K}^1$ may then be viewed as a vertical projection onto the set depicted on Figure 3.2, followed by the retraction onto the edges away from the red centers. The singular set would then look like vertical lines starting from the red centers.

As a final comment concerning this model construction, we note that it appears to be natural in the context of homology theory. Indeed, the first homology group of $\mathscr{K}^1$ is given by $H_1(\mathscr{K}^1) = \mathbb{Z}^5$, with one cycle generated by the boundary of each face of $\mathscr{K}^1$ except the top one which is the sum of all five others. This is clearly seen on $\overline{Q^3} \setminus \mathcal{S}$: there is one cycle winding around each segment constituting $\mathcal{S}$, each one corresponding to the boundary of one of the five lowest faces of $\overline{Q^3}$, and the sum of all of them corresponds to the boundary of the top face. This suggests that our construction is somehow adapted to the homology of $\mathscr{K}^1$. Moreover, this can be used to prove that the set $\mathcal{S}$ must have

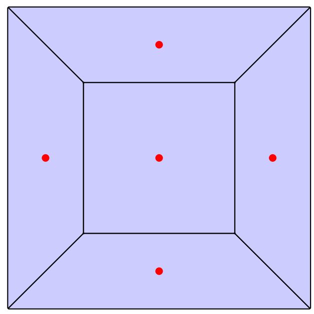

Figure 3.2 – Planar view of the faces of Q^3 without the top one

exactly five connected components *inside of* Q^3, so that our construction is optimal in this sense.

Having at our disposal the smooth retraction g is a first step towards the proof of the density of the class of maps with uncrossed singular set $\mathscr{R}^{\mathrm{uncr}}_{m-\lfloor sp \rfloor-1}$ in $W^{s,p}$, but we are not done yet. As the more rigid class $\mathscr{R}^{\mathrm{rig}}_{m-\lfloor sp \rfloor-1}(Q^m;\mathscr{N})$ is dense in $W^{s,p}(Q^m;\mathscr{N})$, it suffices to show that every map that belongs to $\mathscr{R}^{\mathrm{rig}}_{m-\lfloor sp \rfloor-1}(Q^m;\mathscr{N})$ may be approximated in $W^{s,p}$ by maps in $\mathscr{R}^{\mathrm{uncr}}_{m-\lfloor sp \rfloor-1}(Q^m;\mathscr{N})$. Using a dilation argument if necessary, we may furthermore assume that the restriction of the singular set of those maps to $\overline{Q^m}$ is the dual skeleton of a cubication of $\overline{Q^m}$. However, as it is constructed above, the map g only uncrosses the singularities inside *one* cube, not the full set of singularities of a map in $\mathscr{R}^{\mathrm{rig}}_{m-\lfloor sp \rfloor-1}$. Moreover, this procedure comes without any guarantee that the modified map is close to the original one in the $W^{s,p}$ distance.

As the general constructions are quite involved, the remaining of this section is devoted to some particular cases to explain the main ideas in a more simple setting, allowing for less involved notation and illustrative figures. We start by presenting the full approximation procedure in $W^{1,p}(Q^3;\mathscr{N})$ with $1 \leq p < 2$, which corresponds to the case of *line* singularities. This is the content of Proposition 3.3.1 below. We then briefly explain the additional difficulties that arise when moving towards the general case. For this purpose, we explain how to obtain a smooth construction suited to the full range $0 < s < +\infty$, and we also sketch the topological part of our construction to uncross *plane* singularities in Q^3. The proof of Theorem 3.1.2 in the general case is postponed to Section 3.3.2.

Proposition 3.3.1. *Let* $u \in \mathscr{R}^{\mathrm{rig}}_1(Q^3;\mathscr{N})$ *and* $1 \leq p < 2$. *There exists a sequence* $(u_n)_{n\in\mathbb{N}}$ *in* $W^{1,p}(Q^3;\mathscr{N})$ *such that* $u_n \to u$ *in* $W^{1,p}(Q^3;\mathscr{N})$ *and such that each* u_n *is locally Lipschitz outside of a* 1*-dimensional Lipschitz submanifold* $\mathscr{S}_{u_n}$ *of* Q^3.

To avoid technicalities and focus on the core of the argument, we have stated Proposition 3.3.1 with approximating maps being only locally Lipschitz outside of the singular set. In the proof of the general case of our main result, in Section 3.3, we will take care of making the approximating maps smooth and establishing the estimates near the singular set in order to ensure that they belong to the class $\mathscr{R}^{\mathrm{uncr}}$.

Proof. Since $u \in \mathscr{R}_1^{\mathrm{rig}}(Q^3; \mathscr{N})$, we may assume that its singular set $\mathcal{S}_u$ is the dual skeleton $\mathscr{T}^1$ of the 1-skeleton $\mathscr{K}^1$ of a cubication of $\overline{Q^3}$ of inradius η, for some $\eta \in \frac{1}{2\mathbb{N}_*}$. Let $\mathscr{V}^1$ be the vertical part of $\mathscr{T}^1$, that is, $\mathscr{V}^1$ consists of all the lines in $\mathscr{T}^1$ having directing vector $(0,0,1)$. Let also $\mathscr{V}^1_{\mathrm{tr}} = \mathscr{V}^1 \cap \big((-1,1)^2 \times (-1+\eta, 1)\big)$ be the vertical singular set $\mathscr{V}^1$ to which we have truncated the lower extremity. For every $0 < \mu < \frac{1}{2}$, we define $W_\mu = (\mathscr{V}^1_{\mathrm{tr}} + Q^3_{\mu\eta}) \cap Q^3$. We note that the well W_μ contains all the crossings of the singular set $\mathscr{T}^1$. The reader may refer to Figure 3.3 for an illustration of the well W_μ and the singular set $\mathcal{S}_u$.

We uncross the singularities of u in two steps. The first one, of topological nature, consists in replacing u in W_μ by another extension of $u_{|\partial W_\mu}$. This extension in constructed in a way that produces a singular set *without* crossings, but comes with no control on the energy of the resulting map. The second step, of analytical nature, consists in modifying the map obtained in the first step to obtain a better map with a control on the energy.

Step 1. — Uncrossing the singularities.

We construct a Lipschitz map $\Phi^{\mathrm{top}}_\mu \colon Q^3 \to Q^3$ such that $\Phi^{\mathrm{top}}_\mu = \mathrm{id}$ outside of W_μ and $(\Phi^{\mathrm{top}}_\mu)^{-1}(\mathscr{T}^1)$ is a Lipschitz submanifold of Q^3. Assuming that the map Φ^{top}_μ has been constructed, we explain how to conclude Step 1. We define the map $v_\mu \colon Q^3 \setminus \mathcal{S}_\mu \to \mathscr{N}$ by $v_\mu = u \circ \Phi^{\mathrm{top}}_\mu$. Here, $\mathcal{S}_\mu = (\Phi^{\mathrm{top}}_\mu)^{-1}(\mathscr{T}^1)$ is the singular set of v_μ, which is a Lipschitz submanifold of Q^3 by assumption on Φ^{top}_μ. Then, the map v_μ is locally Lipschitz on $Q^3 \setminus \mathcal{S}_\mu$, and it coincides with u outside of W_μ.

We now explain how to construct the map Φ^{top}_μ. The procedure is illustrated on Figure 3.3. For the part of W_μ that lies around each line in $\mathscr{V}^1$, we proceed similarly to what we did on the model case described by Figure 3.1: We choose a projection point slightly above the line, and we use this point to retract radially the part of $\overline{W_\mu}$ onto the corresponding part of ∂W_μ. We note that here, topological operations like closure or boundary are taken inside Q^3. For instance, ∂W_μ denotes the boundary of W_μ in Q^3 with respect to the subspace topology. This avoids having to systematically take the intersection with Q^3 to remove the part of ∂W_μ that would lie in the boundary of Q^3 in $\mathbb{R}^3$.

Carrying out this construction around each part of W_μ produces a smooth retraction of $\overline{W_\mu}$ onto ∂W_μ. Extending this map by identity outside of $\overline{W_\mu}$ yields a Lipschitz map $\Phi^{\mathrm{top}}_\mu \colon Q^3 \to Q^3$ such that $\Phi^{\mathrm{top}}_\mu = \mathrm{id}$ outside of W_μ.

As $\mathscr{T}^1$ is a union of line segments which cross only in W_μ, we know that $\mathscr{T}^1 \cap (Q^3 \setminus W_\mu)$ is a Lipschitz submanifold of Q^3 with boundary, the latter being the finite set of points $\mathscr{T}^1 \cap \partial W_\mu$. On the other hand, by construction of Φ_μ^{top}, the set $((\Phi_\mu^{\mathrm{top}})_{|\overline{W_\mu}})^{-1}(\mathscr{T}^1)$ is a Lipschitz submanifold of Q^3 — actually a set of lines — also with boundary given by the finite set of points $\mathscr{T}^1 \cap \partial W_\mu$. Therefore, we conclude that $\mathcal{S}_\mu$ is a Lipschitz submanifold of Q^3 (without boundary), which is depicted on the second cube in Figure 3.3. This finishes to prove that the map Φ_μ^{top} enjoys all the required properties.

Step 2. — Controlling the energy.

In the second step, we explain how to modify the map v_μ in order to obtain a better map u_μ with controlled energy. This relies on a scaling argument. For this, the key observation is that, as $p < 2$, contracting a Sobolev map to a smaller region decreases its energy in dimension 2. Let $V_\mu = (\mathscr{V}^1 + Q^3_{\mu\eta}) \cap Q^3$ be a neighborhood of inradius $\mu\eta$ of the vertical part of the singular set of u. We note that $W_\mu \subset V_\mu$ (actually, W_μ corresponds to V_μ with its lower part truncated). The region $V_{2\mu} = (\mathscr{V}^1 + Q^3_{2\mu\eta}) \cap Q^3$ is a twice larger neighborhood of the vertical part of the singular set of u. Given $0 < \tau < 1$, we are going to shrink the values of v_μ in V_μ to the small region $V_{\tau\mu} = (\mathscr{V}^1 + Q^3_{\tau\mu\eta}) \cap Q^3$ while keeping v_μ unchanged outside of $V_{2\mu}$. As explained above, choosing τ sufficiently small, we may make the energy of the shrunk map as small as we want on $V_{\tau\mu}$, hence obtaining a new map with controlled energy regardless of the energy of the extension v_μ constructed in first instance. The region $V_{2\mu} \setminus V_{\tau\mu}$ serves as a transition region. The energy on this region remains under control, since we use the values of v_μ outside of V_μ, where it coincides with the original map u.

We start with the model case of one vertical rectangle. Let $R_\mu = (-\mu\eta, \mu\eta)^2 \times (-1, 1)$. Given $v \in W^{1,p}(Q^3; \mathscr{N})$, we define $v_\tau^{\mathrm{sh}} \in W^{1,p}(Q^3; \mathscr{N})$ by

$$v_\tau^{\mathrm{sh}}(x', x_3) = \begin{cases} v(x', x_3) & \text{if } (x', x_3) \in Q^3 \setminus R_{2\mu}; \\ v(\frac{x'}{\tau}, x_3) & \text{if } (x', x_3) \in R_{\tau\mu}; \\ v\Big(\frac{x'}{|x'|}\big(\frac{1}{2-\tau}(|x'| - \tau\mu\eta) + \mu\eta\big), x_3\Big) & \text{otherwise.} \end{cases}$$

Relying on the additivity of the integral and the change of variable theorem, we estimate

$$\int_{R_{2\mu}} |Dv_\tau^{\mathrm{sh}}|^p \lesssim \int_{R_{2\mu} \setminus R_\mu} |Dv|^p + \tau^{2-p} \int_{R_\mu} |Dv|^p.$$

We now turn to the modification of our map v_μ. Applying the above construction to v_μ on each rectangle constituting $V_{2\mu}$, which is nothing else but a translate of $R_{2\mu}$, we obtain a map $u_{\mu,\tau} \in W^{1,p}(Q^3; \mathscr{N})$ such that

(i) $u_{\mu,\tau}$ is locally Lipschitz on $Q^3 \setminus \mathcal{S}_{\mu,\tau}$, where $\mathcal{S}_{\mu,\tau}$ is a Lipschitz submanifold of Q^3;

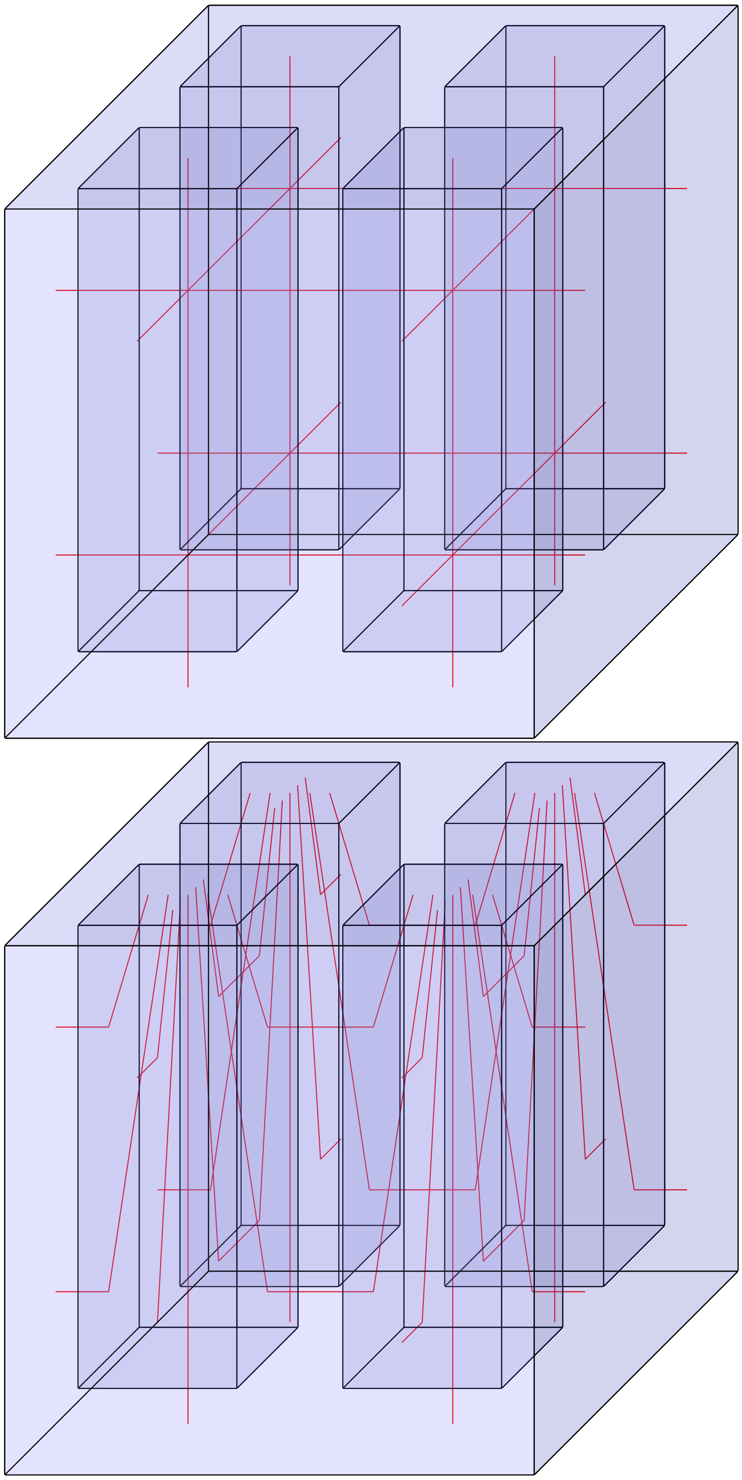

Figure 3.3 – The well W_μ, with singularities before and after uncrossing

(ii) $u_{\mu,\tau} = v_\mu = u$ outside of $V_{2\mu}$;

(iii)

$$\int_{V_{2\mu}} |Du_{\mu,\tau}|^p \lesssim \int_{V_{2\mu}\setminus V_\mu} |Dv_\mu|^p + \tau^{2-p} \int_{V_\mu} |Dv_\mu|^p.$$

Since $p < 2$, we may choose $\tau = \tau_\mu$ sufficiently small, depending on μ, so that

$$\tau^{2-p} \int_{V_\mu} |Dv_\mu|^p \lesssim \int_{V_{2\mu}\setminus V_\mu} |Dv_\mu|^p. \tag{3.3.1}$$

We now let $u_\mu = u_{\mu,\tau_\mu}$. Since $u_\mu = u$ outside of $V_{2\mu}$, we deduce that

$$\int_{Q^3} |Du - Du_\mu|^p \leq \int_{V_{2\mu}} |Du - Du_\mu|^p \lesssim \int_{V_{2\mu}} |Du|^p + \int_{V_{2\mu}\setminus V_\mu} |Dv_\mu|^p + \tau^{2-p} \int_{V_\mu} |Dv_\mu|^p.$$

As $v_\mu = u$ outside of V_μ, we infer from (3.3.1) that

$$\int_{Q^3} |Du - Du_\mu|^p \lesssim \int_{V_{2\mu}} |Du|^p.$$

But $|V_\mu| \to 0$ as $\mu \to 0$, so that Lebesgue's lemma ensures that $Du_\mu \to Du$ in $L^p(Q^3)$ as $\mu \to 0$. On the other hand, since $\mathscr{N}$ is compact, we readily have $u_\mu \to u$ in $L^p(Q^3)$ as $\mu \to 0$. Hence, we conclude that $u_\mu \to u$ in $W^{1,p}(Q^3)$ as $\mu \to 0$. Since u_μ is locally Lipschitz outside of $\mathcal{S}_{\mu,\tau_\mu}$, which is a Lipschitz submanifold of Q^3, this finishes the proof of the proposition. □

In the proof of Proposition 3.3.1 above, the uncrossing map Φ_μ^{top} is only Lipschitz, but not smooth. This is due to the use of a modified version of the radial retraction of a cube minus its origin to its boundary. However, in order to obtain a construction compatible with the higher order regularity of Sobolev mappings in the full range $0 < s < +\infty$, one needs to work with smooth, and not only merely Lipschitz maps. To achieve this, one should replace the retraction onto a boundary by a retraction onto a thick region. To ease the understanding of the general case in Section 3.3.2, we sketch the construction again in the special case of dimension $m = 3$ and line singularities.

We keep the same notation as in the proof of Proposition 3.3.1, and in particular, we work again with the region W_μ. We define the two additional, smaller regions $W_{\mu,\mathrm{in}}$ and $W_{\mu,\mathrm{out}}$ by $W_{\mu,\mathrm{in}} = W_{\underline{\rho}\mu}$ and $W_{\mu,\mathrm{out}} = W_{\overline{\rho}\mu}$, where $0 < \underline{\rho} < \overline{\rho} < 1$ are fixed. This way, we have $W_{\mu,\mathrm{in}} \subset W_{\mu,\mathrm{out}} \subset W_\mu$, and all these regions still contain all the crossings of the singular set $\mathscr{T}^1$.

The starting point is the fact that, for any two open cubes $Q_{\mathrm{in}} \subsetneq Q_{\mathrm{out}}$ centered at 0, denoting $\mathbb{R}^3_- = \mathbb{R}^2 \times (-\infty, 0)$, there exists a smooth diffeomorphism $\Theta \colon \mathbb{R}^3_- \to \mathbb{R}^3_-$ such that $\Theta(\mathbb{R}^3_-) \subset \mathbb{R}^3_- \setminus Q_{\mathrm{in}}$ and $\Theta = \mathrm{id}$ outside of Q_{out}. Such a map is obtained by letting $\Theta(x) = \lambda(x)x$, where $\lambda \colon \mathbb{R}^3_- \to [1, +\infty)$ is suitably chosen, and satisfies in particular $\lambda = 1$ outside of Q_{out}.

Inserting appropriately scaled copies of Θ around each part of W_μ and extending by identity outside produces a smooth map $\Phi^{\mathrm{top}}_\mu \colon Q^3 \to Q^3$ such that $\Phi^{\mathrm{top}}_\mu = \mathrm{id}$ outside of $W_{\mu,\mathrm{out}}$ and $\Phi^{\mathrm{top}}_\mu(Q^3) \subset Q^3 \setminus W_{\mu,\mathrm{in}}$. Thanks to our refined construction, replacing the retraction onto a boundary by a retraction onto the thick region $W_{\mu,\mathrm{out}} \setminus W_{\mu,\mathrm{in}}$, we managed to produce a map which is not only Lipschitz but even smooth, while retaining the key feature that its range has to avoid the crossings in the singular set $\mathscr{T}^1$. Therefore, $(\Phi^{\mathrm{top}}_\mu)^{-1}(\mathscr{T}^1)$ is a smooth submanifold of Q^3, as we needed. We shall build upon this idea, in combination with the constructions sketched below for handling higher dimensional singularities, to prove Theorem 3.1.2 in full generality in the whole range $0 < s < +\infty$.

We now turn to the case of the density of the class $\mathscr{R}^{\mathrm{uncr}}_2(Q^3; \mathscr{N})$, where the maps have plane singularities. Compared to the case of line singularities treated previously, the first topological step consisting in uncrossing the singularities features an additional difficulty, that we explain in this subsection in an informal way, with the help of figures. The precise construction of the topological step, as well as the analytical step in which we improve the construction with a control on the energy and which relies on the same scaling argument as for line singularities, are postponed to Section 3.3.2, where we explain precisely the general tools needed to prove Theorem 3.1.2.

Consider a singular set $\mathscr{T}^2$ for a map u in $\mathscr{R}^{\mathrm{rig}}_2(Q^3; \mathscr{N})$, given by the dual skeleton of the 0-skeleton $\mathscr{K}^0$ of a cubication of $\overline{Q^3}$ having inradius $\eta \in \frac{1}{2\mathbb{N}_*}$. As previously, we let $\mathscr{V}^2$ denote the vertical part of $\mathscr{T}^2$, that is, the union of all hyperplanes which constitute $\mathscr{T}^2$ whose associated vector space contains e_3. This set is made of two unions of parallel planes: the set $\mathscr{T}_{1,3}$ consisting of all the planes in $\mathscr{T}^2$ whose associated vector space is spanned by e_1 and e_3, and the set $\mathscr{T}_{2,3}$ consisting of all the planes in $\mathscr{T}^2$ whose associated vector space is spanned by e_2 and e_3. We also let $\mathscr{V}^2_{\mathrm{tr}} = \mathscr{V}^2 \cap \big((-1,1)^2 \times (-1+\eta, 1)\big)$ be the truncated version of $\mathscr{V}^2$.

As previously, given $0 < \mu < \frac{1}{2}$, we consider $W_\mu = (\mathscr{V}^2_{\mathrm{tr}} + Q^3_{\mu\eta}) \cap Q^3$ a well around $\mathscr{V}^2_{\mathrm{tr}}$. We first uncross the singularities in W_μ as follows. We start with a model construction to uncross two families of parallel planes. We observe that the construction carried out for lines in Q^3 in the proof of Proposition 3.3.1 may also be applied to lines in Q^2. Indeed, it suffices to perform a radial projection around a point outside Q^2 in order to retract $\overline{Q^2}$ onto all its edges except one. This construction can then be applied to uncross two planes (or, more precisely, portions of planes). Assume that one wants to uncross the singularities around the vertical portion of plane $\mathscr{P} = \{x \in Q^3 : x_1 = 0\}$. Consider the line segment $\mathscr{L} = \{0\} \times (-1, 1) \times \{1 + \varepsilon\}$, which is a line segment subparallel to $\mathscr{P}$ and lying slightly above $\mathscr{P}$. For every plane orthogonal to $\mathscr{L}$ determined by $x_2 = t$ with $-1 < t < 1$, one performs the 2-dimensional uncrossing procedure in this plane with respect to the unique point of $\mathscr{L}$ lying in the plane. Otherwise stated, one proceeds to a radial projection around a line segment in the e_2 direction lying slightly above $\mathscr{P}$, viewing the second coordinate variable as a dummy variable. This allows to uncross $\mathscr{P}$ from other planes in horizontal position. The procedure is illustrated in Figure 3.4: the vertical plane around which the well has been dug is uncrossed from the horizontal plane, and both vertical planes are left unchanged.

We may then elaborate on this idea to uncross all singularities in W_μ, as described in Figure 3.5. On the parts of W_μ that do not contain a crossing between two vertical planes (the four darkest parts around the central one in Figure 3.5), we insert a copy of the construction described just above, as in Figure 3.4. We note that the constructions are compatible on the region where two different parts touch — which is a union of vertical line segments — since they coincide with the identity there. On the parts of W_μ around the crossing between orthogonal vertical planes (the central part in Figure 3.5), we finish the construction of our extension by using the radial projection from a point slightly above the crossing, as we did for line singularities. The resulting effect of these glued constructions is to remove all the crossings between horizontal and vertical planes; see Figure 3.5. However, unlike in the case of line singularities, we are not done yet, since there still are crossings between orthogonal vertical planes to remove.

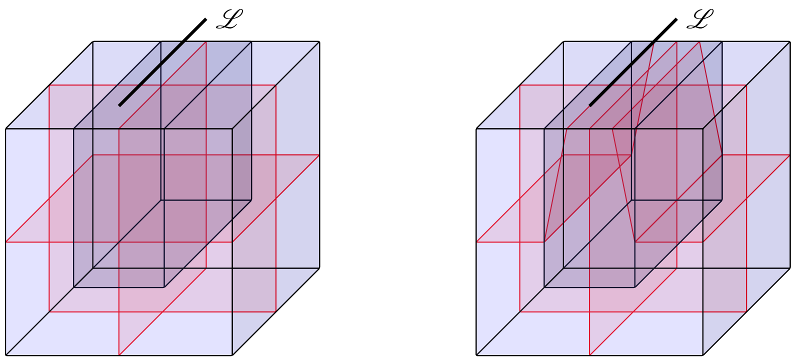

Figure 3.4 – Uncrossing plane singularities in one direction

For this purpose, we use a well in another direction. We consider the truncated set of parallel hyperplanes $\mathscr{H}^2_{\mathrm{tr}} = \mathscr{T}^{1,3} \cap \big((-1+\eta, 1) \times (-1,1)^2\big)$, and the well $H_\mu = (\mathscr{H}^2_{\mathrm{tr}} + Q^3_{\rho\mu\eta}) \cap Q^3$, where $0 < \rho < 1$ is chosen sufficiently small so that H_μ intersects

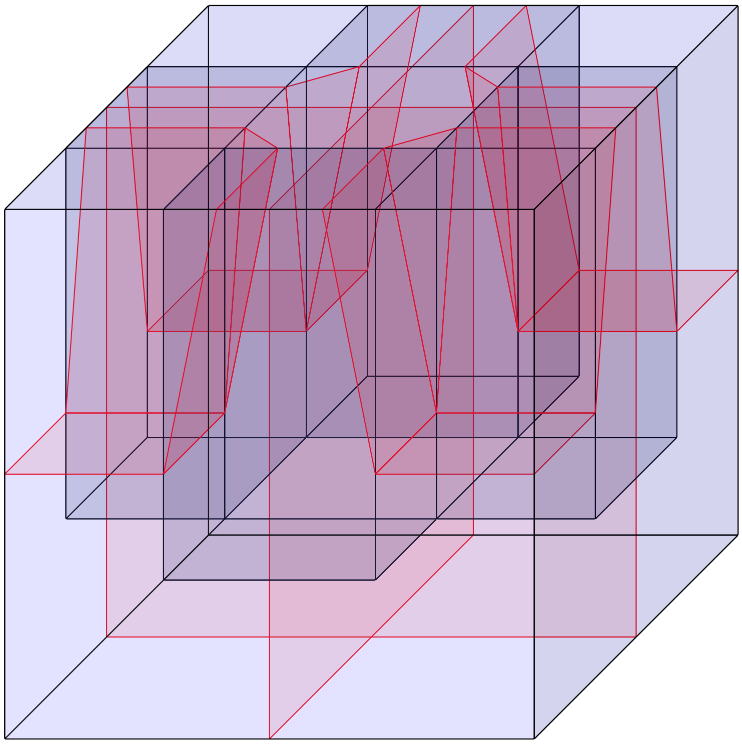

Figure 3.5 – Uncrossing plane singularities around all vertical planes

only the planes in the singular set that have not yet been uncrossed. We note that H_μ contains all the remaining singularities. We then insert a rotated copy of the construction illustrated in Figure 3.4 in each part of the well H_μ around a plane constituting $\mathscr{H}^2_{\mathrm{tr}}$. The procedure is illustrated in Figure 3.6 in the case where there is only one plane in each direction. At the end of this step, the crossings between orthogonal vertical planes have been removed, and therefore no crossings remain.

This concludes our informal presentation of some particular cases of crossings removal. In the next section, we introduce the general version of the two main tools that have been presented here: the topological construction to remove crossings, and the analytical procedure to control the energy on the modified region. These two tools are the key ingredients in the proof of our main result. Concerning the second one, we use the shrinking construction introduced by P. Bousquet, A. Ponce, and J. Van Schaftingen [BPVS15, Section 8]; see also Section 2.7 for the fractional order setting. For the first one, however, we need to perform an ad hoc construction, suited for our purposes. This construction is nevertheless very similar to the thickening procedure introduced in [BPVS15, Section 4] and presented in Section 2.5. As we have seen in our last example with plane singularities, the crossings removal procedure may involve gluing build-

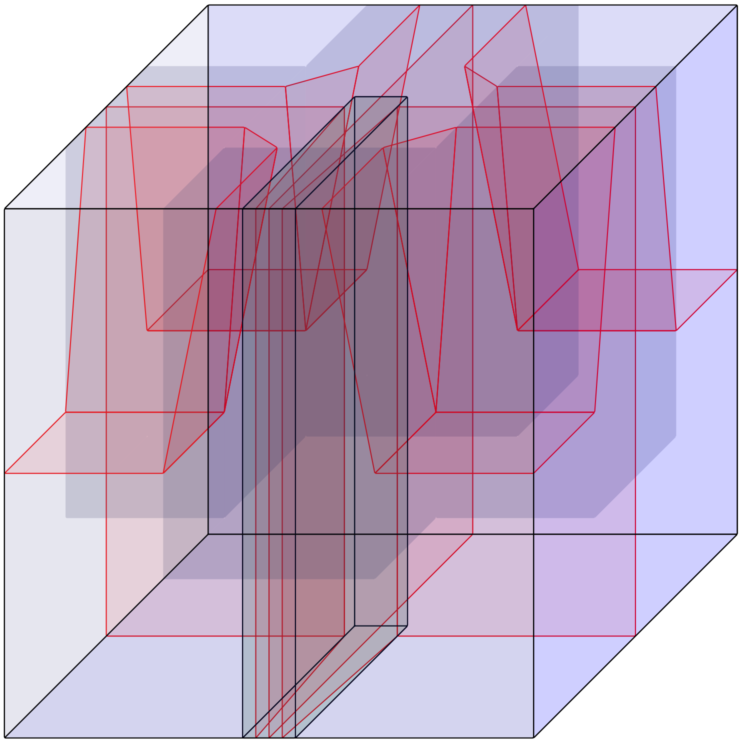

Figure 3.6 – Uncrossing plane singularities between vertical planes

ing blocks in various dimensions and also combining crossings removal procedures in different directions to get rid of all the existing crossings.

3.3.2 The general crossings removal procedure

We now explain how to prove the main result of this chapter, Theorem 3.1.2, in the general case. The argument follows the same two steps as in Proposition 3.3.1: First, we uncross the singularities through a topological procedure, and then we rely on an analytical argument to obtain a control on the energy.

We start by considering the first topological step. This is handled by the following proposition.

Proposition 3.3.2. *Let $d \in \{0, \dots, m-2\}$ and let $\mathscr{T}^{d^*}$ be the dual skeleton of the d-skeleton $\mathscr{K}^d$ of a cubication $\mathscr{K}^m$ of $\overline{Q^m}$ of inradius $\eta > 0$. For every $0 < \mu < 1$, there exists a smooth local diffeomorphism $\Phi\colon Q^m \to Q^m$ such that*

(i) $\mathscr{S}^{d^*} = \Phi^{-1}(\mathscr{T}^{d^*})$ *is a smooth d^*-dimensional submanifold of Q^m;*

(ii) $\Phi = \mathrm{id}$ *outside of* $\mathscr{T}^{d^*} + Q_{\mu\eta}$.

Moreover, Φ can be extended to a smooth local diffeomorphism on a slightly larger open set $\omega \subset \mathbb{R}^m$ such that $Q^m \Subset \omega$.

The proof of Proposition 3.3.2 is similar in its spirit to the thickening construction. However, to have a tool suited for our purposes here, we cannot re-use thickening as such, and we need to proceed to a quite different construction. We also note that our restriction $d \leq m-2$ excludes the case $d = m-1$, where $d^* = 0$, and hence the singular set would have been made of points. But in this case, the classes $\mathscr{R}^{\mathrm{rig}}$, $\mathscr{R}$, and $\mathscr{R}^{\mathrm{uncr}}$ all coincide, so that Theorem 3.1.2 is already contained in Bethuel's theorem and its counterpart for arbitrary s, and requires therefore no additional argument.

Proof. As explained in the last example of Section 3.3.1, the general uncrossing procedure requires to perform successive uncrossing steps in various directions.

Step 1. — Uncrossing singularities in a vertical well.

We let $\mathscr{V}$ be the part of $\mathscr{T}^{d^*}$ consisting only of d^*-planes that have the vertical vector e_m in their associated vector space. We also consider the truncated set of planes $\mathscr{V}_{\mathrm{tr}} = \mathscr{V} \cap ((-1,1)^{m-1} \times (-1+\eta, 1))$. Finally, we let $W_\mu = (\mathscr{V}_{\mathrm{tr}} + Q^m_{\mu\eta}) \cap Q^m$ be a well around $\mathscr{V}_{\mathrm{tr}}$. The well $W_{\mu/2}$ is defined accordingly. We note that $W_{\mu/2}$ contains all the crossings that involve at least one non vertical d^*-plane, i.e., a plane not in $\mathscr{V}$.

Let

$$\frac{\mu}{2} < \underline{\rho}_{d^*-1} < \overline{\rho}_{d^*-1} < \cdots < \underline{\rho}_0 < \overline{\rho}_0 < \mu.$$

We consider $\mathscr{E}^{d^*-1} = \mathscr{V} \cap ((-1,1)^{m-1} \times \{1\})$ the intersection of $\mathscr{V}$ with the top face of Q^m. We note that $\mathscr{E}^{d^*-1}$ is a (d^*-1)-skeleton. For every $l \in \{0, \ldots, d^*-1\}$, we define the rectangles

$$\begin{aligned}
R^l &= (-\mu\eta, \mu\eta)^{m-1-l} \times (-(1-\overline{\rho}_l)\eta, (1-\overline{\rho}_l)\eta)^l \times (-1+\eta-\mu\eta, 1),\\
R^l_{\mathrm{out}} &= (-\overline{\rho}_l\eta, \overline{\rho}_l\eta)^{m-1-l} \times (-(1-\overline{\rho}_l)\eta, (1-\overline{\rho}_l)\eta)^l \times (-1+\eta-\overline{\rho}_l\eta, 1),
\end{aligned}$$

and

$$R^l_{\mathrm{in}} = (-\underline{\rho}_l\eta, \underline{\rho}_l\eta)^{m-1-l} \times (-(1-\overline{\rho}_l)\eta, (1-\overline{\rho}_l)\eta)^l \times (-1+\eta-\underline{\rho}_l\eta, 1).$$

Given an l-face $\sigma^l \in E^l$, we let R_{σ^l} be the rotated copy of R^l positioned so that σ^l corresponds to $\{0\}^{m-l-1} \times (-1,1)^l \times \{1\}$. This way, we note that $R_{\sigma^l} \subset W_\mu$ for every $l \in \{0, \ldots, d^*-1\}$ and every $\sigma^l \in E^l$, and that actually W_μ is made of the union of all such R_{σ^l}. We define similarly $R_{\sigma^l,\mathrm{in}}$ and $R_{\sigma^l,\mathrm{out}}$.

We use as a tool the following special case of the construction from [BPVS15, Proposition 4.3], and that we presented in Proposition 2.5.2: There exists a smooth local diffeomorphism $\Theta_l\colon Q^l_{\mu\eta}\setminus\{0\}\to Q^l_{\mu\eta}$ such that

(i) $\Theta_l(Q^l_{\mu\eta}\setminus\{0\})\subset Q^l_{\mu\eta}\setminus Q^l_{\underline{\rho}_l\eta}$;

(ii) $\Theta_l=\mathrm{id}$ outside of $Q^l_{\overline{\rho}_l\eta}$.

We recall that the map Θ_l is constructed by letting $\Theta_l(x)=\lambda(x)x$ for some well-chosen smooth map $\lambda\colon Q^l_{\mu\eta}\setminus\{0\}\to[1,+\infty)$ such that $\lambda=1$ outside of $Q^l_{\overline{\rho}_l\eta}$. We focus our attention to the restriction of Θ_l to the lower part of $Q^l_{\mu\eta}$, slightly below $\{0\}$. After a suitable distortion of $Q^l_{\mu\eta}$ and addition of dummy variables, this yields a smooth local diffeomorphism $\Psi_l\colon R^l\to R^l$ such that

(i) $\Psi_l(R^l)\subset R^l\setminus R^l_{\mathrm{in}}$;

(ii) $\Psi_l=\mathrm{id}$ outside of R^l_{out}.

Let Ψ_{σ^l} be the map obtained by transporting isometrically Ψ_l to R_{σ^l}, and define $\Phi_l(x)=\Psi_{\sigma^l}(x)$ if $x\in R_{\sigma^l}$. We note that this is well defined. Indeed, if $x\in R_{\sigma^l_1}\cap R_{\sigma^l_2}$, then x is outside of $R_{\sigma^l_1,\mathrm{out}}$ and $R_{\sigma^l_2,\mathrm{out}}$, which implies that $\Psi_{\sigma^l_1}(x)=x=\Psi_{\sigma^l_2}(x)$.

We readily observe that the map Φ_l can be smoothly extended by identity to $Q^m\setminus\bigcup_{\substack{\sigma^{l'}\in E^{l'}\\ l'\in\{0,\dots,l-1\}}}R_{\sigma',\mathrm{in}}$. In particular, this yields $\Phi_l=\mathrm{id}$ outside of W_μ. Moreover, Φ_l has the property that it maps $\bigcup_{\sigma^l\in E^l}R_{\sigma^l}$ outside of $\bigcup_{\sigma^l\in E^l}R_{\sigma^l,\mathrm{in}}$. By induction, this implies that the composition $\Phi_{\mathrm{v}}=\Phi_{d^*-1}\circ\cdots\circ\Phi_0$ is a well-defined smooth local diffeomorphism and maps Q^m outside of $W_{\mu/2}$. We note importantly that the well-defined character of the map relies on the fact that, although Φ_l is not defined on the whole Q^m, it is nevertheless defined on the range of the composition of the previous maps in the induction process. In particular, $\Phi_{\mathrm{v}}^{-1}(\mathscr{T}^{d^*})$ is a finite union of smooth d^*-dimensional submanifolds of Q^m, and as $W_{\mu/2}$ contains all the crossings between d^*-planes in $\mathscr{T}^{d^*}$ involving at least one non vertical one, we deduce that the only submanifolds in $\Phi_{\mathrm{v}}^{-1}(\mathscr{T}^{d^*})$ that intersect correspond to inverse images of vertical d^*-planes. Finally, since the building blocks Θ_l have the form $\Theta_l(x)=\lambda(x)x$, we also find that $\Phi_{\mathrm{v}}^{-1}(\mathscr{V})=\mathscr{V}$.

Step 2. — Uncrossing vertical planes.

It remains to remove the crossings between planes in $\mathscr{V}$. For this purpose, we choose another — non vertical — direction, and we rotate Q^m to make it correspond to the vertical one. We then repeat the exact same construction as in the first step, except that we replace W_μ by $W_{\rho\mu}$ for some $0<\rho<1$ so small that $W_{\rho\mu}$ does not intersect the inverse images under Φ_{v} of d^*-planes of $\mathscr{T}^{d^*}\setminus\mathscr{V}$. The construction should then be modified accordingly, adding the scaling ρ wherever necessary, and this yields another smooth local diffeomorphism $\Phi_{\mathrm{h}}\colon Q^m\to Q^m$ that coincides with the identity outside of $W_{\rho\mu}$

and such that $\Phi_{\mathrm{h}}^{-1}(\mathscr{V})$ is a finite union of smooth d^*-dimensional submanifolds of Q^m. Moreover, only the inverse images coming from planes in the new vertical direction may still cross. Therefore, the map $\Phi_{\mathrm{v}} \circ \Phi_{\mathrm{h}}$ is a smooth local diffeomorphism that coincides with the identity outside of $\mathscr{T}^{d^*} + Q^m_{\mu\eta}$ and such that $(\Phi_{\mathrm{v}} \circ \Phi_{\mathrm{h}})^{-1}(\mathscr{T}^{d^*})$ is a finite union of smooth submanifolds of Q^m, and only the parts coming from d^*-planes aligned with the two chosen directions may still cross.

We pursue this procedure, chosing each time a new direction to be the vertical one, until no crossing remains. This yields the desired map Φ.

Moreover, it is readily observed from our construction, since each building block could have been defined on a slightly larger set, that Φ may be extended to a smooth local diffeomorphism defined on a slightly larger set. □

Concerning the analytical step, it relies on the shrinking construction, and more precisely on Proposition 2.7.1 and the remark below the proposition.

Before proving Theorem 3.1.2, we need one last technical tool. Indeed, our proof involves composing the map $u \in \mathscr{R}^{\mathrm{rig}} \subset \mathscr{R}$ we want to approximate with the maps provided by Propositions 3.3.2 and 2.7.1. The following lemma ensures that the class $\mathscr{R}$ is stable through composition with a local diffeomorphism.

Lemma 3.3.3. *Let $\varepsilon > 0$, and let $\Phi \colon Q^m_{1+\varepsilon} \to \mathbb{R}^m$ be a local diffeomorphism such that $\Phi(Q^m) \subset Q^m$. For every $u \in \mathscr{R}_i(Q^m)$, we have that $u \circ \Phi \in \mathscr{R}_i(Q^m)$.*

Proof. Let $\mathscr{S}$ denote the singular set of u. We may assume that $\mathscr{S} \neq \varnothing$, otherwise the proof is trivial. We may also assume that Φ is actually a local diffeomorphism defined on the whole $\mathbb{R}^m$. Indeed, if this is not the case, we consider a diffeomorphism $\Psi \colon \mathbb{R}^m \to Q^m_{1+\varepsilon}$ such that $\Psi = \mathrm{id}$ on Q^m, and we replace Φ by $\Phi \circ \Psi$, which is a local diffeomorphism on $\mathbb{R}^m$ and coincides with Φ on Q^m. We note importantly that, if Φ has the additional property that $\Phi^{-1}(\mathscr{S})$ is a closedly embedded submanifold of $Q^m_{1+\varepsilon}$ (for the relative topology), then $(\Phi \circ \Psi)^{-1}(\mathscr{S})$ is a closedly embedded submanifold of $\mathbb{R}^m$. This will be important in the sequel, when working with constructions to produce maps in the class $\mathscr{R}^{\mathrm{uncr}}$.

Since Φ is a local diffeomorphism, the map $u \circ \Phi$ is smooth on $Q^m \setminus \tilde{\mathscr{S}}$, where $\tilde{\mathscr{S}} = \Phi^{-1}(\mathscr{S})$ is a finite union of smooth i-dimensional submanifolds of $\mathbb{R}^m$. Moreover, if u extends smoothly on $U \setminus \mathscr{S}$ for some open set $U \subset \mathbb{R}^m$ satisfying $Q^m \Subset U$, then $u \circ \Phi$ extends smoothly on $\Phi^{-1}(U) \setminus \tilde{\mathscr{S}}$, and $\Phi^{-1}(U)$ is an open subset of $\mathbb{R}^m$ satisfying $Q^m \Subset \Phi^{-1}(U)$. It therefore remains to prove the estimates on the derivatives of $u \circ \Phi$.

For this purpose, we first note that, as Φ is defined on the whole $\mathbb{R}^m$, it has bounded derivatives on Q^m. Therefore, the Faà di Bruno formula ensures that, for every $x \in Q^m$

and $j \in \mathbb{N}_*$,

$$|D^j(u \circ \Phi)(x)| \lesssim \sum_{t=1}^{j} |D^t u(\Phi(x))| \lesssim \sum_{t=1}^{j} \frac{1}{\operatorname{dist}(\Phi(x), \mathcal{S})^t} \lesssim \frac{1}{\operatorname{dist}(\Phi(x), \mathcal{S})^j}.$$

We conclude the proof by showing that $\operatorname{dist}(\Phi(x), \mathcal{S}) \gtrsim \operatorname{dist}(x, \tilde{\mathcal{S}})$ for every $x \in Q^m$.

For this purpose, we first note that, by a compactness argument, there exists $\delta > 0$ such that, for every $x \in Q^m$, the restriction of Φ to $\Phi^{-1}(B^m_\delta(\Phi(x)))$ is a diffeomorphism onto $B^m_\delta(\Phi(x))$. Taking δ smaller if necessary, this implies in particular that

$$|x - y| \lesssim |\Phi(x) - \Phi(y)| \quad \text{whenever } |\Phi(x) - \Phi(y)| < \delta. \tag{3.3.2}$$

It suffices to show that $\operatorname{dist}(\Phi(x), \mathcal{S}) \gtrsim \operatorname{dist}(x, \tilde{\mathcal{S}})$ whenever $x \in Q^m$ is such that $\Phi(x)$ is sufficiently close to $\mathcal{S}$. Hence, let $x \in Q^m$ be such that $\operatorname{dist}(\Phi(x), \mathcal{S}) < \delta$. Let $z \in \mathcal{S}$ be such that $|\Phi(x) - z| = \operatorname{dist}(\Phi(x), \mathcal{S})$. In particular, there exists $y \in \Phi^{-1}(B^m_\delta(\Phi(x)))$ such that $\Phi(y) = z$. With this choice, we have $y \in \Phi^{-1}(\mathcal{S}) = \tilde{\mathcal{S}}$ and therefore, due to (3.3.2),

$$\operatorname{dist}(x, \tilde{\mathcal{S}}) \leq |x - y| \lesssim |\Phi(x) - \Phi(y)| = \operatorname{dist}(\Phi(x), \mathcal{S}).$$

This concludes the proof of the lemma. □

Proof of Theorem 3.1.2. Since the more rigid class $\mathcal{R}^{\mathrm{rig}}_{m-[sp]-1}(Q^m; \mathcal{N})$ is dense in the space $W^{s,p}(Q^m; \mathcal{N})$, it suffices to consider $u \in \mathcal{R}^{\mathrm{rig}}_{m-\lfloor sp \rfloor-1}(Q^m; \mathcal{N})$ and to show that it can be approximated by maps in the uncrossed class $\mathcal{R}^{\mathrm{uncr}}_{m-\lfloor sp \rfloor-1}(Q^m; \mathcal{N})$. Let $d = \lfloor sp \rfloor$, and let $\mathcal{T}^{d^*}$ be the dual skeleton of the d-skeleton $\mathcal{K}^d$ of a cubication $\mathcal{K}^m$ of radius $\eta > 0$ of $\overline{Q^m}$, chosen so that $\mathcal{T}^{d^*}$ coincides with the singular set of u. Recall that, as already explained, we may limit ourselves to consider maps such that their singular set is placed like this. Also, recall that we may assume that $d \leq m - 2$, as if $d = m - 1$ there is nothing to prove.

For the sake of conciseness, we let $\mathcal{A}_\mu = Q^m \cap (\mathcal{T}^{d^*} + Q^m_{2\mu\eta})$. Given $0 < \mu < \frac{1}{2}$, we start by applying Proposition 3.3.2 to obtain a map $\Phi^{\mathrm{top}}_\mu \colon Q^m \to Q^m$ such that, defining $u^{\mathrm{top}}_\mu = u \circ \Phi^{\mathrm{top}}_\mu$, we have that

(i) $\mathcal{S}^{\mathrm{top}}_\mu = (\Phi^{\mathrm{top}}_\mu)^{-1}(\mathcal{T}^{d^*})$ is a smooth d^*-dimensional submanifold of Q^m;

(ii) $u^{\mathrm{top}}_\mu = u$ outside of $\mathcal{T}^{d^*} + Q^m_{\mu\eta}$;

(iii) $u^{\mathrm{top}}_\mu \in W^{s,p}(Q^m)$.

Item (iii) above is a consequence of the fact that $u^{\mathrm{top}}_\mu \in \mathcal{R}_{m-\lfloor sp \rfloor-1}(Q^m; \mathcal{N})$, as $u \in \mathcal{R}_{m-\lfloor sp \rfloor-1}(Q^m; \mathcal{N})$ and using Lemma 3.3.3.

Since $d+1 = \lfloor sp \rfloor + 1 > sp$ and thanks to (ii) and (iii) above, we may now invoke the remark that follows Proposition 2.7.1 on u^{top}_{μ} and u, with $\varepsilon = \mu$, to deduce the existence of a smooth local diffeomorphism $\Phi^{\mathrm{sh}}_{\mu} \colon Q^m \to Q^m$ such that, letting $u^{\mathrm{sh}}_{\mu} = u^{\mathrm{top}}_{\mu} \circ \Phi^{\mathrm{sh}}_{\mu}$, we have that $u^{\mathrm{sh}}_{\mu} \in W^{s,p}(Q^m)$ with

(i) if $0 < s < 1$, then

$$|u^{\mathrm{sh}}_{\mu} - u|_{W^{s,p}(Q^m)} \lesssim |u|_{W^{s,p}(\mathscr{A}_{\mu})} + (\mu\eta)^{-s}\|u\|_{L^p(\mathscr{A}_{\mu})} + \mu;$$

(ii) if $s \geq 1$, then for every $j \in \{1, \dots, k\}$,

$$\|\mathrm{D}^j u^{\mathrm{sh}}_{\mu} - \mathrm{D}^j u\|_{L^p(Q^m)} \lesssim \sum_{i=1}^{j} (\mu\eta)^{i-j}\|\mathrm{D}^i u\|_{L^p(\mathscr{A}_{\mu})} + \mu;$$

(iii) if $s \geq 1$ and $\sigma \neq 0$, then for every $j \in \{1, \dots, k\}$,

$$|\mathrm{D}^j u^{\mathrm{sh}}_{\mu} - \mathrm{D}^j u|_{W^{\sigma,p}(Q^m)} \lesssim \sum_{i=1}^{j} \Big((\mu\eta)^{i-j-\sigma}\|\mathrm{D}^i u\|_{L^p(\mathscr{A}_{\mu})} + (\mu\eta)^{i-j}|\mathrm{D}^i u|_{W^{\sigma,p}(\mathscr{A}_{\mu})} \Big) + \mu;$$

(iv) for every $0 < s < +\infty$,

$$\|u^{\mathrm{sh}}_{\mu} - u\|_{L^p(Q^m)} \lesssim \|u\|_{L^p(\mathscr{A}_{\mu})} + \mu.$$

Since Φ^{sh}_{μ} is a local diffeomorphism, we know that $(\Phi^{\mathrm{sh}}_{\mu})^{-1}(\mathscr{S}^{\mathrm{top}}_{\mu})$ is a smooth d^*-dimensional submanifold of Q^m. Moreover, $\Phi^{\mathrm{top}}_{\mu}$ and Φ^{sh}_{μ} satisfy the assumptions of Lemma 3.3.3. This shows that $u^{\mathrm{sh}}_{\mu} \in \mathscr{R}^{\mathrm{uncr}}_{m-\lfloor sp \rfloor - 1}(Q^m; \mathscr{N})$ for every $0 < \mu < \frac{1}{2}$, and it therefore only remains to prove the $W^{s,p}$ convergence $u^{\mathrm{sm}}_{\mu} \to u$ as $\mu \to 0$ to conclude the proof. To accomplish this, we verify that the quantities in the right-hand side of (i) to (iv) converge to 0 as $\mu \to 0$. We draw the attention of the reader to the fact that the reasoning follows the same lines as the proof of Proposition 2.8.3.

We first observe the following estimate on the measure of $\mathscr{A}_{\mu}$:

$$|\mathscr{A}_{\mu}| \lesssim (\mu\eta)^{d+1}. \tag{3.3.3}$$

By Lebesgue's lemma, we deduce that the quantities $|u|_{W^{s,p}(\mathscr{A}_{\mu})}$ and $\|u\|_{L^p(\mathscr{A}_{\mu})}$, that appear on (i) and (iv) respectively, indeed tend to 0 as $\mu \to 0$. Moreover, when $0 < s < 1$, using the fact that $u \in L^{\infty}(Q^m)$ by the compactness of $\mathscr{N}$, we have

$$\|u\|_{L^p(\mathscr{A}_{\mu})} \lesssim |\mathscr{A}_{\mu}|^{\frac{1}{p}} \lesssim (\mu\eta)^{\frac{d+1}{p}}.$$

Therefore,

$$(\mu\eta)^{-s}\|u\|_{L^p(\mathscr{A}_\mu)} \lesssim (\mu\eta)^{\frac{d+1-sp}{sp}},$$

which converges to 0 as $\mu \to 0$ because of the fact that $sp < d+1$.

We now consider estimates (ii) and (iii), when $s \geq 1$. Observe that, since $u \in W^{s,p}(Q^m) \cap L^\infty(Q^m)$, the Gagliardo–Nirenberg inequality implies that $\mathrm{D}^i u \in L^{sp/i}(Q^m)$ for every $i \in \{1, \ldots, k\}$. Hence, Hölder's inequality and (3.3.3) ensure that

$$\|\mathrm{D}^i u\|_{L^p(\mathscr{A}_\mu)} \leq |\mathscr{A}_\mu|^{\frac{s-i}{sp}}\|\mathrm{D}^i u\|_{L^{sp/i}(\mathscr{A}_\mu)} \lesssim (\mu\eta)^{\frac{(d+1)(s-i)}{sp}}\|\mathrm{D}^i u\|_{L^{sp/i}(\mathscr{A}_\mu)}.$$

Therefore, we deduce that

$$(\mu\eta)^{i-j}\|\mathrm{D}^i u\|_{L^p(\mathscr{A}_\mu)} \lesssim (\mu\eta)^{\frac{(d+1)(s-i)-(j-i)sp}{sp}}\|\mathrm{D}^i u\|_{L^{sp/i}(\mathscr{A}_\mu)} \lesssim (\mu\eta)^{\frac{(j-i)(d+1-sp)}{sp}}\|\mathrm{D}^i u\|_{L^{sp/i}(\mathscr{A}_\mu)}.$$

As $sp < d+1$, the exponent of $\mu\eta$ is positive, which implies that the right-hand side converges to 0 as $\mu \to 0$. This handles estimate (ii).

If $s \geq 1$ and $\sigma \neq 0$, the same reasoning leads to

$$(\mu\eta)^{i-j-\sigma}\|\mathrm{D}^i u\|_{L^p(\mathscr{A}_\mu)} \lesssim (\mu\eta)^{\frac{(\sigma+j-i)(d+1-sp)}{sp}}\|\mathrm{D}^i u\|_{L^{sp/i}(\mathscr{A}_\mu)},$$

which also goes to 0 as $\mu \to 0$. Similarly, by interpolation, using Lemma 2.6.2, we find

$$\begin{aligned}|D^i u|_{W^{\sigma,p}(\mathscr{A}_\mu)} &\lesssim |\mathscr{A}_\mu|^{\frac{s-i-\sigma}{sp}}\|\mathrm{D}^i u\|^{1-\sigma}_{L^{sp/i}(\mathscr{A}_\mu)}\|\mathrm{D}^{i+1}u\|^{\sigma}_{L^{sp/i+1}(Q^m)}\\ &\lesssim (\mu\eta)^{\frac{(s-i-\sigma)(d+1)}{sp}}\|\mathrm{D}^i u\|^{1-\sigma}_{L^{sp/i}(\mathscr{A}_\mu)}\end{aligned}$$

for every $i \in \{1, \ldots, j-1\}$. Therefore,

$$(\mu\eta)^{i-j}|\mathrm{D}^i u|_{W^{\sigma,p}(\mathscr{A}_\mu)} \lesssim (\mu\eta)^{\frac{(j-i)(d+1-sp)}{sp}}\|\mathrm{D}^i u\|^{1-\sigma}_{L^{sp/i}(\mathscr{A}_\mu)},$$

which once more goes to 0 as $\mu \to 0$. This finishes to handle the second term in estimate (iii) when $i < j$. The second term for $i = j$ is simply $|\mathrm{D}^j u|_{W^{\sigma,p}(\mathscr{A}_\mu)}$, which converges to 0 due to the Lebesgue lemma.

All cases being covered, this finishes to prove that $u^{\mathrm{sh}}_\mu \to u$ as $\mu \to 0$, which concludes the proof of the theorem. □

As a concluding remark, we note that our method uses in a explicit way the fact that the domain is a cube. However, the argument can be adapted to any domain which has a shape allowing to evacuate crossings as we did for the cube. For instance, consider the ball with a hole $B_2^m \setminus B_1^m \subset \mathbb{R}^m$. One may use a decomposition into cells that are diffeomorphic to cubes and arranged in a radial way, and evacuate crossings between lines along the radial direction to deduce the density of $\mathscr{R}_1^{\mathrm{uncr}}(B_2^m \setminus B_1^m; \mathscr{N})$ in $W^{s,p}(B_2^m \setminus B_1^m; \mathscr{N})$ when $\lfloor sp \rfloor = m - 2$. The idea of the construction is illustrated on Figure 3.7 in dimension $m = 2$, where we have represented the singular set of the map in the radial equivalent of the class $\mathscr{R}^{\mathrm{rig}}$ to be approximated in red, and the wells used to uncross the singularities in dark blue. We shall not attempt to present a detailed argument, since it would require to adapt the whole proof of the density of class $\mathscr{R}^{\mathrm{rig}}$ in a radial version, which would considerably increase the length of this text. We therefore keep this observation as a remark, and not a theorem with precise statement and proof. On the other hand, on the same domain, the method does not seem to work to uncross plane singularities for instance, since there is no second direction along which to evacuate the remaining crossings after the first uncrossing step has been performed.

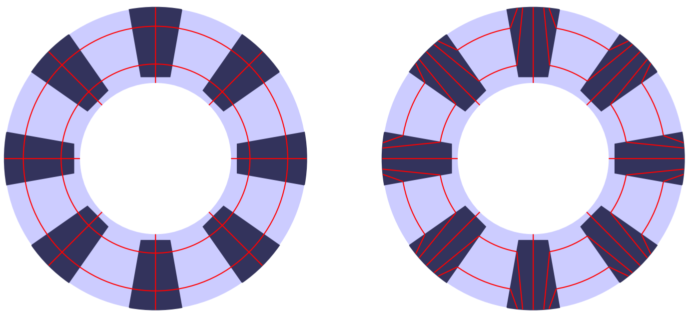

Figure 3.7 – Radial uncrossing procedure

Nevertheless, this particular situation does not provide a counterexample, since one could extend the map to be approximated inside the hole by homogeneous extension, and then apply the technique we introduced to uncross the singularities of the extended map on B_2^m. However, such a straightforward extension argument cannot be implemented on a general domain. Actually, there does not seem to be a direct way to solve the case of a general domain using the technique we introduced for Q^m as such.

It is not clear to us what should be the general situation. It could be that the class $\mathscr{R}^{\mathrm{uncr}}$ is always dense in $W^{s,p}$, but that the proof for a general domain requires an adaptation of our argument or even a new idea. It could also be that there are some new obstructions that arise, stemming for instance from the topology of the domain, in the spirit of the work of Hang F. and Lin F. [HL03a]. This motivates us to conclude on the following open problem.

Question 3.3.4. Is it true that $\mathscr{R}^{\mathrm{uncr}}_{m-\lfloor sp \rfloor -1}(\Omega; \mathscr{N})$ is always dense in $W^{s,p}(\Omega; \mathscr{N})$ for any domain $\Omega \subset \mathbb{R}^m$ sufficiently smooth?

Chapter 4

Generic non-uniqueness of minimizing harmonic maps from a ball to a sphere

Résumé

Dans ce chapitre, nous mettons en œuvre certaines des techniques que nous avons présentées jusqu'ici pour étudier les applications harmoniques minimisantes. Plus spécifiquement, nous étudions une question de non-unicité générique pour les applications harmoniques minimisantes de $\mathbb{B}^3$ vers $\mathbb{S}^2$. Nous montrons que toute donnée au bord peut être modifiée en une autre donnée au bord qui admet plusieurs minimiseurs de l'énergie de Dirichlet, et ce en effectuant seulement une faible modification par rapport à la norme $W^{1,p}$ pour $p < 2$. Ceci renforce une remarque due à K. Mazowiecka et P. Strzelecki, et est également dans l'esprit d'un résultat d'unicité générique dû à F. Almgren et E. Lieb. Le nouvel ingrédient principal est une construction d'homotopie, qui répond à une variante simplifiée d'une question délicate concernant l'existence d'un contrôle en norme pour les homotopies entre applications $W^{1,p}$, et repose notamment sur la technique du *shrinking* qui a déjà prouvé sa grande utilité dans les deux chapitres qui précèdent. Cela correspond à un travail en collaboration avec K. Mazowiecka.

Abstract

In this chapter, we apply some of the techniques that we have presented until now to study minimizing harmonic maps. More specifically, we study non-uniqueness for minimizing harmonic maps from $\mathbb{B}^3$ to $\mathbb{S}^2$. We show that every boundary map can be modified to a boundary map that admits multiple minimizers of the Dirichlet energy by a small $W^{1,p}$-change for $p < 2$. This strengthens a remark by K. Mazowiecka and P. Strzelecki, and is also in line with a generic uniqueness result from F. Almgren and E. Lieb. The main novel ingredient is a homotopy construction, which is the answer to an easier variant of a challenging question regarding the existence of a norm control for homotopies between $W^{1,p}$ maps, and relies notably on the shrinking technique that has already proved its great usefulness in the two previous chapters. This corresponds to a joint work with K. Mazowiecka.

4.1 Introduction

As we already explained in the introduction of this thesis, minimizing harmonic maps from $\mathbb{B}^3$ to $\mathbb{S}^2$ are defined as mappings with the least Dirichlet energy

$$E(u) = \mathcal{E}^{1,2}(u, \mathbb{B}^3) = \int_{\mathbb{B}^3} |\mathrm{D}u|^2 \,\mathrm{d}x \tag{4.1.1}$$

among maps $u \in W^{1,2}(\mathbb{B}^3; \mathbb{S}^2)$ with fixed boundary datum $u_{|\partial\mathbb{B}^3} = \varphi \in W^{1/2,2}(\partial\mathbb{B}^3; \mathbb{S}^2)$.

For such maps φ, we also define the space

$$W^{1,2}_{\varphi}(\mathbb{B}^3; \mathbb{S}^2) = \{v \in W^{1,2}(\mathbb{B}^3; \mathbb{S}^2) \colon v = \varphi \text{ on } \partial\mathbb{B}^3 \text{ in the trace sense}\}$$

and note that this space is always nonempty. For instance, for a given smooth boundary datum $\varphi \in C^\infty(\partial\mathbb{B}^3; \mathbb{S}^2)$, one can easily construct an extension $u \in W^{1,2}(\mathbb{B}^3; \mathbb{S}^2)$ of φ, simply by considering $u(x) = \varphi\big(\frac{x}{|x|}\big)$. More generally, any boundary map $\varphi \in W^{1/2,2}(\partial\mathbb{B}^3; \mathbb{S}^2)$ admits an extension $u \in W^{1,2}(\mathbb{B}^3; \mathbb{S}^2)$; see [HL87, Theorem 6.2]. Once again, we emphasize that this is not an immediate consequence of the analogue property of linear Sobolev spaces. For example, there exists a boundary datum $\varphi \in W^{1/2,2}(\partial\mathbb{B}^3; \mathbb{S}^1)$ which has *no* extension $u \in W^{1,2}(\mathbb{B}^3; \mathbb{S}^1)$; see [HL87, 6.3]. As we briefly mentioned in Section 1.4.1, this is related to the so-called *extension of traces problems*, which stands along density problems among the highly non-trivial questions regarding the properties of Sobolev spaces of mappings into manifolds.

Minimizing harmonic maps satisfy the following system of Euler–Lagrange equations

$$\begin{cases} -\Delta u = |\mathrm{D}u|^2 u & \text{in } \mathbb{B}^3, \\ \quad\; u = \varphi & \text{on } \partial\mathbb{B}^3. \end{cases} \tag{4.1.2}$$

It is known that for every non-constant boundary datum, the system (4.1.2) admits infinitely many solutions; see [Riv93]. Minimizers of (4.1.1) are not the only solutions to (4.1.2) (see, e.g., [HKL88, Section 3]). However, even in the class of minimizing harmonic maps, we do not have uniqueness for a given boundary datum $\varphi \colon \mathbb{B}^3 \to \mathbb{S}^2$. We already mentioned this striking difference as compared to the linear theory of minimizing harmonic (real-valued) functions in the introduction; let us list here a few specific examples.

— In [HK88, Section 3], there is an example of a planar boundary datum which admits two different minimizers, one with values on the southern hemisphere and the other one with values on the northern hemisphere.

— In [HKL90, 2.2. Corollary], there is an example of a boundary datum for which there exists a 1-parameter family of distinct energy minimizing maps.

— In [HL89, Section 5], there is an example of a boundary map which serves as a boundary datum for at least two minimizers, one singular and the other one regular.

— In [AL88, 5.5 Theorem], there is an example of a boundary datum with mirror symmetry for which there are at least two different minimizers without the mirror symmetry.

Nevertheless, in the class of minimizing harmonic maps, we have the following *generic uniqueness* result ([AL88] attributes this theorem to F. Almgren).

Theorem 4.1.1 ([AL88, Theorem 4.1])**.** *Let $\varphi \in W^{1,2}(\mathbb{S}^2;\mathbb{S}^2)$. For every $\varepsilon > 0$, there exists $\psi \in W^{1,2}(\mathbb{S}^2;\mathbb{S}^2)$ such that $\|\varphi - \psi\|_{W^{1,2}(\mathbb{S}^2)} < \varepsilon$ and for which there exists exactly one energy minimizer $u\colon \mathbb{B}^3 \to \mathbb{S}^2$ having boundary datum ψ. Moreover, ψ coincides with φ outside of $B^3_\varepsilon(x) \cap \mathbb{S}^2$, for some $x \in \mathbb{S}^2$.*

In [MS17], K. Mazowiecka and P. Strzelecki suspected that *generic non-uniqueness* occurs, when taking into account small perturbation of the boundary datum in the topology of the space $W^{1,p}$ for $p < 2$. The main result of this chapter is the strengthening of [MS17, Remark 4.1].

Theorem 4.1.2. *Let $\varphi \in C^\infty(\mathbb{S}^2;\mathbb{S}^2)$. For every $\varepsilon > 0$, there exists $\psi \in C^\infty(\mathbb{S}^2;\mathbb{S}^2)$ such that $\|\varphi - \psi\|_{W^{1,p}(\mathbb{S}^2)} < \varepsilon$ which serves as a boundary datum for at least two energy minimizing maps from $\mathbb{B}^3$ to $\mathbb{S}^2$ having a different number of singularities.*

Otherwise stated, Theorem 4.1.2 asserts that boundary data for which non-uniqueness occurs are dense in $W^{1,p}(\mathbb{S}^2;\mathbb{S}^2)$. This strengthens [HL89, Section 5] and [MS17, Remark 4.1], which provide existence of *one* boundary map for which non-uniqueness occurs. To be precise, as it is stated, Theorem 4.1.2 only asserts that boundary data subjected to non-uniqueness are dense in $C^\infty(\mathbb{S}^2;\mathbb{S}^2)$ with respect to the $W^{1,p}$ topology. In turn, $C^\infty(\mathbb{S}^2;\mathbb{S}^2))$ is dense in $W^{1,p}(\mathbb{S}^2;\mathbb{S}^2)$, which ensures the density of boundary data for which non-uniqueness occurs in the whole $W^{1,p}(\mathbb{S}^2;\mathbb{S}^2)$. In the setting of the general answer to the strong density problem, this latter fact follows from the fact that $\pi_{\lfloor p \rfloor}(\mathbb{S}^1) = \pi_1(\mathbb{S}^2) = \{0\}$, but let us mention that this special case was already contained in [BZ88, Theorem 1].

Both Theorem 4.1.1 and Theorem 4.1.2 are in line with the *stability* results: On one hand, it is known that small perturbations of boundary data (for which there is a unique minimizer) in the $W^{1,2}$ norm do not change the number of singularities for corresponding minimizers (see [HL89] for perturbations in the $W^{1,\infty}$ norm, [MMS18] and [Li21] for perturbations in the $W^{1,2}$ norm). On the other hand, small perturbations of the boundary datum in the $W^{1,p}$ norm for $p < 2$ can change the number of singularities for corresponding minimizers [MS17].

We prove Theorem 4.1.2 in Section 4.3. To do so, roughly speaking, we follow an example by R. Hardt and Lin F. [HL89, Section 5]. We start with any smooth boundary datum and use the construction of a boundary map (homotopic to the original one) of [MS17] (see [Maz18] for necessary modifications) for which a *Lavrentiev gap phenomenon* occurs. In Section 4.2, we show that a homotopy between these two maps can be chosen small in $W^{1,p}$-norm for $p < 2$, which is the novelty of this chapter, and prove that within this homotopy, there is a boundary datum with the required properties.

As we explained, our key contribution in this chapter, which allows the transition from the existence to the density of boundary data where non-uniqueness occurs, is the homotopy construction presented in Section 4.2. We conclude this introduction with some extra comments concerning this construction.

Assume that one is given $1 \leq p < 2$ and two maps φ and $\psi \in C^\infty(\mathbb{S}^2; \mathbb{S}^2)$ that have the same topological degree. Therefore, there exists a continuous, and even smooth homotopy connecting φ to ψ. A natural question is whether or not, knowing that φ and ψ are close with respect to the $W^{1,p}$ distance, one can choose the homotopy between φ and ψ to remain close to φ and ψ all along the deformation. More precisely, one could for instance expect that there exists a constant $C > 0$ depending on p such that a homotopy $H \in C^\infty(\mathbb{S}^2 \times [0,1]; \mathbb{S}^2)$ between φ and ψ can be chosen so that

$$\|\varphi - H_t\|_{W^{1,p}(\mathbb{S}^2)} \leq C\|\varphi - \psi\|_{W^{1,p}(\mathbb{S}^2)} \quad \text{for every } 0 \leq t \leq 1. \tag{4.1.3}$$

Here, we recall that H_t stands for the map $H(\cdot, t)$. The question is already interesting if we assume in addition that φ and ψ coincide outside of a small disk. For instance, one could ask whether or not a homotopy such that (4.1.3) holds can be found under the additional assumption that $\varphi = \psi$ outside of a ball of radius r, for some $r > 0$ sufficiently small, possibly depending on the map φ that would be fixed in advance.

We are not able to solve this question, and a precise statement of the problem in a more general context is given as Open Problem 4.2.3. However, we are able to solve a weaker version of this problem, which is nevertheless sufficient for our purposes. Namely, we prove that, if the maps φ and ψ coincide outside of a small ball, then a smooth homotopy between them can be found such that $\|\varphi - H_t\|_{W^{1,p}(\mathbb{S}^2)}$ is controlled, not by the distance between φ and ψ, but by the sum of their norms on a neighborhood of the region where they differ. This is the content of the main result of Section 4.2, Proposition 4.2.1. This allows us to deduce that, for a fixed φ and a given $\varepsilon > 0$, one can choose the radius $r > 0$ sufficiently small such that, for any map ψ sufficiently close to φ such that $\varphi = \psi$ outside of $B^3_r(x)$, a homotopy H connecting φ to ψ can be found such

that

$$\|\varphi - H_t\|_{W^{1,p}(\mathbb{S}^2)} \leq \varepsilon \quad \text{for every } 0 \leq t \leq 1;$$

see Corollary 4.2.2. This is sufficient to prove our main result, Theorem 4.1.2, but does not solve Open Problem 4.2.3, as in our proof the radius $r > 0$ of the ball outside of which the maps φ and ψ are required to coincide has to depend on ε, ruling out the possibility of controlling $\|\varphi - H_t\|_{W^{1,p}(\mathbb{S}^2)}$ uniformly in t solely by $\|\varphi - \psi\|_{W^{1,p}(\mathbb{S}^2)}$ with our argument.

4.2 Homotopy construction

In this section, we consider the general case where $\mathscr{N}$ is a (non necessarily compact) Riemannian manifold. We work on the sphere $\mathbb{S}^m$, but the result may be readily extended to an arbitrary domain, either an open subset of $\mathbb{R}^m$ or a Riemannian manifold $\mathscr{M}$ of dimension m. We also always assume that $p < m$.

Proposition 4.2.1. *Let $\varphi \in C^\infty(\mathbb{S}^m; \mathscr{N})$ and $p < m$. For every $r > 0$, for every $x \in \mathbb{S}^m$, and every $\psi \in C^\infty(\mathbb{S}^m; \mathscr{N})$ homotopic to φ and satisfying $\varphi = \psi$ on $\mathbb{S}^m \setminus B_r(x)$, there exists a homotopy $H \in C^\infty(\mathbb{S}^m \times [0,1]; \mathscr{N})$ from φ to ψ such that*

$$\sup_{0 \leq t \leq 1} \|\varphi - H_t\|_{W^{1,p}(\mathbb{S}^m)} \leq C(\|\varphi\|_{W^{1,p}(B_{2r}(x))} + \|\psi\|_{W^{1,p}(B_{2r}(x))}),$$

for some constant $C > 0$ depending only on m and p.

This proposition can be used in combination with Lebesgue's lemma to obtain a homotopy which remains close to φ in $W^{1,p}$. Indeed, choosing r sufficiently small, depending on φ, we may ensure that $\|\varphi\|_{W^{1,p}(B_{2r}(x))}$ is as small as we want, uniformly with respect to r. Since $\|\psi\|_{W^{1,p}(B_{2r}(x))} \leq \|\varphi\|_{W^{1,p}(B_{2r}(x))} + \|\varphi - \psi\|_{W^{1,p}(\mathbb{S}^m)}$, assuming in addition that $\|\varphi - \psi\|_{W^{1,p}(\mathbb{S}^m)}$ is small, we can make $\sup_{0 \leq t \leq 1} \|\varphi - H_t\|_{W^{1,p}(\mathbb{S}^m)}$ as small as we want. This yields the following corollary.

Corollary 4.2.2. *Let $\varphi \in C^\infty(\mathbb{S}^m, \mathscr{N})$ and $p < m$. For every $\varepsilon > 0$, there exists $r > 0$ sufficiently small, depending on φ, and there exists $\delta > 0$ such that, for every $x \in \mathbb{S}^m$ and every $\psi \in C^\infty(\mathbb{S}^m; \mathscr{N})$ homotopic to φ and satisfying $\varphi = \psi$ on $\mathbb{S}^m \setminus B_r(x)$ and $\|\varphi - \psi\|_{W^{1,p}(\mathbb{S}^m)} \leq \delta$, there exists a homotopy $H \in C^\infty(\mathbb{S}^m \times [0,1]; \mathscr{N})$ from φ to ψ such that*

$$\sup_{0 \leq t \leq 1} \|\varphi - H_t\|_{W^{1,p}(\mathbb{S}^m)} \leq \varepsilon.$$

Proof of Proposition 4.2.1. Let $G \in C^\infty(\mathbb{S}^m \times [0,1]; \mathscr{N})$ be any homotopy connecting φ to

ψ with $G_0 = \varphi$ and $G_1 = \psi$. Since $\varphi = \psi$ outside of $B_r(x)$, we may assume that G is stationary outside of $B_r(x)$, i.e., for each $t \in [0,1]$, we have $G_t = \varphi = \psi$ on $\mathbb{S}^m \setminus B_r(x)$. This claim can be proved with a by-hand construction, that we sketch below. We denote by $\hat{x}$ the point at the antipode of x. Let $\Psi\colon \mathbb{S}^m \to \mathbb{S}^m$ be a smooth map such that $\Psi = \mathrm{id}$ outside of $B_r(x)$, and such that Ψ maps the annulus $B_r(x) \setminus \overline{B_{r/2}}(x)$ diffeomorphically onto the annulus $\mathbb{S}^m \setminus (\overline{B_{r/2}}(x) \cup \{\hat{x}\})$, the circle $\partial B_{r/2}(x)$ onto $\{\hat{x}\}$, and the ball $B_{r/2}(x)$ diffeomorphically onto $\mathbb{S}^m \setminus \{\hat{x}\}$. It is readily observed that $\Psi \sim \mathrm{id}$, through a homotopy stationary outside of $B_r(x)$. Therefore, the maps $u \circ \Psi$ and $v \circ \Psi$ are homotopic to u and v respectively, through a homotopy stationary outside of $B_r(x)$. Now, given a homotopy G' connecting u to v, a homotopy G'' connecting $u \circ \Psi$ to $v \circ \Psi$ can be constructed by prescribing that G'' is stationary outside of $B_r(x)$, by letting $G''_t = G'_t \circ \Psi$ on $\overline{B_{r/2}}(x)$ — which corresponds to rescaling G' from $\mathbb{S}^m \setminus \{\hat{x}\}$ to $B_r(x)$ — and extending smoothly on the annulus $B_r(x) \setminus \overline{B_{r/2}}(x)$. This is readily done by combining the observations that (i) $u \circ \Psi$ and $v \circ \Psi$ coincide also on $B_r(x) \setminus B_{r/2}(x)$ and are constant on $\partial B_{r/2}(x)$; and (ii) G''_t is constant on $\partial B_{r/2}(x)$. The required homotopy G stationary outside of $B_r(x)$ is then obtained by patching the three above homotopies, from u to $u \circ \Psi$, from $u \circ \Psi$ to $v \circ \Psi$, and from $v \circ \Psi$ to v.

We consider $\tau > 0$ to be chosen sufficiently small at a later stage. We are going to rescale G, φ, and ψ from $B_r(x)$ to a smaller ball $B_\tau(x)$, while keeping them unchanged outside of $B_{2r}(x)$. This corresponds to a model case of the shrinking procedure, but for the sake of making this chapter self-contained, we present the complete argument. More specifically, let $(\Phi_t)_{0 \le t \le 1}$ be a family of smooth diffeomorphisms of $\mathbb{S}^m$ such that $\Phi_t = \mathrm{id}$ outside of $B_{2r}(x)$ and such that, on $B_{2r}(x)$, in the local chart given by the exponential map around x, Φ_t is expressed as

$$\begin{cases} \frac{rx}{(1-t)r+t\tau} & \text{if } |x| \le (1-t)r + t\tau, \\ \frac{x}{|x|}\bigl(\frac{r}{2r-(1-t)r-t\tau}(|x| - (1-t)r - t\tau) + r\bigr) & \text{if } (1-t)r + t\tau \le |x| \le 2r. \end{cases}$$

We define $H \in C^\infty(\mathbb{S}^m \times [0,1]; \mathscr{N})$ by

$$H_t = \begin{cases} \varphi \circ \Phi_{3t} & \text{if } 0 \le t \le \frac{1}{3}, \\ G_{3(t-1/3)} \circ \Phi_1 & \text{if } \frac{1}{3} \le t \le \frac{2}{3}, \\ \psi \circ \Phi_{1-3(t-2/3)} & \text{if } \frac{2}{3} \le t \le 1. \end{cases}$$

Of course, H is a homotopy from φ to ψ. It remains to show that, if $\tau > 0$ is suitably small, then H satisfies the required estimate.

For $0 \le t \le \frac{1}{3}$, we note that $\varphi - H_t = 0$ outside $B_{2r}(x)$. We readily obtain bounds on the Jacobian and the derivatives of Φ_t, so that the change of variable theorem combined with $n - p > 0$ implies that

$$\|\varphi - H_t\|_{W^{1,p}(\mathbb{S}^m)} \leq \|\varphi\|_{W^{1,p}(B_{2r}(x))} + \|\varphi \circ \Phi_{3t}\|_{W^{1,p}(B_{2r}(x))} \lesssim \|\varphi\|_{W^{1,p}(B_{2r}(x))}.$$

Similarly, for $\frac{2}{3} \leq t \leq 1$, we have

$$\begin{aligned}\|\varphi - H_t\|_{W^{1,p}(\mathbb{S}^m)} &\leq \|\varphi\|_{W^{1,p}(B_{2r}(x))} + \|\psi \circ \Phi_{3t}\|_{W^{1,p}(B_{2r}(x))} \\ &\lesssim \|\varphi\|_{W^{1,p}(B_{2r}(x))} + \|\psi\|_{W^{1,p}(B_{2r}(x))}.\end{aligned}$$

Concerning $\frac{1}{3} \leq t \leq \frac{2}{3}$, we estimate

$$\begin{aligned}&\|\varphi - H_t\|_{W^{1,p}(\mathbb{S}^m)} \\ &\quad\leq \|\varphi\|_{W^{1,p}(B_{2r}(x))} + \|G_{3(t-1/3)} \circ \Phi_1\|_{W^{1,p}(B_{2r}(x))} \\ &\quad\lesssim \|\varphi\|_{W^{1,p}(B_{2r}(x))} + \|G_{3(t-1/3)}\|_{W^{1,p}(B_{2r}(x)\setminus B_r(x))} + \tau^{\frac{m-p}{p}}\|G_{3(t-1/3)}\|_{W^{1,p}(B_{2r}(x))}.\end{aligned}$$

Since the homotopy G has been assumed to be stationary outside of $B_r(x)$, we know that $\|G_{3(t-1/3)}\|_{W^{1,p}(B_{2r}(x)\setminus B_r(x))} = \|\varphi\|_{W^{1,p}(B_{2r}(x)\setminus B_r(x))}$. On the other hand, by compactness, we have

$$\sup_{0\leq t\leq 1}\|G_t\|_{W^{1,p}(B_{2r}(x))} \leq C_1$$

for some possibly large constant $C_1 > 0$. We may assume that either $\|\varphi\|_{W^{1,p}(B_{2r}(x))} \neq 0$ or $\|\psi\|_{W^{1,p}(B_{2r}(x))} \neq 0$. Indeed, if $\|\varphi\|_{W^{1,p}(B_{2r}(x))} = 0 = \|\psi\|_{W^{1,p}(B_{2r}(x))}$, this implies that both φ and ψ are identically zero — note that this may only happen if $0 \in \mathscr{N}$ — and we may directly conclude by choosing H to be constantly zero. As $p < m$, we may therefore choose $\tau > 0$ sufficiently small, depending on C_1, so that

$$\tau^{\frac{m-p}{p}}\|G_{3(t-1/3)}\|_{W^{1,p}(B_{2r}(x))} \leq \|\varphi\|_{W^{1,p}(B_{2r}(x))} + \|\psi\|_{W^{1,p}(B_{2r}(x))} \quad \text{for every } \frac{1}{3} \leq t \leq \frac{2}{3}.$$

Hence, we deduce that

$$\|\varphi - H_t\|_{W^{1,p}(\mathbb{S}^m)} \lesssim \|\varphi\|_{W^{1,p}(B_{2r}(x))} + \|\psi\|_{W^{1,p}(B_{2r}(x))} \quad \text{for every } \frac{1}{3} \leq t \leq \frac{2}{3}.$$

This concludes the proof. □

In Corollary 4.2.2, both the $\delta > 0$ controlling $\|\varphi - \psi\|_{W^{1,p}(\mathbb{S}^m)}$ *and* the $r > 0$ depend on ε. A very natural question is whether or not one may find a homotopy H so that $\sup_{0\leq t\leq 1}\|\varphi - H_t\|_{W^{1,p}(\mathbb{S}^m)}$ is controlled only by $\|\varphi - \psi\|_{W^{1,p}(\mathbb{S}^m)}$. More precisely, we formulate the following open question (cf. [MS17, Problem, p.11]).

Question 4.2.3. Let $\varphi \in C^\infty(\mathbb{S}^m; \mathscr{N})$. Does there exist some $r > 0$, possibly depending on φ, such that for every $x \in \mathbb{S}^m$ and every $\psi \in C^\infty(\mathbb{S}^m; \mathscr{N})$ homotopic to φ and satisfying $\varphi = \psi$ on $\mathbb{S}^m \setminus B_r(x)$, there exists a homotopy $H \in C^\infty(\mathbb{S}^m \times [0,1]; \mathscr{N})$ from φ to ψ such that

$$\sup_{0 \leq t \leq 1} \|\varphi - H_t\|_{W^{1,p}(\mathbb{S}^m)} \leq \omega(\|\varphi - \psi\|_{W^{1,p}(\mathbb{S}^m)}),$$

where ω is a modulus of continuity satisfying $\omega(t) \to 0$ as $t \to 0$.

One may expect ω to be linear in t, but any modulus of continuity would already be of interest. The question is already interesting for maps $\mathbb{S}^2 \to \mathbb{S}^2$.

4.3 Proof of the generic non-uniqueness

Proof of Theorem 4.1.2. As we explained in the introduction of the chapter, the argument essentially follows the strategy devised by R. Hardt and Lin F. [HL89, Section 5], which relies on a number of classical results from the theory of harmonic maps, and incorporates as a new ingredient the homotopy construction from Section 4.2.

More specifically, fix $\varepsilon > 0$ and $\varphi \in C^\infty(\mathbb{S}^2; \mathbb{S}^2)$. We note first that, by Theorem 4.1.1 combined with Hölder's inequality, we may find another mapping $\varphi_0 \in C^\infty(\mathbb{S}^2; \mathbb{S}^2)$ which admits exactly one energy minimizer $u_0 \colon \mathbb{B}^3 \to \mathbb{S}^2$ among all maps having boundary datum φ_0, and such that φ_0 differs from φ only on a set $B_{\varepsilon/2}(x_0)$ for some $x_0 \in \mathbb{S}^2$ and is such that

$$\|\varphi - \varphi_0\|_{W^{1,p}(\mathbb{S}^2)} < \frac{\varepsilon}{2}. \tag{4.3.1}$$

We recall that, combining the regularity result [SU82, Theorem II] with the boundary regularity [SU83, Theorem 2.7] of R. Schoen and K. Uhlenbeck, u_0 can have only a finite number of singularities; let us denote this number by M (possibly $M = 0$).

Next, we apply Corollary 4.2.2 to $\varphi_0 \in C^\infty(\mathbb{S}^2; \mathbb{S}^2)$. We obtain the existence of a $\delta = \delta(\varepsilon) > 0$ and an $r = r(\varphi_0, \varepsilon) > 0$ such that for any $\psi \in C^\infty(\mathbb{S}^2; \mathbb{S}^2)$ that differs from φ_0 only on the set $B_r(x_0)$ and such that $\|\varphi_0 - \psi\|_{W^{1,p}(\mathbb{S}^2)} < \delta$, there exists a homotopy $H \in C^\infty(\mathbb{S}^2 \times [0,1]; \mathbb{S}^2)$ with

$$\sup_{0 \leq t \leq 1} \|\varphi_0 - H_t\|_{W^{1,p}(\mathbb{S}^2)} < \frac{\varepsilon}{2}. \tag{4.3.2}$$

Let $\varepsilon_1 = \min\{\delta, r, \frac{\varepsilon}{2}\}$. By [Maz18, Theorem 2.3.1], we construct $\varphi_1 \in C^\infty(\mathbb{S}^2; \mathbb{S}^2)$ with the following properties:

(*i*) $\deg \varphi_0 = \deg \varphi_1$;

(*ii*) $\|\varphi_0 - \varphi_1\|_{W^{1,p}} < \varepsilon_1$ and $\varphi_0 = \varphi_1$ except on $B_{\varepsilon_1}(x)$ for some point $x \in \mathbb{S}^2$;

(*iii*) φ_1 admits only one energy minimizer $u_1 : \mathbb{B}^3 \to \mathbb{S}^2$ having at least $M + 1$ singularities.

To be precise, the statement [Maz18, Theorem 2.3.1] gives only that $\mathscr{H}^2(\{x \in \mathbb{S}^2 : \varphi_0(x) \neq \varphi_1(x)\}) < \varepsilon_1$, but following the lines of the proof, we may deduce that $\varphi_0 = \varphi_1$ except on $B_{\varepsilon_1}(x)$ for some point $x \in \mathbb{S}^2$.

Now, let us take the homotopy H_t between φ_0 and φ_1 constructed in Corollary 4.2.2. Let

$$\tau = \sup\left\{t \in [0,1] : \begin{array}{l}\text{each energy minimizer with boundary datum } H_t\\ \text{has at most } M \text{ singular points in } \mathbb{B}^3\end{array}\right\}.$$

We argue like in [MS17, Remark 4.1] (which is a modified argument from [HL89, Section 5]). For the convenience of the reader, we state here the main lines of the reasoning. First, we note that from the Stability Theorem [HL89], see also [MMS18, Theorem 8.9], we have $\tau \in (0,1)$.

Now take $s_n \nearrow \tau$ and a sequence of minimizing harmonic maps $u_n \in W^{1,2}(\mathbb{B}^3; \mathbb{S}^2)$ with $u_i\big|_{\partial\mathbb{B}^3} = H_{s_n}$ and $\operatorname{card}(\operatorname{sing} u_n) \leq M$. Let us also take $t_n \searrow \tau$ and a sequence of minimizing harmonic maps $v_n \in W^{1,2}(\mathbb{B}^3; \mathbb{S}^2)$ with $v_n\big|_{\partial\mathbb{B}^3} = H_{t_n}$ and $\operatorname{card}(\operatorname{sing} v_n) > M$. Since $\sup_n([H_{s_n}]_{W^{1,2}(\mathbb{S}^2)} + [H_{t_n}]_{W^{1,2}(\mathbb{S}^2)}) < +\infty$, we may deduce from the strong convergence of minimizers, see [AL88, Theorem 1.2 (4)] (see also [MMS18, Theorem 6.1 (3)]), that, up to a subsequence, we have

$$\begin{aligned} u_n &\to u \quad \text{strongly in } W^{1,2}(\mathbb{B}^3; \mathbb{S}^2),\\ v_n &\to v \quad \text{strongly in } W^{1,2}(\mathbb{B}^3; \mathbb{S}^2), \end{aligned}$$

and both u and v are energy minimizers with $u\big|_{\partial\mathbb{B}^3} = v\big|_{\partial\mathbb{B}^3} = H_\tau$. We claim that $\operatorname{card}(\operatorname{sing} u) \leq M$. Indeed, assume on the contrary that $\operatorname{card}(\operatorname{sing} u) > M$. Then, by [AL88, Theorem 1.8 (2)] (see also [MMS18, Theorem 2.10]), we would obtain that for each $y \in \operatorname{sing} u$ and for sufficiently large n, there would exist $y_n \in \operatorname{sing} u_n$ with $y_n \to y$ as $n \to +\infty$, a contradiction.

Moreover, $\operatorname{card}(\operatorname{sing} v) > M$. To see this, let us again assume by contradiction that $\operatorname{card}(\operatorname{sing} v) \leq M$. Let now $z_{n,j} \in \operatorname{sing} v_n$ for $j \in \{1, \dots, M+1\}$ be distinct singular points of v_n. Now let us observe that for sufficiently large n, we know that that H_{t_n} and H_τ are close in C^∞. Hence, by uniform boundary regularity [AL88, Theorem 1.10 (2)] (see also [MMS18, Theorem 7.4]), there is a uniform neighborhood of the boundary $\partial\mathbb{B}^3$ which contains no singularities of v and v_n, say $\operatorname{dist}(z, \partial\mathbb{B}^3) \geq \lambda > 0$ for any $z \in \bigcup_n \operatorname{sing} v_n \cup \operatorname{sing} v$. Since singular points converge to singular points, we deduce from [AL88, Theorem 1.8 (1)] (see also [MMS18, Theorem 2.5]) that for each j, we have $z_{n,j} \to z_j$ as $n \to +\infty$ and $z_j \in \operatorname{sing} v$. The only possibility for $\operatorname{card}(\{z_1, \ldots, z_{M+1}\}) < M+1$ is that two singularities of v_n converge to the same singularity of v. This, however, is impossible, because by the uniform distance between singularities [AL88, Theorem 2.1] (see also [MMS18, Theorem 2.12]), there exists a universal constant C (independent of the minimizer) such that no singularity can occur next to $z_{n,j}$ at a distance $C \operatorname{dist}(z_{n,j}, \partial\mathbb{B}^3) \geq C\lambda$.

Hence, $H_\tau \colon \mathbb{S}^2 \to \mathbb{S}^2$ serves as a boundary condition for at least two minimizers u and v having a different number of singularities. Combining (4.3.2) with (4.3.1), we obtain

$$\|\varphi - H_\tau\|_{W^{1,p}(\mathbb{S}^2)} \leq \|\varphi - \varphi_0\|_{W^{1,p}(\mathbb{S}^2)} + \|\varphi_0 - H_\tau\|_{W^{1,p}(\mathbb{S}^2)} < \frac{\varepsilon}{2} + \varepsilon_1 \leq \varepsilon.$$

This finishes the proof. $\square$

Chapter 5

Analytical obstructions to the weak approximation of Sobolev mappings into manifolds

Résumé

Dans ce chapitre, on étudie le problème de l'approximation faible, qui correspond à la question (Q4). Pour tout naturel $p \geq 2$, on construit une variété riemannienne compacte $\mathscr{N}$ telle que, si $\dim \mathscr{M} > p$, alors il existe une application dans l'espace de Sobolev $W^{1,p}(\mathscr{M};\mathscr{N})$ qui n'est pas une limite faible d'applications lisses à valeurs dans $\mathscr{N}$, en raison d'un mécanisme d'obstruction analytique. Pour $p = 4d - 1$, la variété cible peut être choisie comme étant la sphère $\mathbb{S}^{2d}$, grâce à une construction en produit de Whitehead d'applications avec un invariant de Hopf non trivial, généralisant ainsi le résultat de F. Bethuel pour $p = 4d - 1 = 3$. Les ingrédients clés pour ce faire sont une construction périodique d'application dont l'énergie relaxée croît de façon superlinéaire par rapport à leur énergie de Sobolev, un résultat précis de formation de bulles pour quantifier la concentration d'énergie pour des suites faiblement convergentes d'applications lisses, et des estimations soigneuses d'énergie combinée autour des multiples singularités des applications que nous construisons. Ces résultats s'étendent aux espaces de Sobolev d'ordre supérieur $W^{s,p}$, avec $s \in \mathbb{R}$, $s \geq 1$, $sp \in \mathbb{N}$, et $sp \geq 2$. Cela correspond à un travail en collaboration avec J. Van Schaftingen.

Abstract

This chapter is concerned with the weak approximation problem, which corresponds to question (Q4). For any integer $p \geq 2$, we construct a compact Riemannian manifold $\mathcal{N}$ such that if $\dim \mathcal{M} > p$, there is a map in the Sobolev space of mappings $W^{1,p}(\mathcal{M};\mathcal{N})$ which is not a weak limit of smooth maps into $\mathcal{N}$ due to a mechanism of analytical obstruction. For $p = 4d - 1$, the target manifold can be taken to be the sphere $\mathbb{S}^{2d}$ thanks to the construction by Whitehead product of maps with nontrivial Hopf invariant, generalizing the result by F. Bethuel for $p = 4d - 1 = 3$. The key ingredients in this endeavor are a periodic construction or mappings whose relaxed energy blows up superlinearly with respect to their Sobolev energy, a precise bubbling result to quantify the energy concentration for weakly converging

sequences of smooth maps, and careful joint energy estimates around the multiples singularities of the maps we construct. The results extend to higher order Sobolev spaces $W^{s,p}$, with $s \in \mathbb{R}$, $s \geq 1$, $sp \in \mathbb{N}$, and $sp \geq 2$. This corresponds to a joint work with J. Van Schaftingen.

5.1 Introduction

This chapter is entirely dedicated to the construction of families of counterexamples to the weak approximation, namely Theorems 1.4.13 and 1.4.14 for a general domain $\mathscr{M}$. More specifically, we first prove the following result which shows that, for *any* integer $p \geq 2$, analytical obstructions to the weak approximation may occur — recall that, when $p = 1$, the weak approximation property always holds as a consequence of Hajłasz's result.

Theorem 5.1.1. *For every* $p \in \mathbb{N} \setminus \{0, 1\}$, *there exists a compact manifold* $\mathscr{N}$ *such that if* $\dim \mathscr{M} > p$, *then*

$$H^{1,p}_W(\mathscr{M}; \mathscr{N}) \subsetneq W^{1,p}(\mathscr{M}; \mathscr{N}).$$

In particular, Theorem 5.1.1 is the *first* instance of the failure of the weak approximation property when $\mathscr{M}$ is a ball for $p \neq 3$.

The manifold $\mathscr{N}$ is defined explicitly, depending on p; it retracts onto the p-dimensional skeleton of a $(p+1)$-dimensional torus. Moreover, one can show that $\mathscr{N}$ can be obtained as

$$\mathscr{N} \simeq (\mathbb{T}^{p+1} \setminus \mathbb{B}^{p+1}) \times \mathbb{S}^{l-1} \cup_\partial \mathbb{S}^p \times \mathbb{B}^l,$$

where $l \in \mathbb{N}_*$ is a parameter involved in the construction. There is a trade-off in the choice of the parameter l. On the one hand, one can choose it to be minimal, that is, take $l = 1$, in order to obtain a target of dimension as small as possible. In this case, it can be observed that $\mathscr{N}$ is actually a connected sum of two tori:

$$\mathscr{N} \simeq \mathbb{T}^{p+1} \# \mathbb{T}^{p+1}.$$

On the other hand, one can try to choose l in order to have as much information as possible on the topology of the resulting target $\mathscr{N}$. In this case, this leads to taking l large. For instance, choosing $l > p$, one can ensure that $\pi_2(\mathscr{N}) \simeq \cdots \simeq \pi_{p-1}(\mathscr{N}) \simeq \{0\}$

while $\pi_1(\mathscr{N})$ is abelian and $\pi_p(\mathscr{N})$ is finitely generated as a $\pi_1(\mathscr{N})$-module but not as a $\mathbb{Z}$-module. The precise description of the target $\mathscr{N}$ that we construct, as well as the proof of its topological properties, will be given in Proposition 5.3.9 and Remark 5.3.10.

Let us explain how Theorem 5.1.1 fits into the existing weak approximation results. So far, there are three strategies to prove the weak density. The first one is to deduce it from the strong appproximation; this works when $\pi_p(\mathscr{N}) = \{0\}$, which will not be the case in the setting of Theorem 5.1.1. A second strategy relies on a controlled almost retraction [Haj94] (see also [BZ88]); for topological reasons, the restriction $\pi_1(\mathscr{N}) = \dots = \pi_{p-1}(\mathscr{N}) = \{0\}$ is essential. The last class of methods, of which we gave a sketch in Section 1.4.6 in the introduction, is based first on the construction and analysis of a topological singular set and of minimal connections, and second on the elimination of the topological singularities via a dipole construction; see e.g. [BCL86, ABL88, Bet90, ABO03, CO19] for mappings into the sphere, and [PR03] for more general targets. Although there is no evidence that the assumption $\pi_1(\mathscr{N}) = \dots = \pi_{p-1}(\mathscr{N}) = \{0\}$ should be essential, it simplifies the setting considerably, since one then only has to consider the charges in $\pi_p(\mathscr{N})$ without having to take into account their interplay with lower-dimensional phenomena. In a situation like ours, where the only nontrivial lower homotopy group is $\pi_1(\mathscr{N})$, one could naturally try to use the universal covering $\widetilde{\mathscr{M}}$ for which $\pi_1(\widetilde{\mathscr{M}}) = \dots = \pi_{p-1}(\widetilde{\mathscr{M}}) = \{0\}$. Although from the point of view of homotopy theory, the $(p-1)$-dimensional skeleton of $\widetilde{\mathscr{M}}$ is homotopically equivalent to a point, there is no reason to believe that there is in general a reasonable quantitative control on the resulting homotopy that could be used for analytical constructions. Actually, in our situation, the non-simple connectedness of $\mathscr{N}$ can be exploited to thwart the such weak approximation scheme. More details about the phenomenon at work here will be given after Proposition 5.3.7, where the precise topological properties of the target manifold we construct are proved.

Even though Theorem 5.1.1 covers all the integer exponents $p \geq 2$, the resulting manifold $\mathscr{N}$ is not as simple as the sphere $\mathbb{S}^2$ in Bethuel's result [Bet20]. Using a variant of our construction, we also recover Bethuel's counterexample, and show that it is actually part of an infinite family.

Theorem 5.1.2. *For every $d \in \mathbb{N} \setminus \{0\}$, if $\dim \mathscr{M} > 4d-1$, then*

$$H^{1,4d-1}_W(\mathscr{M};\mathbb{S}^{2d}) \subsetneq W^{1,4d-1}(\mathscr{M};\mathbb{S}^{2d}).$$

Before we delve to technical matters, let us give a sketch of the proof, to highlight the main ideas behind. For a given integer $p \geq 2$, the basic construction in our proof of Theorem 5.1.1 defines $u\colon \mathbb{R}^{p+1} \setminus \Sigma \to \widetilde{\mathscr{M}_0}$ as the (singular) retraction of $\mathbb{R}^{p+1} \setminus \Sigma$ to

$\widetilde{\mathscr{M}_0}$, where $\widetilde{\mathscr{M}_0}$ is the p-dimensional component of the decomposition of $\mathbb{R}^{p+1}$ into cubes with vertices in $\mathbb{Z}^{p+1}$ and $\Sigma = (\mathbb{Z} + 1/2)^{p+1}$ is the corresponding 0-dimensional dual complex. More specifically, u is defined on every cube centered at the point $\sigma \in \Sigma$ as $u(x) = \sigma + (x - \sigma)/(2|x - \sigma|_\infty)$.

For every $\ell \in \mathbb{N}$, we have by periodicity of u

$$\int_{[0,\ell]^{p+1}} |\mathrm{D}u|^p = \ell^{p+1} \int_{[0,1]^{p+1}} |\mathrm{D}u|^p.$$

The relaxed energy of a Sobolev map w on a domain $\mathscr{M}$ is defined as

$$\begin{aligned}\mathcal{E}^{1,p}_{\mathrm{rel}}(w, \mathscr{M}) \\ = \inf\Big\{\liminf_{k\to+\infty} \mathcal{E}^{1,p}(u_k, \mathscr{M})\colon u_k \to w \text{ a.e. and } u_k \in W^{1,p}(\mathscr{M}; \widetilde{\mathscr{M}_0}) \cap C(\mathscr{M}; \widetilde{\mathscr{M}_0})\Big\},\end{aligned} \tag{5.1.1}$$

where we recall that the Sobolev energy is defined by

$$\mathcal{E}^{1,p}(w, \mathscr{M}) = \int_{\mathscr{M}} |\mathrm{D}w|^p.$$

Although our definition of the relaxed energy is not identical to the usual definition where the infimum is taken over sequences of smooth maps, they are nonetheless equivalent, as any continuous Sobolev mapping may always be *strongly* approximated by smooth maps by a classical regularization and reprojection argument, and the equivalence follows then from a diagonal argument, since the convergence in measure is metrizable.

If we can show that the relaxed energy $\mathcal{E}^{1,p}_{\mathrm{rel}}(u, [0,\ell]^{p+1})$ of u on $[0,\ell]^{p+1}$ grows faster than the Sobolev energy $\mathcal{E}^{1,p}(u, [0,\ell]^{p+1})$ as $\ell \to +\infty$, the obstruction to the weak approximation will follow from a nonlinear uniform boundedness principle [HL03b, Theorem 9.6] (see also [MVS19]), which is a kind of nonlinear counterpart of the classical Banach–Steinhaus theorem in functional analysis.

In order to achieve this, we will compare $\mathcal{E}^{1,p}_{\mathrm{rel}}(u, [0,5\ell]^{p+1})$ to $\mathcal{E}^{1,p}_{\mathrm{rel}}(u, [0,\ell]^{p+1})$. A simple additivity and translation argument shows that

$$\mathcal{E}^{1,p}_{\mathrm{rel}}(u, [0,5\ell]^{p+1}) \geq 5^{p+1} \mathcal{E}^{1,p}_{\mathrm{rel}}(u, [0,\ell]^{p+1}); \tag{5.1.2}$$

we are going to strengthen this inequality. For this purpose, we consider a sequence $(u_n)_{n\in\mathbb{N}}$ in $W^{1,p}([0,5\ell]^{p+1}; \widetilde{\mathscr{M}_0}) \cap C([0,5\ell]^{p+1}; \widetilde{\mathscr{M}_0})$ realizing the infimum in (5.1.1). By a classical Fatou and Fubini–Tonelli argument, the sequence converges weakly on the boundary ∂Q of some cube Q with the same center as $[0, 5\ell]^{p+1}$ and edge length between 3ℓ and 5ℓ, with

$$\liminf_{n\to+\infty} \int_{\partial Q} |\mathrm{D}u_n|^p \leq \frac{C}{\ell} \liminf_{n\to+\infty} \int_{[0,5\ell]^{p+1}} |\mathrm{D}u_n|^p. \tag{5.1.3}$$

The sequence $(u_{n|\partial Q})_{n\in\mathbb{N}}$ is thus a sequence of maps homotopic to a constant converging to $u_{|\partial Q}$ which is not homotopic to a constant. Using the fact that $\dim \partial Q = p$, we show in Section 5.2 that, when n is sufficiently large, there is a finite family of disjoint balls such that u_n and u are homotopic outside these balls. (Although such bubbling phenomena are ubiquitous in the analysis of Sobolev and harmonic mappings, we had to develop a statement providing all the information we need in Proposition 5.2.1.) The gap in homotopy classes between u_n and u needs thus to be compensated by the homotopical charge beared on the small balls. In geometric terms, every singular point should be engulfed in the image of u_n on some singular ball. By the isoperimetric theorem, if $u_n = b$ on $\partial B^{p+1}_\rho(a) \cap \partial Q$, the number of singularities engulfed in the image of a small ball $B^{p+1}_\rho(a) \cap \partial Q$ is controlled by

$$\Bigl(\int_{B^{p+1}_\rho(a)\cap\partial Q} \mathcal{J}u_n\Bigr)^{1+\frac{1}{p}} \leq \Bigl(\int_{B^{p+1}_\rho(a)\cap\partial Q} |\mathrm{D}u_n|^p\Bigr)^{1+\frac{1}{p}}$$

(where $\mathcal{J}u_n = |\det(\mathrm{D}u_n)|$ is the Jacobian of u_n), leading to a total contribution to the energy $\mathcal{E}^{1,p}(u_n, \partial Q)$ of those small disks containing ℓ^{p+1} singularities of the order of ℓ^p, and thus by (5.1.3) to a contribution ℓ^{p+1} to the energy $\mathcal{E}^{1,p}(u_n, [0, 5\ell]^{p+1})$.

In order to transform this idea into an improvement of (5.1.2), we localize the estimate on the number of engulfed singularities: assuming that $B^{p+1}_\rho(a) \subseteq [0,\ell]^{p+1}$, the number of singularities in the cube $[2\ell, 3\ell]^{p+1}$ can be controlled by

$$\Bigl(\int_{B^{p+1}_\rho(a)\cap\partial Q} |\mathrm{D}u_n|^p - \int_{B^{p+1}_\rho(a)\cap\partial Q} |\mathrm{D}(\Pi\circ u_n)|^p\Bigr)^{1+\frac{1}{p}},$$

where Π is a suitable retraction from $\widetilde{\mathscr{M}_0}$ to $[0,\ell]^{p+1} \cap \widetilde{\mathscr{M}_0}$. For such a retraction, we have

$$\liminf_{n\to+\infty} \int_{[0,\ell]^{p+1}\cap\partial Q} |\mathrm{D}(\Pi\circ u_n)|^p \geq \mathcal{E}^{1,p}_{\mathrm{rel}}(u, [0,\ell]^{p+1} \cap \partial Q).$$

Combining these ingredients, one gets

$$\mathcal{E}^{1,p}_{\mathrm{rel}}(u, [0, 5\ell]^{p+1}) \geq 5^{p+1}\mathcal{E}^{1,p}_{\mathrm{rel}}(u, [0,\ell]^{p+1}) + c\ell^{p+1}$$

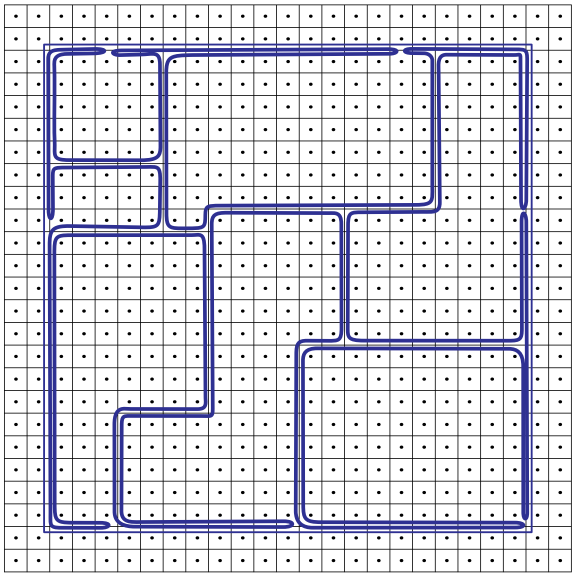

Figure 5.1 – On a generic square, a smooth map u_n approximating u should take at most points a value close to u while engulfing at the other points all the singularities through the creation of bubbles.

with $c > 0$, from which it follows that

$$\lim_{\ell \to +\infty} \frac{\mathcal{E}^{1,p}_{\mathrm{rel}}(u, [0, \ell]^{p+1})}{\ell^{p+1}} = +\infty.$$

If $\widetilde{\mathscr{M}_0}$ were compact and a manifold, we would have our conclusion by the uniform boundedness principle. In order to remedy these issues, we first consider the space $\mathscr{N}_0 = \widetilde{\mathscr{M}_0}/\mathbb{Z}^{p+1}$. By construction, $\widetilde{\mathscr{M}_0}$ is the universal covering of $\mathscr{N}_0$, and the latter is compact. Since $p \geq 2$, the results of the analysis above can be transferred to get a mapping $v\colon \mathbb{R}^{p+1} \to \mathscr{N}_0$ such that

$$\lim_{\ell \to +\infty} \frac{\mathcal{E}^{1,p}_{\mathrm{rel}}(v, [0, \ell]^{p+1})}{\ell^{p+1}} = +\infty.$$

While $\mathscr{N}_0$ is a compact simplicial complex, it is not yet a manifold. In order to fix this last issue, we define explicitly a manifold containing $\mathscr{N}_0$ and retracting to $\mathscr{N}_0$. (Alternatively, this could be done in a more abstract fashion, constructing a simplicial embedding of $\mathscr{N}_0$ in a Euclidean space, giving it a tubular neighborhood, and endowing the boundary of the latter with a smooth structure.)

Concerning Theorem 5.1.2, a quite delicate part of Bethuel's proof is the construction of mappings having a prescribed number of singularities of the same degree, starting from the spaghetton map on $\mathbb{S}^3$ and then defining a map on $\mathbb{R}^4$ by a Gordian cut through a suitable deformation.

The presentation is substantially simplified and the transfer to higher dimension is eased by noting that there is a periodic map $u \in W^{1,4d-1}_{\mathrm{loc}}(\mathbb{R}^{4d};\mathbb{S}^{2d})$ constructed by Whitehead product. This map can be constructed thanks to periodic smooth maps $\mathbb{R}^{2d} \to \mathbb{S}^{2d}$ which are constant on the boundary of unit cubes and have Brouwer degree 1 in these cubes, so that the resulting homotopy class is the Whithead product of the $(2d)$-dimensional homotopy classes and has Hopf invariant 2. When $d = 1$, we have a short construction of a map having the same properties as Bethuel's. Having Hopf invariant 2 is essential for covering the full range of $d \in \mathbb{N} \setminus \{0\}$, since maps having Hopf invariant 1 only exist when $d \in \{1, 2, 4\}$ [Ada60].

At a more technical level, this periodic character of the maps we construct and the arrangement of the singularities on a regular grid eliminate the need of relying on the notion of scans introduced by R. Hardt and T. Rivière [HR03, HR08] to analyze the concentration, and allow using instead more elementary arguments.

The lower estimate on the relaxed energy is performed thanks to a branched transport argument, as in [Bet20].

We close this introduction with a short word on extensions of the results in this chapter to higher order Sobolev spaces of mappings. To the best of our knowledge, no analytical obstruction to the weak approximation was known until now for $s \neq 1$. In Section 5.5, we prove that both our main results admit a natural counterpart for any $s \geq 1$. Namely, we prove the following theorems.

Theorem 5.1.3. *For every $1 \leq s < +\infty$ and $1 \leq p < +\infty$ such that $sp \in \mathbb{N} \setminus \{0, 1\}$, there exists a manifold $\mathscr{N}$ such that if $\dim \mathscr{M} > sp$, then*

$$H^{s,p}_W(\mathscr{M};\mathscr{N}) \subsetneq W^{s,p}(\mathscr{M};\mathscr{N}).$$

It will appear in the proof that the manifold $\mathscr{N}$ only depends on the number $sp \geq 2$.

Theorem 5.1.4. *For every $n \in \mathbb{N} \setminus \{0\}$, if $\dim \mathscr{M} > 4d - 1$, for every $1 \leq s < +\infty$ and $1 \leq p < +\infty$ such that $sp = 4d - 1$, then*

$$H^{s,p}_W(\mathscr{M};\mathbb{S}^{2d}) \subsetneq W^{s,p}(\mathscr{M};\mathbb{S}^{2d}).$$

Our results are restricted to $s \geq 1$ since our proof relies on the Gagliardo–Nirenberg inequality (5.5.2) to use the fact that $H^{s,p}_W(\mathscr{M};\mathscr{N}) \subseteq H^{1,sp}_W(\mathscr{M};\mathscr{N})$. Since this procedure is not available for $0 < s < 1$, the study of this case is not a byproduct of our method, although several of the ideas developed in this work could be useful.

5.2 Bubbling of sequences of Sobolev mappings

In this section, we take as a target space a set $\mathscr{N} \subset \mathbb{R}^\nu$ which has the property of being a *uniform Lipschitz neighborhood retract*, that is, there exists $\iota > 0$ such that there is a Lipschitz-continuous retraction $\Pi_{\mathscr{N}} \colon \mathscr{N} + B^\nu_\iota \to \mathscr{N}$. This includes the particular case where $\mathscr{N}$ is an embedded smooth compact manifold, but also more general cell complexes. For instance, one may take $\mathscr{N}$ to be a skeleton of $\mathbb{R}^\nu$, which will be crucial for us in the sequel.

The goal of this section is to prove the following *bubbling proposition*.

Proposition 5.2.1. *Let $\mathscr{M}$ be a compact manifold and let $p = \dim \mathscr{M}$. For every $\varepsilon \in (0, +\infty)$ and $M \in (0, +\infty)$, there exists $\delta \in (0, +\infty)$ such that, given $\rho_0 \in (0, \delta)$ and $u, v \in W^{1,p}(\mathscr{M}; \mathscr{N}) \cap C(\mathscr{M}; \mathscr{N})$ satisfying*

$$\int_{\mathscr{M}} |\mathrm{D}u|^p \le M, \tag{5.2.1}$$

$$\int_{\mathscr{M}} |\mathrm{D}v|^p \le M, \tag{5.2.2}$$

$$\sup_{a \in \mathscr{M}} \int_{B_{\rho_0}(a)} |\mathrm{D}u|^p \le \delta^p, \tag{5.2.3}$$

and

$$\int_{\mathscr{M}} |u - v| \le \rho_0^p \delta, \tag{5.2.4}$$

there exist $w \in W^{1,p}(\mathscr{M}; \mathscr{N}) \cap C(\mathscr{M}; \mathscr{N})$, $J \in \mathbb{N}$, $a_1, \dots, a_J \in \mathscr{M}$, $\rho_1, \dots, \rho_J \in (0, \rho_0)$, and $b_1, \dots, b_J \in \mathscr{N}$ such that

(*i*) *the balls $B_{\rho_j}(a_j)$ are disjoint,*

(*ii*) $\sum_{j=1}^J \rho_i \le \rho_0$,

(*iii*) $b_j \in u(\partial B_{\rho_j}(a_j))$,

(*iv*) $w_{|\partial B_{\rho_j/2}(a_j)} = b_j$,

(*v*) $w(B_{\rho_j}(a_j) \setminus B_{\rho_j/4}(a_j)) \subseteq B_\varepsilon(b_j)$,

(*vi*) $w = v$ *on* $\mathscr{M} \setminus \bigcup_{j=1}^J B_{\rho_j}(a_j)$,

(*vii*) $\displaystyle\sum_{j=1}^J \int_{B_{\rho_j}(a_j) \setminus B_{\rho_j/4}(a_j)} |\mathrm{D}w|^p \le \varepsilon^p$,

(viii) *for every* $x \in B_{\rho_j/4}(a_j)$, $w(x) = v(4x)$ *in exponential coordinates,*

(ix) w *is homotopic to* v,

(x) w_0 *is homotopic to* u, *where*

$$w_0 = \begin{cases} w & \text{in } \mathscr{M} \setminus \bigcup_{j=1}^{J} B_{\rho_j/2}(a_j), \\ b_j & \text{in } B_{\rho_j/2}(a_j). \end{cases} \tag{5.2.5}$$

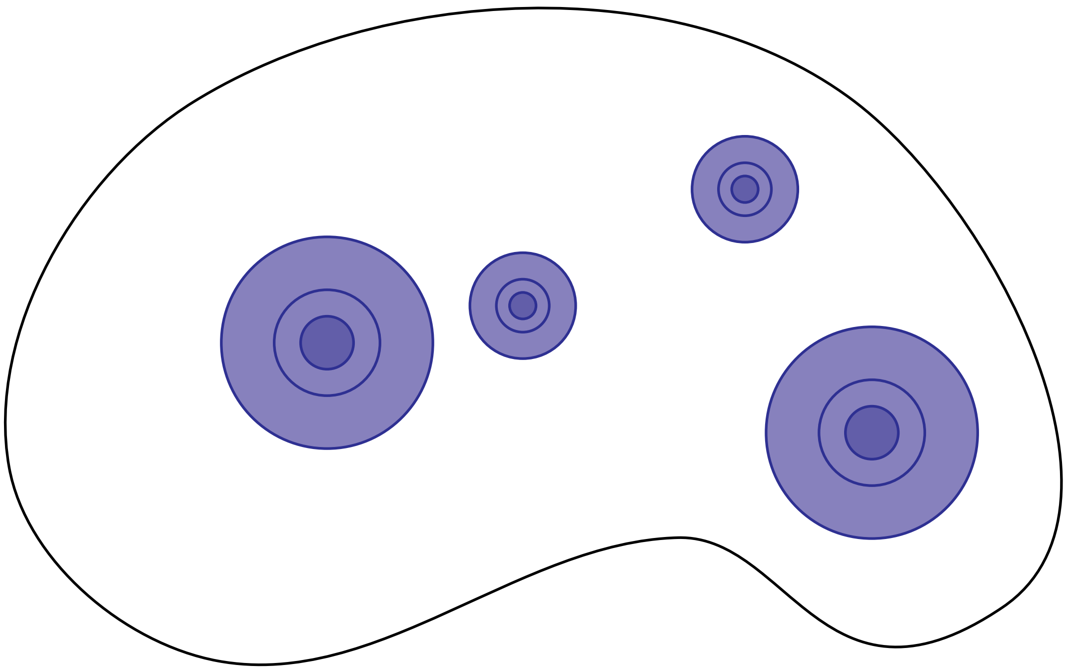

Figure 5.2 – Given u and v, the map w is constructed so that it coincides with v outside the larger balls, it is constant on the intermediate sphere and it is a rescaling of v on the smaller balls; the map w_0 defined similarly outside the intermediate balls and constant inside those is homotopic to u.

Results similar to Proposition 5.2.1 are well established for sequences of harmonic maps [SU81, Proposition 4.3 & Theorem 4.4], [NT95, Theorem 2]. Also in works related to density questions in Sobolev spaces of mappings, bubbling constructions have proved their usefulness in numerous places. Examples include, but not only, [HR08, Proposition 3.4], [Bet20, Remark 1], and also the study of the relaxed energy, see for instance [GMS98b, Theorem 3.1.5.1]. The estimate (5.2.14) in the proof generalizes the Courant–Lebesgue lemma [Jos84, Lemma 3.1]. The main conceptual difference between most of the above quoted results and our proposition is that, most often, bubbling statements are formulated as limiting results, stating that, if a sequence $(u_n)_{k\in\mathbb{N}}$ converges weakly in $W^{1,p}$ to a limiting map u, then the gradients $|Du_n|^p$ converge weakly as measures to $|Du|^p$ plus a weighted sum of Dirac masses, which account for the formation and concentration of bubbles around isolated points.

Our statement in Proposition 5.2.1, on the contrary, is concerned with two *fixed* maps, stating that, if they are taken to be sufficiently close and with suitable control on their energy, then they are homotopic to each other, upon removing a finite number of bubbles which are located in small balls. This quite sharp — and inevitably more complex — statement will be needed to locate precisely enough the bubbles when applying Proposition 5.2.1.

Before we move to the technical ingredients required in the proof of Proposition 5.2.1 and then to the proof itself, we give an informal sketch of the argument, in order to allow the reader to have in mind the main lines of the proof.

The first step is to choose a finite collection of disjoint balls that contains the region of $\mathscr{M}$ where the energy of v is concentrated. Letting these balls grow exponentially and merge for a well-chosen time, we may further take them so that the energy of u and v as well as the integral distance between u and v are controlled on their boundary. Thanks to the Morrey–Sobolev embedding $W^{1,p}(\partial\mathbb{B}^p) \hookrightarrow C^0(\partial\mathbb{B}^p)$, u and v both take their values on a common small ball on the boundary of these enlarged balls.

The map w is then constructed from v as follows. Outside of the selected balls $B_{\rho_j}(a_j)$, the map v is left unchanged. The values of v inside the balls are then shrunk by a linear change of variable in exponential coordinates to make them fit into the four times smaller balls $B_{\rho_j/4}(a_j)$. Finally, since v takes values in a small ball $B_\varepsilon(b_j) \subseteq \mathscr{N}$ on $\partial B_{\rho_j}(a_j)$, we may use an interpolation between the values of v and b_j plus a reprojection procedure to fill in the annulus $B_{\rho_j}(a_j) \setminus B_{\rho_j/4}(a_j)$ with values in $B_\varepsilon(b_j)$ so that w is constantly equal to b_j on $\partial B_{\rho_j/2}(a_j)$. This already shows assertions (i) to (ix) in Proposition 5.2.1.

The proof that u is homotopic to the map w_0 defined through (5.2.5) relies essentially on a VMO (vanishing mean oscillation) criterion for homotopy. Indeed, since the balls $B_{\rho_j}(a_j)$ have been chosen to contain the region of energy concentration of v, the map w_0 can be proved to have small energy on all balls of radius ρ_0. On the other hand, the map u satisfies the same property by assumption. This combined with the integral estimate of the distance between u and v allows to apply the VMO criterion for homotopy to deduce that u and w_0 are homotopic as continuous maps.

The following proposition will allow us to perform the growing of balls procedure mentioned at the beginning of the above sketch.

Proposition 5.2.2 (Growing balls)**.** *Given a Riemannian manifold $\mathscr{M}$ and given a collection* *of balls $B_{\rho_1}(a_1), \dots, B_{\rho_J}(a_J) \subset \mathscr{M}$, there exist balls*

$$B_{\rho_1(t)}(a_1(t)), \dots, B_{\rho_{J(t)}(t)}(a_{J(t)}(t))$$

such that for every $t \in [0, +\infty)$,

(*i*) *the balls $\overline{B_{\rho_1(t)}}(a_1(t)), \dots, \overline{B_{\rho_{J(t)}(t)}}(a_{J(t)}(t))$ are disjoint,*

(*ii*) $\displaystyle\bigcup_{j=1}^{J} B_{\rho_j}(a_j) \subseteq \bigcup_{j=1}^{J(t)} B_{\rho_j(t)}(a_j(t))$,

(*iii*) $\displaystyle\sum_{j=1}^{J(t)} \rho_j(t) \le \mathrm{e}^t \sum_{j=1}^{J} \rho_j$,

(*iv*) *if $f : \mathscr{M} \to [0, +\infty]$ is Borel-measurable, then*

$$\int_0^{+\infty} \Bigl(\sum_{j=1}^{J(t)} \rho_j(t) \int_{\partial B_{\rho_j(t)}(a_j(t))} f \Bigr) \,\mathrm{d}t \le \int_{\mathscr{M}} f.$$

Even though there is no upper bound on t appearing in the statement of Proposition 5.2.2, if $\mathscr{M}$ is bounded, then for t sufficiently large, one will inevitably have $\partial B_{\rho_j(t)}(a_j) = \varnothing$, and the statement will give no information about such t. Also, the conclusions are the most useful when $\rho_j(t)$ is sufficiently small to have $\partial B_{\rho_j(t)}(a_j(t))$ diffeomorphic to a sphere.

Proof of Proposition 5.2.2. The proof follows the Euclidean case [SS07, Theorem 4.2] (see also [San98, Jer99]). Roughly speaking, one defines for $t \in (0, T_1)$

$$a_j(t) = a_j \qquad \text{and} \qquad \rho_j(t) = \mathrm{e}^t \rho_j,$$

in such a way that the closed balls $\overline{B_{\rho_1(t)}}(a_1(t)), \dots, \overline{B_{\rho_{J(t)}(t)}}(a_{J(t)}(t))$ are disjoint for $t < T_1$ and not for $t = T_1$. The assertions (i), (ii), and (iii) are immediate. For (iv), we have by the coarea formula and a change of variable

$$\begin{aligned}\int_{B_{\rho_j(T_1)}(a_j) \setminus B_{\rho_j(0)}(a_j)} f &= \int_{\rho_j}^{\rho_j \mathrm{e}^{T_1}} \Bigl(\int_{\partial B_r(a_j)} f \Bigr) \,\mathrm{d}r \\ &= \int_0^{T_1} \rho_j(t) \Bigl(\int_{\partial B_{\rho_j(t)}(a_j)} f \Bigr) \,\mathrm{d}t.\end{aligned}$$

To continue the construction, we apply Lemma 5.2.3 sufficiently many times to the collection $\overline{B_{\rho_1(t)}}(a_1(t)), \dots, \overline{B_{\rho_{J(t)}(t)}}(a_{J(t)}(t))$ to get a disjoint collection. Repeating this procedure at most J times, we get the required collection of families of balls. □

Lemma 5.2.3 (Merging balls)**.** *If $\mathscr{M}$ is a complete Riemannian manifold, and if $\overline{B_{\rho_0}}(a_0) \cap \overline{B_{\rho_1}}(a_1) \neq \emptyset$, then there exists a ball $B_\rho(a)$ such that*

$$B_\rho(a) \supseteq B_{\rho_0}(a_0) \cup B_{\rho_1}(a_1)$$

and

$$\rho \leq \rho_0 + \rho_1.$$

Proof. This is again a classical argument (see for example [SS07, Lemma 4.1]). If $B_{\rho_0}(a_0) \subseteq B_{\rho_1}(a_1)$ or $B_{\rho_1}(a_1) \subseteq B_{\rho_0}(a_0)$ we can take $a = a_1$ and $\rho = \rho_1$, or $a = a_0$ and $\rho = \rho_0$. Otherwise, we take a point a on a minimising geodesic from a_0 to a_1 such that $\rho_0 + d(a_0, a) = \rho_1 + d(a_1, a)$ and $\rho = (\rho_0 + d(a_0, a_1) + \rho_1)/2$. □

As already mentioned, we will also use a VMO criterion for homotopy, whose statement is as follows.

Proposition 5.2.4. *There exist $\theta \in (0, +\infty)$ and $\rho_* \in (0, +\infty)$ such that, if $u_0, u_1 \in C(\mathscr{M}; \mathscr{N}) \cap W^{1,p}(\mathscr{M}; \mathscr{N})$ and $\rho \in (0, \rho_*)$ satisfy the condition that for every $a \in \mathscr{M}$,*

$$\frac{1}{\rho^p} \int_{B_\rho(a)} |u_0 - u_1| \leq \theta$$

and

$$\int_{B_\rho(a)} |\mathrm{D}u_0|^p + |\mathrm{D}u_1|^p \leq \theta^p,$$

then u_0 and u_1 are homotopic.

Even though the statement of Proposition 5.2.4 might not be found under this exact form in the literature, it relies on classical arguments that go back to the work of R. Schoen and K. Uhlenbeck [SU83], and H. Brezis and L. Nirenberg [BN95]. Moreover, this kind of argument will be of key importance in Chapter 6 that follows.

Proof of Proposition 5.2.4. Define

$$u_j^r(x) = \fint_{B_r(x)} u_j.$$

Since $\mathscr{M}$ is a Riemannian manifold, its injectivity radius is positive; we assume that it is $2\rho_*$ with $\rho_* \in (0, +\infty)$; in particular, for every $a \in \mathscr{M}$, the exponential map at a

is uniformly controlled on $B_{\rho_*}(a)$. Therefore, by the Poincaré–Wirtinger inequality, for every $0 < r \le \rho < \rho_*$, we have

$$\operatorname{dist}(u_j^r(x), \mathscr{N}) \lesssim \fint_{B_r(x)} \fint_{B_r(x)} |u_j(y) - u_j(z)| \, \mathrm{d}y \, \mathrm{d}z \lesssim \left(\int_{B_r(a)} |\mathrm{D}u_j|^p \right)^{\frac{1}{p}} \lesssim \theta,$$

whereas by the triangle inequality

$$|u_0^\rho(x) - u_1^\rho(x)| \lesssim \frac{1}{\rho^p} \int_{B_\rho(x)} |u_1 - u_0| \lesssim \theta.$$

Hence, if θ was chosen sufficiently small, we get the required homotopy. Indeed, one first goes from u_0 to u_0^ρ via the u_0^r, then from u_0^ρ to u_1^ρ via linear interpolation plus reprojection, and finally from u_1^ρ to u_1 via the u_1^r. More precisely, one may e.g. use $H\colon [0,1] \times \mathscr{M} \to \mathscr{N}$ defined by

$$H(t,x) = \begin{cases} u_0^{3t\rho}(x) & \text{if } 0 \le t \le 1/3, \\ \Pi_{\mathscr{N}}((3t-1)u_1^\rho(x) + (2-3t)u_0^\rho(x)) & \text{if } 1/3 \le t \le 2/3, \\ u_1^{3(1-t)\rho}(x) & \text{if } 2/3 \le t \le 1, \end{cases}$$

where $\Pi_{\mathscr{N}}$ is the Lipschitz-continuous retraction onto $\mathscr{N}$. □

Proof of Proposition 5.2.1. Since Proposition 5.2.1 depends on quite a few parameters — either part of the statement, or specific to the proof — that may depend from each other, we adopt the following convention. All constraints for the various parameters involved in the proof will be displayed inside boxes. Then, at the end of the proof, we give the exact relations that explain how and in which order to choose those parameters so that they satisfy the required constraints.

We again assume that the injectivity radius of $\mathscr{M}$ is $2\rho_*$ with $\rho_* \in (0, +\infty)$, which implies a uniform control on the exponential map on any ball $B_{\rho_*}(a)$, and we assume that $\boxed{\rho_0 \le \rho_*.}$

Given $\eta \in (0, +\infty)$ and $\rho \in (0, +\infty)$ to be chosen later on, we consider the set

$$A = \left\{ a \in \mathscr{M} \colon \int_{B_\rho(a)} |\mathrm{D}v|^p \ge \eta^p \right\}, \tag{5.2.6}$$

and a maximal subset $A_* \subseteq A$ such that the balls $\{B_\rho(a)\}_{a \in A_*}$ are disjoint. In particular, we have by disjointness and by (5.2.2)

$$\operatorname{card}(A_*) \leq \frac{1}{\eta^p} \sum_{a \in A_*} \int_{B_\rho(a)} |\mathrm{D}v|^p \leq \frac{1}{\eta^p} \int_{\mathscr{M}} |\mathrm{D}v|^p \leq \frac{M}{\eta^p}. \tag{5.2.7}$$

On the other hand, by maximality of A_*, $A \subseteq \bigcup_{a \in A_*} B_{2\rho}(a)$, so that for every $a \in \mathscr{M} \setminus \bigcup_{a' \in A_*} B_{2\rho}(a')$, $a \notin A$. (If $A_* = \varnothing$, the following arguments remain valid albeit unnecessarily complicated in this trivial case.)

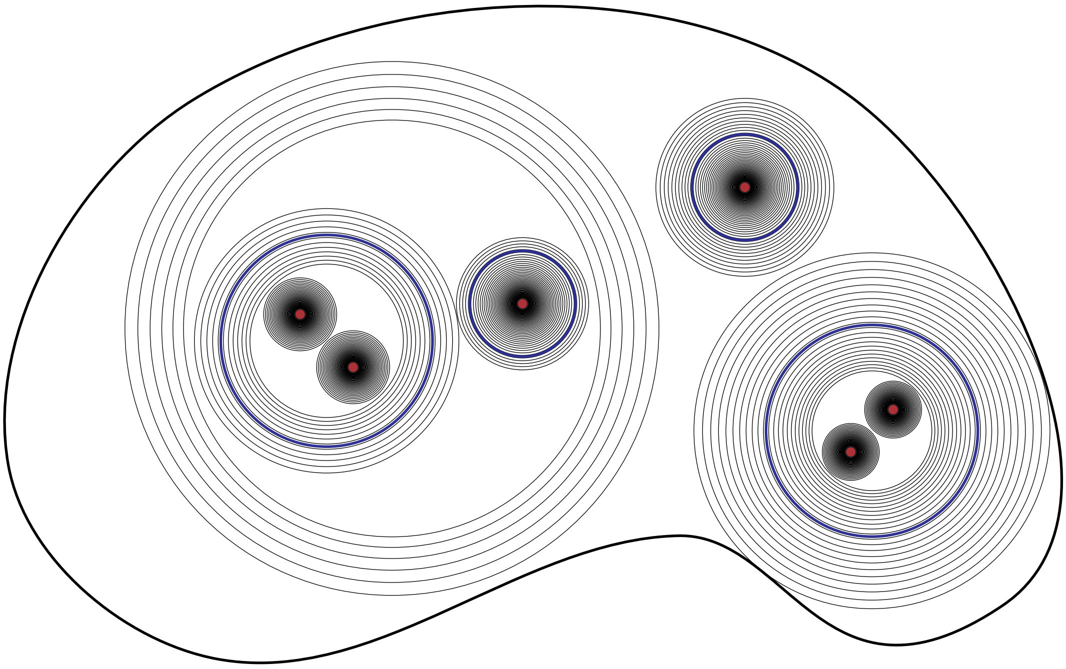

Figure 5.3 – Thanks to an averaging argument, the spheres $\partial B_{\rho_j}(a_j)$ (in blue) can be chosen out of the growing balls generated by the balls $B_\rho a$ (in red) so that u and v have a small Sobolev energy and are at small average distance on them.

Applying Proposition 5.2.2 to $(B_{2\rho}(a))_{a \in A_*}$, we get a finite collection

$$B_{\rho_1(t)}(a_1(t)), \dots, B_{\rho_{J(t)}(t)}(a_{J(t)}(t))$$

for $t \in (0, +\infty)$; by Proposition 5.2.2 (iv), it satisfies for every $T_* \in (0, +\infty)$

$$\int_0^{T_*} \Bigl(\sum_{j=1}^{J(t)} \rho_j(t) \int_{\partial B_{\rho_j(t)}(a_j(t))} |\mathrm{D}u|^p\Bigr) \,\mathrm{d}t \le \int_{\mathscr{M}} |\mathrm{D}u|^p \le M,$$

$$\int_0^{T_*} \Bigl(\sum_{j=1}^{J(t)} \rho_j(t) \int_{\partial B_{\rho_j(t)}(a_j(t))} |\mathrm{D}v|^p\Bigr) \,\mathrm{d}t \le \int_{\mathscr{M}} |\mathrm{D}v|^p \le M,$$

and

$$\int_0^{T_*} \Bigl(\sum_{j=1}^{J(t)} \rho_j(t) \int_{\partial B_{\rho_j(t)}(a_j(t))} |u - v|\Bigr) \,\mathrm{d}t \le \int_{\mathscr{M}} |u - v| \le \rho_0^p \delta,$$

in view of (5.2.1), (5.2.2), and (5.2.4). There exists thus $t_* \in (0, T_*)$ such that, if we set $J = J(t_*)$, and for $j \in \{1, \dots, J\}$, $a_j = a_j(t_*)$ and $\rho_j = \rho_j(t_*)$, then we have

$$\sum_{j=1}^{J} \rho_j \int_{\partial B_{\rho_j}(a_j)} |\mathrm{D}u|^p \le \frac{3M}{T_*}, \tag{5.2.8}$$

$$\sum_{j=1}^{J} \rho_j \int_{\partial B_{\rho_j}(a_j)} |\mathrm{D}v|^p \le \frac{3M}{T_*}, \tag{5.2.9}$$

and

$$\sum_{j=1}^{J} \rho_j \int_{\partial B_{\rho_j}(a_j)} |u - v| \le \frac{3\rho_0^p \delta}{T_*}. \tag{5.2.10}$$

We observe now that

$$A \subseteq \bigcup_{a \in A_*} B_{2\rho}(a) \subseteq \bigcup_{j=1}^{J} B_{\rho_j}(a_j) \tag{5.2.11}$$

and, by (5.2.7),

$$\max_{j \in \{1,\dots,J\}} \rho_j \le \sum_{j=1}^{J} \rho_j \le \frac{2\mathrm{e}^{T_*} \rho M}{\eta^p} \le \rho_0 \le \rho_*, \tag{5.2.12}$$

so that (ii) holds, provided

$$\boxed{2\rho M \mathrm{e}^{T_*} \leq \rho_0 \eta^p.} \tag{5.2.13}$$

By the Morrey–Sobolev embedding, we have for every $j \in \{1, \dots, J\}$ and x, $y \in \partial B_{\rho_j}(a_j)$, in view of (5.2.8), (5.2.12), and by the choice of ρ_* in terms of the injectivity radius,

$$|u(x) - u(y)| \lesssim \left(\int_{\partial B_{\rho_j}(a_j)} |\mathrm{D}u|^p \right)^{\frac{1}{p}} \rho_j^{\frac{1}{p}} \leq \frac{C_1 M^{1/p}}{T_*^{1/p}}. \tag{5.2.14}$$

We also have for every $j \in \{1, \dots, J\}$ and every $x, y \in \partial B_{\rho_j}(a_j)$, by (5.2.14), (5.2.9), (5.2.10), by (5.2.12) and the conditions on ρ_*, since $\rho \leq \rho_j$,

$$\begin{aligned} &|u(x) - v(y)| \\ &\leq \sup_{z \in \partial B_{\rho_j}(a_j)} |u(x) - u(z)| + \inf_{z \in \partial B_{\rho_j}(a_j)} |u(z) - v(z)| + \sup_{z \in \partial B_{\rho_j}(a_j)} |v(z) - v(y)| \\ &\lesssim \left(\int_{\partial B_{\rho_j}(a_j)} |\mathrm{D}u|^p \right)^{\frac{1}{p}} \rho_j^{\frac{1}{p}} + \frac{1}{\rho_j^{p-1}} \int_{\partial B_{\rho_j}(a_j)} |u - v| + \left(\int_{\partial B_{\rho_j}(a_j)} |\mathrm{D}v|^p \right)^{\frac{1}{p}} \rho_j^{\frac{1}{p}} \\ &\leq \frac{2C_1 M^{1/p}}{T_*^{1/p}} + \frac{C_2 \rho_0^p \delta}{\rho^p T_*}. \end{aligned} \tag{5.2.15}$$

Using the fact that $\mathscr{N}$ is a uniform Lipschitz neighborhood retract, we fix $\tau \in (0, \varepsilon)$ such that for every $b \in \mathscr{N}$, $\Pi_{\mathscr{N}}$ is well-defined on $\bar{B}_\tau(b)$, and such that for every $z \in \overline{B_\tau}(b)$ and $t \in [0, 1]$,

$$|b - \Pi_{\mathscr{N}}((1 - t)z + tb)| \leq \varepsilon.$$

If

$$\boxed{\frac{2C_1 M^{1/p}}{T_*^{1/p}} + \frac{C_2 \rho_0^p \delta}{\rho^p T_*} \leq \tau,} \tag{5.2.16}$$

taking $b_j \in u(\partial B_{\rho_j}(a_j))$, for every $x \in \partial B_{\rho_j}(a_j)$, we have in view of (5.2.14) and (5.2.15),

$$|u(x) - b_j| \leq \tau \qquad \text{and} \qquad |v(x) - b_j| \leq \tau.$$

We can thus define now, in exponential coordinates, by (5.2.12) and the choice of ρ_*,

$$w(x) = \begin{cases} v(x) & \text{if } x \in \mathscr{M} \setminus \bigcup_{j=1}^{J} B_{\rho_j}(a_j), \\ \Pi_{\mathscr{N}}((2\frac{|x|}{\rho_j} - 1)v(\frac{\rho_j}{|x|}x) + (2 - 2\frac{|x|}{\rho_j})b_j) & \text{if } x \in B_{\rho_j}(a_j) \setminus B_{\rho_j/2}(a_j), \\ \Pi_{\mathscr{N}}((2 - 4\frac{|x|}{\rho_j})v(\frac{\rho_j}{|x|}x) + (4\frac{|x|}{\rho_j} - 1)b_j) & \text{if } x \in B_{\rho_j/2}(a_j) \setminus B_{\rho_j/4}(a_j), \\ v(4x) & \text{if } x \in B_{\rho_j/4}(a_j). \end{cases}$$

Since v is continuous, we have $\operatorname{tr}_{\partial B_{\rho_j}(a_j)} v = v|_{\partial B_{\rho_j}(a_j)}$, so that $w \in W^{1,p}(\mathscr{M};\mathscr{N})$. Properties (iii), (iv), (v), (vi), and (viii) clearly hold by construction of w. Moreover, the map defined in exponential coordinates as

$$H(t,x) = \begin{cases} v(x) & \text{if } x \in \mathscr{M} \setminus \bigcup_{j=1}^{J} B_{\rho_j}(a_j), \\ \Pi_{\mathscr{N}}\big((2\frac{|x|}{\rho_j} - 1)v(\frac{\rho_j}{|x|}x) + (2 - 2\frac{|x|}{\rho_j})b_j\big) & \text{if } x \in B_{\rho_j}(a_j) \setminus B_{(1-t/2)\rho_j}(a_j), \\ \Pi_{\mathscr{N}}\big((5 - 3t - 4\frac{|x|}{\rho_j})v(\frac{\rho_j}{|x|}x) \\ \qquad +(4\frac{|x|}{\rho_j} + 3t - 4)b_j\big) & \text{if } x \in B_{(1-t/2)\rho_j}(a_j) \setminus B_{(1-3t/4)\rho_j}(a_j), \\ v(x/(1-3t/4)) & \text{if } x \in B_{(1-3t/4)\rho_j}(a_j), \end{cases}$$

can be checked to be a homotopy between the maps w and v, so that (ix) holds.

Via an integration of (5.2.15), we have for every $j \in \{1, \dots, J\}$,

$$\begin{aligned} \int_{\partial B_{\rho_j}(a_j)} |v - b_j|^p \lesssim \rho_j^p \int_{\partial B_{\rho_j}(a_j)} |\mathrm{D}u|^p + \rho_j^{p-1}\Big(\frac{1}{\rho_j^{p-1}} \int_{\partial B_{\rho_j}(a_j)} |u - v|\Big)^p \\ + \rho_j^p \int_{\partial B_{\rho_j}(a_j)} |\mathrm{D}v|^p, \end{aligned}$$

and thus, by the conditions on ρ_* and $\rho \le \rho_j$

$$\begin{aligned} \int_{B_{\rho_j}(a_j) \setminus B_{\rho_j/4}(a_j)} |\mathrm{D}w|^p &\lesssim \rho_j \int_{\partial B_{\rho_j}(a_j)} |\mathrm{D}v|^p + \frac{1}{\rho_j^{p-1}} \int_{\partial B_{\rho_j}(a_j)} |v - b_j|^p \\ &\lesssim \rho_j \int_{\partial B_{\rho_j}(a_j)} |\mathrm{D}v|^p + \Big(\frac{\rho_j}{\rho^p} \int_{\partial B_{\rho_j}(a_j)} |u - v|\Big)^p \\ &\quad + \rho_j \int_{\partial B_{\rho_j}(a_j)} |\mathrm{D}u|^p. \end{aligned}$$

Therefore, in view of (5.2.8), (5.2.9), and (5.2.10), we find

$$\sum_{j=1}^{J} \int_{B_{\rho_j}(a_j) \setminus B_{\rho_j/4}(a_j)} |Dw|^p \leq \frac{C_3 M}{T_*} + C_4 \Big(\frac{\rho_0^p \delta}{\rho^p T_*} \Big)^p . \tag{5.2.17}$$

In particular, if

$$\boxed{\frac{C_3 M}{T_*} + C_4 \Big(\frac{\rho_0^p \delta}{\rho^p T_*} \Big)^p \leq \varepsilon^p ,} \tag{5.2.18}$$

then (vii) follows from (5.2.17).

We now prove that u and w_0 are homotopic, relying on Proposition 5.2.4. Since $w_0 = w$ on $B_{\rho_j}(a_j) \setminus B_{\rho_j/4}(a_j)$ and $w_0 = b$ on $B_{\rho_j/4}(a_j)$, we have by (5.2.17)

$$\sum_{j=1}^{J} \int_{B_{\rho_j}(a_j)} |Dw_0|^p \leq \sum_{j=1}^{J} \int_{B_{\rho_j}(a_j) \setminus B_{\rho_j/4}(a_j)} |Dw|^p \leq \frac{C_3 M}{T_*} + C_4 \Big(\frac{\rho_0^p \delta}{\rho^p T_*} \Big)^p . \tag{5.2.19}$$

Given $a \in \mathscr{M}$, we have, either $B_{\rho/2}(a) \subseteq \bigcup_{j=1}^{J} B_{\rho_j}(a_j)$, and thus, by (5.2.19),

$$\int_{B_{\rho/2}(a)} |Dw_0|^p \leq \frac{C_3 M}{T_*} + C_4 \Big(\frac{\rho_0^p \delta}{\rho^p T_*} \Big)^p ,$$

or there exists $a' \in \mathscr{M} \setminus \bigcup_{j=1}^{J} B_{\rho_j}(a_j)$ such that $B_{\rho/2}(a) \subseteq B_{\rho}(a')$, and thus, by (5.2.19) again

$$\begin{aligned} \int_{B_{\rho/2}(a)} |Dw_0|^p &\leq \int_{B_{\rho}(a') \setminus \bigcup_{j=1}^{J} B_{\rho_j}(a_j)} |Dv|^p + \sum_{j=1}^{J} \int_{B_{\rho_j}(a_j)} |Dw_0|^p \\ &\leq \eta^p + \frac{C_3 M}{T_*} + C_4 \Big(\frac{\rho_0^p \delta}{\rho^p T_*} \Big)^p \end{aligned}$$

in view of (5.2.11), (5.2.6) and (5.2.17). In particular, if

$$\boxed{\eta^p + \frac{C_3 M}{T_*} + C_4 \Big(\frac{\rho_0^p \delta}{\rho^p T_*} \Big)^p \leq \theta^p ,} \tag{5.2.20}$$

then

$$\int_{B_{\rho/2}(a)} |\mathrm{D}w_0|^p \leq \theta^p.$$

In order to verify the assumptions of Proposition 5.2.4, we actually need to work with a modified version of u, constructed from u by a similar process to the one used to construct w_0 from v. More specifically, given $\sigma_1, \dots, \sigma_J$ such that $0 < \sigma_j < \rho_j/4$, we define the map $u_0 \colon \mathscr{M} \to \mathscr{N}$ by

$$u_0(x) = \begin{cases} u(x) & \text{if } x \in \mathscr{M} \setminus \bigcup_{j=1}^J B_{\rho_j}(a_j), \\ \Pi_{\mathscr{N}}\big((2\frac{|x|}{\rho_j} - 1)u(\frac{\rho_j}{|x|}x) + \big(2 - 2\frac{|x|}{\rho_j}\big)b_j\big) & \text{if } x \in B_{\rho_j}(a_j) \setminus B_{\rho_j/2}(a_j), \\ b_j & \text{if } x \in B_{\rho_j/2}(a_j) \setminus B_{2\sigma_j}(a_j), \\ \Pi_{\mathscr{N}}\big((2 - \frac{|x|}{\sigma_j})u(\frac{\rho_j}{|x|}x) + \big(\frac{|x|}{\sigma_j} - 1\big)b_j\big) & \text{if } x \in B_{2\sigma_j}(a_j) \setminus B_{\sigma_j}(a_j), \\ u(\frac{\rho_j}{\sigma_j}x) & \text{if } x \in B_{\sigma_j}(a_j). \end{cases}$$

We observe that the map u is homotopic to u_0. Therefore, it suffices to show that w_0 is homotopic to u_0.

As we did for v above, we have for every $j \in \{1, \dots, J\}$ that, by (5.2.14),

$$\int_{\partial B_{\rho_j}(a_j)} |u(x) - b_j|^p \lesssim \rho_j^p \int_{\partial B_{\rho_j}(a_j)} |\mathrm{D}u|^p,$$

and thus, by (5.2.12) and the conditions on ρ_*,

$$\int_{B_{\rho_j}(a_j) \setminus B_{\sigma_j}(a_j)} |\mathrm{D}u_0|^p \lesssim \rho_j \int_{\partial B_{\rho_j}(a_j)} |\mathrm{D}u|^p.$$

Therefore, by (5.2.8),

$$\sum_{j=1}^J \int_{B_{\rho_j}(a_j) \setminus B_{\sigma_j}(a_j)} |\mathrm{D}u_0|^p \leq \frac{C_5 M}{T_*}.$$

We also have, by (5.2.12), the conditions on ρ_*, and by the assumption (5.2.4),

$$\int_{B_{\sigma_j}(a_j)} |\mathrm{D}u_0|^p \leq C_6 \int_{B_{\rho_j}(a_j)} |\mathrm{D}u|^p \leq C_6 \delta^p,$$

with $C_6 \geq 1$. (In the case where $\mathscr{M}$ is flat, one can take $C_6 = 1$; the constant comes from the bound on the geometry of $\mathscr{M}$ on scales below ρ_*.)

Given $a \in \mathscr{M}$, we consider two cases. If $B_{\rho/2}(a) \subseteq B_{\rho_j}(a_j)$, then

$$\int_{B_{\rho/2}(a)} |\mathrm{D}u_0|^p \leq \frac{C_5 M}{T_*} + C_6 \delta^p,$$

provided (5.2.13) holds. Otherwise, since $\rho \leq \rho_j/2$ and $\sigma_j \leq \rho_j/4$, $B_{\rho/2}(a) \cap B_{\sigma_j}(a_j) = \varnothing$, and thus by (5.2.3),

$$\int_{B_{\rho/2}(a)} |\mathrm{D}u_0|^p \leq \frac{C_5 M}{T_*} + \delta^p,$$

provided $2\rho \leq \rho_0$, which follows from the condition (5.2.13), assuming without loss of generality that

$$\boxed{\eta^p \leq M.} \tag{5.2.21}$$

If

$$\boxed{\frac{C_5 M}{T_*} + C_6 \delta^p \leq \theta^p,} \tag{5.2.22}$$

we then have

$$\int_{B_{\rho/2}(a)} |\mathrm{D}u_0|^p \leq \theta^p.$$

Finally, since $u_0 = u$ and $w_0 = v$ outside of the balls $B_{\rho_j}(a_j)$, we find that

$$\begin{aligned}\int_{B_{\rho/2}(a)} |u_0 - w_0| &\leq \int_{\mathscr{M}} |u - v| + C_7 \rho^p \varepsilon + \sum_{j=1}^{J} C_8 \sigma_j^p \\ &\leq \rho_0^p \delta + C_7 \rho^p \varepsilon + C_8 \sum_{j=1}^{J} \sigma_j^p .\end{aligned}$$

We note that here C_8 depends on u and v through their L^∞ norm. If

$$\rho_0^p \delta + C_7 \rho^p \varepsilon + C_8 \sum_{j=1}^{J} \sigma_j^p \leq \frac{\rho^p \theta}{2^p},$$

we have

$$\int_{B_{\rho/2}(a)} |u_0 - w_0| \le \frac{\rho^p \theta}{2^p}. \tag{5.2.23}$$

If

$$\boxed{\frac{\rho_0^p}{\rho^p}\delta + C_7 \varepsilon \le \frac{\theta}{2^{p+1}},} \tag{5.2.24}$$

then $\sigma_1, \dots, \sigma_k$ can be chosen sufficiently small, depending on u and v, so that (5.2.23) holds. By Proposition 5.2.4, w_0 is homotopic to u_0 and thus to u, proving (x).

It remains to show that η, ρ, and T_* satisfying the conditions (5.2.13), (5.2.16), (5.2.18), (5.2.20), (5.2.21), (5.2.22), and (5.2.24) can be found. We first assume without loss of generality that

$$\varepsilon \le \frac{\theta}{2^{p+2}C_7} \qquad \text{and} \qquad \rho_0 \le \rho_*.$$

We next choose successively

$$\eta = \min\Big(M^{1/p}, \frac{\theta}{3^{1/p}}, \frac{\varepsilon}{2^{1/p}}\Big), \quad T_* = \max\Big(\frac{(4C_1)^p M}{\tau^p}, \frac{C_3 M}{\eta^p}, \frac{2C_5 M}{\theta^p}\Big),$$
$$\lambda = \frac{\eta^p}{2Me^{T_*}}, \quad \rho = \lambda\rho_0, \quad \delta = \min\Big(\frac{T_*\lambda^p\tau}{2C_2}, \frac{T_*\lambda^p\eta}{(C_4)^{1/p}}, \frac{\theta}{(2C_6)^{1/p}}, \frac{\theta\lambda^p}{2^{p+2}}\Big),$$

and check that all the conditions are satisfied. □

5.3 Analytical obstruction for integer exponent

This section is devoted to the proof of Theorem 5.1.1. We start by explaining some tools that will be crucial to construct a suitable family of Sobolev mappings with values into a skeleton of $\mathbb{R}^n$ and prove that their relaxed energy grows superlinearly with respect to their Sobolev energy. We then explain the procedure to transfer these constructions, first to a compact skeleton, and then to a compact manifold without boundary, yielding the proof of Theorem 5.1.1 via the nonlinear uniform boundedness principle.

5.3.1 Conical joint estimate on the Brouwer degree

Our goal in this section is to establish an analytic estimate of the joint Brouwer degrees of a map $f \in C^1(\mathbb{S}^d; \mathbb{R}^{d+1})$ with respect to a finite number of points in $\mathbb{R}^{d+1}$ that are avoided by f.

We first recall that Brouwer's topological degree of a map $f \in C^1(\mathbb{S}^d; \mathbb{S}^d)$ can be computed by the formula

$$\deg(f) = \frac{\displaystyle\int_{\mathbb{S}^d} (\det \mathrm{D} f) w \circ f}{\displaystyle\int_{\mathbb{S}^d} w}, \tag{5.3.1}$$

for any weight function $w \in C(\mathbb{S}^d; \mathbb{R})$ with $\int_{\mathbb{S}^d} w \neq 0$. (The determinant in (5.3.1) is computed on the tangent space with the orientation induced by the canonical orientation on the ambient space $\mathbb{R}^{d+1}$.)

Actually, formula (5.3.1) is still valid for computing the degree of a continuous map which is not C^1, but merely $W^{1,d}$, see [BC83, BN95]. This kind of considerations will also be crucial in Chapter 6. Therefore, from now on, we will work with continuous and $W^{1,d}$ maps — this notably avoids some technical issues when working with smooth maps between cell complexes. For every open set $G \subset \mathbb{S}^d$, taking $w = w_n$, where $(w_n)_{n \in \mathbb{N}}$ is a sequence approximating the characteristic function of G, and letting $k \to +\infty$, we get the estimate

$$|\deg(f)| \leq \frac{1}{\mathscr{H}^d(G)} \int_{f^{-1}(G)} |\mathrm{D} f|^d. \tag{5.3.2}$$

Given $f \in W^{1,d}(\mathbb{S}^d; \mathbb{R}^{d+1} \setminus \{0\}) \cap C(\mathbb{S}^d; \mathbb{R}^{d+1} \setminus \{0\})$, we define

$$\deg f = \deg(f/|f|). \tag{5.3.3}$$

We say that a set $\mathscr{C} \subseteq \mathbb{R}^{d+1}$ is a cone whenever, for every $t \in (0, +\infty)$ and $x \in \mathscr{C}$, $tx \in \mathscr{C}$. If $\mathscr{C} \subseteq \mathbb{R}^{d+1}$ is an open cone, then it follows from (5.3.2) and (5.3.3) that

$$|\deg(f)| \leq \frac{1}{\mathscr{H}^d(\mathscr{C} \cap \mathbb{S}^d)} \int_{f^{-1}(\mathscr{C})} \frac{|\mathrm{D} f|^d}{|f|^d}, \tag{5.3.4}$$

since $|\mathrm{D}(f/|f|)| \leq |\mathrm{D} f|/|f|$ everywhere on $\mathbb{S}^d$.

We will use the degree with respect to a point $\sigma \in \mathbb{R}^{d+1}$, defined for $f \in C(\mathbb{S}^d; \mathbb{R}^{d+1} \setminus \{\sigma\})$ as

$$\deg_\sigma(f) = \deg(f - \sigma).$$

We are now going to give a joint estimate on the degrees with respect to a finite set of points $\Sigma \subseteq \mathbb{R}^{d+1}$. We start by observing that (5.3.4) implies that

$$\begin{aligned}\sum_{\sigma\in\Sigma} |\deg_\sigma(f)| &\le \frac{1}{\mathscr{H}^d(\mathscr{C}\cap \mathbb{S}^d)} \sum_{\sigma\in\Sigma} \int_{f^{-1}(\mathscr{C}+\sigma)} \frac{|\mathrm{D}f|^d}{|f-\sigma|^d} \\ &\le \frac{1}{\mathscr{H}^d(\mathscr{C}\cap \mathbb{S}^d)} \Big(\int_{f^{-1}(\mathscr{C}+\Sigma)} |\mathrm{D}f|^d\Big) \sup_{x\in\mathbb{S}^d} \sum_{\sigma\in\Sigma} \frac{1}{|f(x)-\sigma|^d}.\end{aligned} \tag{5.3.5}$$

We now estimate the sum that appears on the last line of (5.3.5).

Proposition 5.3.1. *If* $\Sigma \subseteq \mathbb{Z}^{d+1}$, $y \in \mathbb{R}^{d+1}$, *and if* $\operatorname{dist}(y,\Sigma) \ge 1/2$, *then*

$$\sum_{\sigma\in\Sigma} \frac{1}{|y-\sigma|^d} \le C(\operatorname{card}(\Sigma))^{\frac{1}{d+1}},$$

where the constant $C > 0$ *depends only on* d.

Proof. We have

$$\begin{aligned}\sum_{\sigma\in\Sigma} \frac{1}{|y-\sigma|^d} &= \int_0^{+\infty} \operatorname{card}(\Sigma\cap B_r^{d+1}(y)) \frac{d}{r^{d+1}}\,\mathrm{d}r \\ &\le C_1 \int_0^{+\infty} \min(\operatorname{card}(\Sigma), r^{d+1}) \frac{1}{r^{d+1}}\,\mathrm{d}r \\ &= C_1\Big(\int_0^{(\#\Sigma)^{1/(d+1)}} \mathrm{d}r + \int_{(\operatorname{card}(\Sigma))^{1/(d+1)}}^{+\infty} \frac{\operatorname{card}(\Sigma)}{r^{d+1}}\,\mathrm{d}r\Big) \\ &= C_1\big(1+\tfrac{1}{d}\big)(\operatorname{card}(\Sigma))^{\frac{1}{d+1}}.\end{aligned}$$

In the first inequality above, we have used the fact that

$$\operatorname{card}(\Sigma\cap B_r^{d+1}(y)) \le \mathcal{L}^n(B^{d+1}_{r+\sqrt{d+1}/2}(y)) = \mathcal{L}^n(\mathbb{B}^{d+1})(r+\sqrt{d+1}/2)^{d+1} \le C_2 r^{d+1}$$

provided $r \ge 1/2$, whereas $\operatorname{card}(\Sigma\cap B_r^{d+1}(y)) = 0$ when $r < 1/2$. □

Injecting Proposition 5.3.1 in (5.3.5), we obtain the following conical joint estimate on the degrees.

Proposition 5.3.2. *Given a finite set* $\Sigma \subseteq \mathbb{Z}^{d+1}$, *an open cone* $\mathscr{C} \subset \mathbb{R}^{d+1}$, *and a map* $f \in W^{1,n-1}(\mathbb{S}^d;\mathbb{R}^{d+1}\setminus(B^{d+1}_{1/2}+\Sigma)) \cap C(\mathbb{S}^d;\mathbb{R}^{d+1}\setminus(B^{d+1}_{1/2}+\Sigma))$, *one has*

$$\Big(\sum_{\sigma\in\Sigma} |\deg_\sigma(f)|\Big)^{1-\frac{1}{d+1}} \le \frac{C}{\mathscr{H}^d(\mathscr{C}\cap\mathbb{S}^d)} \int_{f^{-1}(\mathscr{C}+\Sigma)} |\mathrm{D}f|^d,$$

where the constant $C > 0$ depends only on d.

Proof. Without loss of generality, we assume that for every $\sigma \in \Sigma$, $\deg_\sigma(f) \neq 0$. We then have by (5.3.5) and Proposition 5.3.1,

$$\begin{aligned}\sum_{\sigma\in\Sigma} |\deg_\sigma(f)| &\lesssim \frac{(\operatorname{card}(\Sigma))^{\frac{1}{d+1}}}{\mathscr{H}^d(\mathscr{C}\cap\mathbb{S}^d)} \int_{f^{-1}(\mathscr{C}+\Sigma)} |Df|^d \\ &\lesssim \frac{1}{\mathscr{H}^d(\mathscr{C}\cap\mathbb{S}^{n-1})} \Big(\sum_{\sigma\in\Sigma}|\deg_\sigma(f)|\Big)^{\frac{1}{d+1}} \int_{f^{-1}(\mathscr{C}+\Sigma)} |Df|^d,\end{aligned}$$

and the conclusion follows. □

5.3.2 Lower bounds on energies on spheres

In this section, we combine our conical joint degrees estimate with our bubbling construction to give a lower estimate on the Sobolev energy of approximating sequences in different homotopy classes.

We first get a lower bound in a general setting in terms of joint degree differences between two continuous Sobolev maps.

Proposition 5.3.3. *Given a finite set $\Sigma \subseteq \mathbb{Z}^{d+1}$, open cones $\mathscr{C}_1, \dots, \mathscr{C}_I \subseteq \mathbb{R}^{d+1}$, and open sets $G_1, \dots, G_I \subseteq \mathbb{R}^{d+1}$ such that for every $i \in \{1, \dots, I\}$, $G_i \cap (\mathscr{C}_i + \Sigma) = \emptyset$, there exists a constant C such that, given $u \in W^{1,d}(\mathbb{S}^d; \mathbb{R}^{d+1} \setminus (B^{d+1}_{1/2} + \Sigma)) \cap C(\mathbb{S}^d; \mathbb{R}^{d+1} \setminus (B^{d+1}_{1/2} + \Sigma))$ satisfying*

$$u(\mathbb{S}^d) \subseteq \bigcup_{i=1}^{I} G_i,$$

for every $M \in (0, +\infty)$, there exists $\eta \in (0, +\infty)$ such that, if $v \in W^{1,d}(\mathbb{S}^d; \mathbb{R}^{d+1} \setminus (B^{d+1}_{1/2} + \Sigma)) \cap C(\mathbb{S}^d; \mathbb{R}^{d+1} \setminus (B^{d+1}_{1/2} + \Sigma))$, if

$$\int_{\mathbb{S}^d} |Dv|^d \le M,$$

and if

$$\int_{\mathbb{S}^d} |u - v| \le \eta,$$

then

$$\Big(\sum_{\sigma\in\Sigma}|\deg_\sigma(u)-\deg_\sigma(v)|\Big)^{1-\frac{1}{d+1}} \le C\sum_{i=1}^{I}\int_{u^{-1}(G_i)\cap v^{-1}(\mathscr{C}_i+\Sigma)}|\mathrm{D}v|^d.$$

Moreover, the constant $C>0$ depends only on d and on $\mathscr{H}^d(\mathscr{C}_i\cap\mathbb{S}^d)$.

Proof. We choose ε sufficiently small so that for every $x\in\mathbb{S}^d$, there exists $i\in\{1,\dots,I\}$ for which $B^{d+1}_{2\varepsilon}(u(x))\subseteq G_i$ (in other words, 2ε is the Lebesgue number of the covering of the compact set by $u(\mathbb{S}^d)$ by $G_1,\dots,G_I$).

Without loss of generality, we may also assume that

$$\int_{\mathbb{S}^d}|\mathrm{D}u|^d\le M.$$

We take ρ_0 such that for every $x\in\mathbb{S}^d$, $u(B_{\rho_0}(x))\subseteq B^{d+1}_\varepsilon(u(x))$. Invoking Lebesgue's lemma, we may furthermore assume that ρ_0 has been chosen so that $\rho_0\le\delta$ and

$$\sup_{a\in\mathbb{S}^d}\int_{B_{\rho_0}(a)}|\mathrm{D}u|^d\le\delta^d,$$

where $\delta\in(0,+\infty)$ is given by Proposition 5.2.1. Finally, we let $\eta=\rho_0^d\delta$. Let us note that ρ_0, and therefore η, depend on u via the use of Lebesgue's lemma. On the other hand, the constants that will appear in the proof, and therefore the final constant C, do not depend on u.

Let w be given by Proposition 5.2.1. Defining $w_j=w_{|B_{\rho_j/2}(a_j)}$, we observe that for every $\sigma\in\Sigma$,

$$\deg_\sigma(v)-\deg_\sigma(u)=\deg_\sigma(w_0)+\sum_{j=1}^{J}\deg_\sigma(w_j)-\deg_\sigma(u)=\sum_{j=1}^{J}\deg_\sigma(w_j).$$

It follows thus by the triangle inequality and sublinearity that

$$\Big(\sum_{\sigma\in\Sigma}|\deg_\sigma(v)-\deg_\sigma(u)|\Big)^{1-\frac{1}{d+1}}\le\Big(\sum_{j=1}^{J}\sum_{\sigma\in\Sigma}|\deg_\sigma(w_j)|\Big)^{1-\frac{1}{d+1}}\le\sum_{j=1}^{J}\Big(\sum_{\sigma\in\Sigma}|\deg_\sigma(w_j)|\Big)^{1-\frac{1}{d+1}}.$$

By the choice of ρ_0 and of ε, for every $j\in\{1,\dots,J\}$, there is $i\in\{1,\dots,I\}$ such that

$$u(B_{\rho_j}(a_j))\subseteq B^{d+1}_\varepsilon(u(a_j))\subseteq G_i. \tag{5.3.6}$$

Therefore,

$$w_j(B_{\rho_j/2}(a_j) \setminus B_{\rho_j/4}(a_j)) \subseteq B^{d+1}_{2\varepsilon}(u(a_j)) \subseteq G_i,$$

where we have used assertions (v) and (iii) in Proposition 5.2.1. It follows thus from Proposition 5.3.2 that

$$\begin{aligned}\Big(\sum_{\sigma\in\Sigma}|\deg_\sigma(w_j)|\Big)^{1-\frac{1}{d+1}} &\leq C_1\int_{w_j^{-1}(\mathscr{C}_i+\Sigma)}|\mathrm{D}w_j|^d\\ &= C_1\int_{B_{\rho_j/4}(a_j)\cap w_j^{-1}(\mathscr{C}_i+\Sigma)}|\mathrm{D}w_j|^d\\ &\leq C_2\int_{B_{\rho_j}(a_j)\cap v^{-1}(\mathscr{C}_i+\Sigma)}|\mathrm{D}v|^d,\end{aligned}$$

since (j) $w_j(B_{\rho_j}(a_j) \setminus B_{\rho_j/4}(a_j)) \subseteq G_i$, (jj) $G_i \cap (\mathscr{C}_i + \Sigma) = \emptyset$, and (jjj) $v(4x) = w_j(x)$ on $B_{\rho_j/4}(a_j)$. We note that actually, in order to apply Proposition 5.3.2, one needs to view the map w_j, which is originally defined as a map on the disk $B_{\rho_j/2}(a_j)$ with constant value on the boundary, as a map defined on the sphere $\mathbb{S}^d$, which can be done extending w_j to the exterior of the ball by a constant. We also note that C_1 depends on the measure of $\mathscr{C}_i \cap \mathbb{S}^d$ through the use of Proposition 5.3.2, while C_2 is a purely geometric constant that depends only on d.

It follows thus, using also (5.3.6), that

$$\begin{aligned}\sum_{\substack{j\in\{1,\dots,J\}\\ B_\varepsilon(u(a_j))\subseteq G_i}}\Big(\sum_{\sigma\in\Sigma}|\deg_\sigma(w_j)|\Big)^{1-\frac{1}{d+1}} &\leq C_2\sum_{j=1}^{J}\int_{u^{-1}(G_i)\cap B_{\rho_j}(a_j)\cap v^{-1}(\mathscr{C}_i+\Sigma)}|\mathrm{D}v|^d\\ &\leq C_2\int_{u^{-1}(G_i)\cap v^{-1}(\mathscr{C}_i+\Sigma)}|\mathrm{D}v|^d.\end{aligned}$$

The conclusion follows by summing the above estimate over all $i \in \{1, \dots, I\}$. □

In order to get a lower estimate on sequences of maps on spheres related to our counterexample, we will choose some specific sets $\mathscr{C}_i$ and G_i in Proposition 5.3.3. We define the cube

$$Q_\ell = [0, \ell]^{d+1}.$$

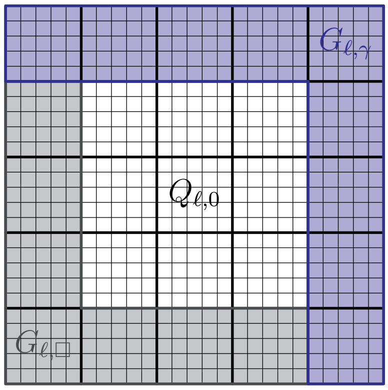

Figure 5.4 – The sets $Q_{\ell,0}$, $G_{\ell,\square}$, and $G_{\ell,\gamma}$ for $\gamma = (-1,-1)$. The colored cubes on the boundary form the set $G_{\ell,\square}$, and the cubes that are further colored in blue form the set $G_{\ell,\gamma}$ for $\gamma = (-1,-1)$.

For every $\alpha \in \mathbf{A} = \{-2,-1,0,1,2\}^{d+1}$, we set

$$Q_{\ell,\alpha} = Q_\ell + \ell\alpha + (2\ell,\dots,2\ell),$$

so that

$$Q_{5\ell} = \bigcup_{\alpha\in\mathbf{A}} Q_{\ell,\alpha}, \tag{5.3.7}$$

and the interiors of the $(Q_{\ell,\alpha})_{\alpha\in\mathbf{A}}$ are mutually disjoint. We also define, for every $\gamma \in \mathbf{\Gamma} = \{-1,1\}^{d+1}$ and for every $\gamma \in \mathbf{\Gamma}$, the cone

$$C_\gamma = \{(x_1,\dots,x_{d+1}) \in \mathbb{R}^{d+1} \colon \text{for every } i \in \{1,\dots,d+1\},\ \gamma_i x_i > 0\}.$$

Setting

$$\mathbf{A}_\square = \{\alpha \in \mathbf{A} \colon \max_{1\le i\le d+1} |\alpha_i| = 2\}, \qquad G_{\ell,\square} = \operatorname{int}\Big(\bigcup_{\alpha\in A_\square} Q_{\ell,\alpha}\Big),$$

$$\mathbf{A}_\gamma = \{\alpha \in \mathbf{A} \colon \min_{1\le i\le d+1} \alpha_i\gamma_i = -2\}, \qquad G_{\ell,\gamma} = \operatorname{int}\Big(\bigcup_{\alpha\in A_\gamma} Q_{\ell,\alpha}\Big),$$

and viewing $Q_{5\ell}$ as made of 5^{d+1} cubes of sidelength ℓ according to (5.3.7), we observe that $\mathbf{A}_\square$ is the set of all indices α such that the cube $Q_{\ell,\alpha}$ lies on the boundary of $Q_{5\ell}$, while $G_{\ell,\square}$ is made of the interior of the union of all such cubes. The set of indices $\mathbf{A}_\gamma$ is a refinement of $\mathbf{A}_\square$, corresponding to cubes that lie on the boundary of $Q_{5\ell}$ on some specific faces, indicated by the multi-index γ — more precisely, $\mathbf{A}_\gamma$ selects the cubes on the boundary of $Q_{5\ell}$ that lie at the opposite of at least one face indicated by γ (see Figure 5.4). We observe that

$$G_{\ell,\square} = \bigcup_{\gamma \in \mathbf{\Gamma}} G_{\ell,\gamma}.$$

Indeed, given $x \in G_{\ell,\square} \cap Q_{\ell,\alpha}$, we have $x \in G_{\ell,\gamma}$ with any $\gamma \in \mathbf{\Gamma}$ such that for every $i \in \{1, \dots, d+1\}$, $\alpha_i \gamma_i \neq 2$.

We now introduce some notation for the centers of the cubes of our decomposition. More precisely, we define

$$\Sigma_\ell = Q_{\ell,0} \cap (\mathbb{Z} + 1/2)^{d+1}.$$

Otherwise stated, if we consider the standard decomposition of $\mathbb{R}^{d+1}$ into unit cubes, such that the origin is a vertex of a cube, then Σ_ℓ is the set of all centers of those cubes that lie inside to $Q_{\ell,0}$. We observe that, for every $\gamma \in \mathbf{\Gamma}$ and every $\alpha \in \mathbf{A}_\gamma$, it holds that

$$\operatorname{dist}_\infty(y, Q_{\ell,\alpha}) \geq \ell \quad \text{for every } y \in \mathscr{C}_\gamma + \Sigma_\ell. \tag{5.3.8}$$

Indeed, by definition, there exists an index $i \in \{1, \dots, d+1\}$ such that $\alpha_i \gamma_i = -2$. Assume without loss of generality that $\gamma_i = 1$. Then, for every $y \in \mathscr{C}_\gamma + \Sigma_\ell$ and $z \in Q_{\ell,\alpha}$, we have $y_i \geq 2\ell + 1/2$ while $z_i \leq \ell$, so that

$$|y - z|_\infty \geq y_i - z_i \geq \ell,$$

which proves our claim.

Finally, we define a retraction from $\mathbb{R}^{d+1}$ into $Q_{\ell,\alpha}$. More specifically, given any $\alpha \in \mathbf{A}$ and $x \in \mathbb{R}^n$, we let

$$\Theta_{\ell,\alpha}(x) = \begin{cases} x & \text{if } x \in Q_{\ell,\alpha}, \\ c_{\ell,\alpha} + \frac{\ell(x - c_{\ell,\alpha})}{2|x - c_{\ell,\alpha}|_\infty} & \text{otherwise,} \end{cases}$$

where $c_{\ell,\alpha}$ is the center of the cube $Q_{\ell,\alpha}$.

We can now state and prove the main result of this section, which is a lower bound on the energy gap of approximating sequences that will be the key ingredient in order to prove the required estimate to strengthen inequality (5.1.2), according to the strategy described in the introduction.

Proposition 5.3.4. *Given* $u \in W^{1,d}(\mathbb{S}^d; \mathbb{R}^{d+1} \setminus (B^{d+1}_{1/2} + \Sigma_\ell)) \cap C(\mathbb{S}^d; \mathbb{R}^{d+1} \setminus (B^{d+1}_{1/2} + \Sigma_\ell))$ *such* *that*

$$u(\mathbb{S}^d) \subseteq G_{\ell,\square},$$

$M \in (0, +\infty)$*, and* $(E_\alpha)_{\alpha \in \mathbf{A}_\square}$ *a family of subsets of* $\mathbb{S}^d$ *such that for every* $\gamma \in \mathbf{\Gamma}$*,*

$$u^{-1}(G_{\ell,\gamma}) \subseteq \bigcup_{\alpha \in \mathbf{A}_\gamma} E_\alpha, \tag{5.3.9}$$

there exists $\eta \in (0, +\infty)$ *such that, for every* $v \in W^{1,n-1}(\mathbb{S}^d; \mathbb{R}^{d+1} \setminus (B^{d+1}_{1/2} + \Sigma_\ell)) \cap C(\mathbb{S}^d; \mathbb{R}^{d+1} \setminus (B^{d+1}_{1/2} + \Sigma_\ell))$ *satisfying*

$$\int_{\mathbb{S}^d} |\mathrm{D}v|^d \leq M$$

and

$$\int_{\mathbb{S}^d} |u - v| \leq \eta,$$

we have

$$\Bigl(\sum_{\sigma \in \Sigma_\ell} |\deg_\sigma(u) - \deg_\sigma(v)| \Bigr)^{1 - \frac{1}{d+1}} \leq C \sum_{\alpha \in \mathbf{A}_\square} \int_{E_\alpha} |\mathrm{D}v|^d - |\mathrm{D}(\Theta_{\ell,\alpha} \circ v)|^d,$$

where the constant $C > 0$ *depends only on* n*. Moreover, the above statement remains valid if the domain* $\mathbb{S}^d$ *is replaced by the boundary* ∂Q *of any cube* $Q \subset \mathbb{R}^{d+1}$*, with a constant* $C > 0$ *independent of the choice of the cube.*

Proof. Given an open set $\Omega \subseteq \mathbb{S}^d$ and $v \in W^{1,d}(\mathbb{S}^d; \mathbb{R}^{d+1})$, we have for every $x \in \Omega$

$$|\mathrm{D}(\Theta_{\ell,\alpha} \circ v)(x)| \leq \frac{|\mathrm{D}v(x)|}{1 + 2\,\mathrm{dist}_\infty(v(x), Q_{\ell,\alpha})/\ell},$$

and therefore

$$|\mathrm{D}v(x)|^d - |\mathrm{D}(\Theta_{\ell,\alpha} \circ v)(x)|^d \geq \Bigl(1 - \frac{1}{(1 + 2\,\mathrm{dist}_\infty(v(x), Q_{\ell,\alpha})/\ell)^d} \Bigr) |\mathrm{D}v(x)|^d. \tag{5.3.10}$$

Hence, for every $\alpha \in \mathbf{A}_\gamma$, we deduce from (5.3.8) and (5.3.10) that

$$\int_{E_\alpha \cap v^{-1}(\mathscr{C}_\gamma + \Sigma_\ell)} |\mathrm{D}v|^d \leq \frac{1}{1 - 3^{-d}} \int_{E_\alpha \cap v^{-1}(\mathscr{C}_\gamma + \Sigma_\ell)} |\mathrm{D}v|^d - |\mathrm{D}(\Theta_{\ell,\alpha} \circ v)|^d. \tag{5.3.11}$$

On the other hand, Proposition 5.3.3 ensures that we can choose $\eta \in (0, +\infty)$ such that

$$\begin{aligned}\Big(\sum_{\sigma\in\Sigma_\ell}|\deg_\sigma(u)-\deg_\sigma(v)|\Big)^{1-\frac{1}{d+1}} &\lesssim \sum_{\gamma\in\mathbf{\Gamma}}\int_{u^{-1}(G_{\ell,\gamma})\cap v^{-1}(\mathscr{C}_\gamma+\Sigma_\ell)}|\mathrm{D}v|^d\\ &\lesssim \sum_{\gamma\in\mathbf{\Gamma}}\sum_{\alpha\in\mathbf{A}_\gamma}\int_{E_\alpha\cap v^{-1}(\mathscr{C}_\gamma+\Sigma_\ell)}|\mathrm{D}v|^d,\end{aligned} \tag{5.3.12}$$

in view of (5.3.9). We deduce from (5.3.11) and (5.3.12) that

$$\begin{aligned}\Big(\sum_{\sigma\in\Sigma_\ell}|\deg_\sigma(u)-\deg_\sigma(v)|\Big)^{1-\frac{1}{d+1}} &\leq C_1\sum_{\gamma\in\mathbf{\Gamma}}\sum_{\alpha\in\mathbf{A}_\square}\int_{E_\alpha}|\mathrm{D}v|^d-|\mathrm{D}(\Theta_{\ell,\alpha}\circ v)|^d\\ &\leq C_1 2^{d+1}\sum_{\alpha\in\mathbf{A}_\square}\int_{E_\alpha}|\mathrm{D}v|^d-|\mathrm{D}(\Theta_{\ell,\alpha}\circ v)|^d,\end{aligned}$$

since $\operatorname{card}(\mathbf{\Gamma}) = 2^{d+1}$. Moreover, since $\mathscr{H}^d(\mathscr{C}_\gamma\cap\mathbb{S}^d)$ only depends on d, the constant C_1 only depends on d.

This proves the proposition when the domain is $\mathbb{S}^d$. The fact that the statement remains valid if the domain is instead the boundary of a cube follows from the fact that there is a bi-Lipschitz transformation between $\mathbb{S}^d$ and the boundary of a unit cube in $\mathbb{R}^{d+1}$, along with the scaling invariance of the $W^{1,d}$ energy in dimension d. □

5.3.3 Lower bound on the relaxed energy

In this section, we give the final and key estimate that allows to prove that the maps we construct indeed lead to an analytical obstruction to weak density. Given $d \geq 1$, we define the set

$$\widetilde{\mathscr{N}_0} = \{(x_1,\dots,x_{d+1})\in\mathbb{R}^{d+1}\colon \text{there exists } i\in\{1,\dots,d+1\} \text{ such that } x_i\in\mathbb{Z}\}.$$

We observe that $\widetilde{\mathscr{N}_0}$ is actually the d-skeleton of the standard decomposition of $\mathbb{R}^{d+1}$ into unit cubes, such that the origin is a vertex of one cube.

Proposition 5.3.5. *Let $u \in W^{1,d}(G_{\ell,\square};\widetilde{\mathscr{N}_0})\cap C(G_{\ell,\square}\setminus(\mathbb{Z}+1/2)^{d+1};\widetilde{\mathscr{N}_0})$ be such that, for every $\alpha\in\mathbf{A}_\square$,*

$$u(Q_{\ell,\alpha})\subseteq Q_{\ell,\alpha}. \tag{5.3.13}$$

Assume also that $v_n\in C(Q_{5\ell};\widetilde{\mathscr{N}_0})$ and that $(v_{n|G_{\ell,\square}})_{n\in\mathbb{N}}$ converges weakly to u in $W^{1,d}(G_{\ell,\square};\widetilde{\mathscr{N}_0})$.

Then,

$$\liminf_{n\to+\infty} \sum_{\alpha\in\mathbf{A}_\square} \int_{Q_{\ell,\alpha}} |Dv_n|^d - |D(\Theta_{\ell,\alpha}\circ v_n)|^d \geq C\ell^{d+1}, \tag{5.3.14}$$

for some constant $C > 0$ *depending only on* d*.*

Proof. We consider Q^t to be the cube of edge length $t \in (0, 5\ell)$ and same center as $Q_{5\ell}$. For almost every $t \in (3\ell, 5\ell)$, the map $u_{|\partial Q^t}$ is continuous, and by (5.3.13), for every $\sigma \in \Sigma_\ell$, $\deg_\sigma(u_{|\partial Q^t}) = 1$. On the other hand, we know that $\deg_\sigma(v_{n\,|\partial Q^t}) = 0$. Hence,

$$\sum_{\sigma\in\Sigma_\ell} |\deg_\sigma(u_{|\partial Q^t}) - \deg_\sigma(v_{n\,|\partial Q^t})| = \ell^{d+1}.$$

By Fubini–Tonelli's theorem and Fatou's lemma, we can assume up to a subsequence that

$$\begin{aligned}\liminf_{k\to+\infty} \sum_{\alpha\in\mathbf{A}_\square} \int_{\partial Q^t\cap Q_{\ell,\alpha}} & |Dv_n|^d - |D(\Theta_{\ell,\alpha}\circ v_k)|^d \\ & \leq \frac{1}{\ell}\liminf_{k\to+\infty} \sum_{\alpha\in\mathbf{A}_\square} \int_{Q_{\ell,\alpha}} |Dv_n|^d - |D(\Theta_{\ell,\alpha}\circ v_n)|^d, \end{aligned} \tag{5.3.15}$$

that

$$\sup_{n\in\mathbb{N}} \int_{\partial Q^t} |Dv_n|^d < +\infty,$$

and that

$$\lim_{n\to+\infty} \int_{\partial Q^t} |v_n - v| = 0.$$

Taking $E_\alpha = \partial Q^t \cap Q_{\ell,\alpha}$, we observe that (5.3.9) is indeed satisfied. Therefore, if n is sufficiently large depending on η, we deduce that Proposition 5.3.4 applies and yields

$$\ell^d \lesssim \liminf_{n\to+\infty} \sum_{\alpha\in\mathbf{A}_\square} \int_{\partial Q^t\cap Q_{\ell,\alpha}} |Dv_n|^d - |D(\Theta_{\ell,\alpha}\circ v_n)|^d.$$

The conclusion (5.3.14) follows then from (5.3.15). □

We recall the definition of the relaxed energy of a map $u \in W^{1,p}(\mathscr{M}, \widetilde{\mathscr{M}_0})$ as

$$\begin{aligned}\mathcal{E}^{1,p}_{\mathrm{rel}}&(u, \mathscr{M})\\ &= \inf\Bigl\{\liminf_{n\to+\infty} \mathcal{E}^{1,p}(v_n, \mathscr{M}) \colon v_n \to u \text{ a.e. and } v_n \in W^{1,p}(\mathscr{M}; \widetilde{\mathscr{M}_0}) \cap C^0(\mathscr{M}; \widetilde{\mathscr{M}_0})\Bigr\}.\end{aligned}$$

Using the tools constructed before, we may finally give a lower bound on the relaxed energy of maps with values into $\widetilde{\mathscr{N}_0}$. More precisely, we work with the map u defined as the retraction from $\mathbb{R}^{d+1}$ to $\widetilde{\mathscr{N}_0}$.

Proposition 5.3.6. *For every $d \geq 1$,*

$$\liminf_{\ell\to+\infty} \frac{\mathcal{E}^{1,d}_{\mathrm{rel}}(u, Q_\ell)}{\ell^{d+1}\ln \ell} > 0.$$

Proof. If $v \in W^{1,d}(Q_{5\ell}; \widetilde{\mathscr{N}_0})$ and $v_n \in W^{1,p}(Q_{5\ell}; \widetilde{\mathscr{N}_0}) \cap C^0(Q_{5\ell}; \widetilde{\mathscr{N}_0})$ are such that $v_n \to v$ almost everywhere, then

$$\begin{aligned}\int_{Q_{5\ell}} |\mathrm{D}v_n|^d &= \sum_{\alpha\in\mathbf{A}} \int_{Q_{\ell,\alpha}} |\mathrm{D}v_n|^d\\ &= \sum_{\alpha\in\mathbf{A}\setminus\mathbf{A}_\square} \int_{Q_{\ell,\alpha}} |\mathrm{D}v_n|^d\\ &\quad + \sum_{\alpha\in\mathbf{A}_\square} \int_{Q_{\ell,\alpha}} |\mathrm{D}(\Theta_{\ell,\alpha}\circ v_n)|^d\\ &\quad + \sum_{\alpha\in\mathbf{A}_\square} \int_{Q_{\ell,\alpha}} |\mathrm{D}v_n|^d - |\mathrm{D}(\Theta_{\ell,\alpha}\circ v_n)|^d.\end{aligned}$$

If $v(Q_{\ell,\alpha}) \subseteq Q_{\ell,\alpha}$, we have $\Theta_{\ell,\alpha}\circ v_{n|Q_{\ell,\alpha}} \to v_{|Q_{\ell,\alpha}}$ almost everywhere, and thus, invoking Proposition 5.3.5, we deduce that

$$\mathcal{E}^{1,d}_{\mathrm{rel}}(v, Q_{5\ell}) \geq \sum_{\alpha\in\mathbf{A}} \mathcal{E}^{1,d}_{\mathrm{rel}}(v, Q_{\ell,\alpha}) + C\ell^{d+1}.$$

In particular, if we take $v = u$, we have

$$\mathcal{E}^{1,d}_{\mathrm{rel}}(u, Q_{5\ell}) \geq 5^{d+1}\mathcal{E}^{1,d}_{\mathrm{rel}}(u, Q_\ell) + C\ell^{d+1},$$

or equivalently,

$$\frac{\mathcal{E}^{1,d}_{\mathrm{rel}}(u, Q_{5\ell})}{(5\ell)^{d+1}} \geq \frac{\mathcal{E}^{1,d}_{\mathrm{rel}}(u, Q_\ell)}{\ell^{d+1}} + \frac{C}{5^{d+1}}.$$

It follows by induction that

$$\frac{\mathcal{E}^{1,d+1}_{\mathrm{rel}}(u, Q_{5^m})}{5^{m(d+1)}} \geq \frac{Cm}{5^{d+1}},$$

and the conclusion follows. □

As explained in the introduction, if $\widetilde{\mathcal{N}_0}$ was a compact Riemannian manifold, we would be done with the proof of our main result. In the next section, we explain how to remedy the non compactness issue, by taking a suitable quotient space of $\widetilde{\mathcal{N}_0}$.

5.3.4 Analytical obstruction with values into a skeleton

In this section, we explain how to construct a sequence of Sobolev mappings whose relaxed energy grows superlinearly with respect to its Sobolev energy when the target is not a manifold but merely a skeleton of the $(d+1)$-dimensional torus $\mathbb{T}^{d+1}$. Later on, in Section 5.3.5, we will explain how to move from a skeleton to a legitimate compact manifold without boundary, while preserving all the required properties of our construction.

We define $\mathcal{N}_0$ to be the d-skeleton of the torus $\mathbb{T}^{d+1}$, or equivalently the quotient

$$\mathcal{N}_0 = \widetilde{\mathcal{M}_0}/\mathbb{Z}^{d+1}.$$

In particular, $\mathcal{N}_0$ can be isometrically embedded into a Euclidean space, as a subset of the compact Riemannian manifold $\mathbb{T}^{d+1} \subseteq \mathbb{R}^{2(d+1)}$. It enjoys the following properties.

Proposition 5.3.7. *The quotient map $\pi\colon \widetilde{\mathcal{N}_0} \to \mathcal{N}_0$ is a universal cover, and*

$$\pi_1(\mathcal{N}_0) = \mathbb{Z}^{d+1},$$

$$\pi_d(\mathcal{N}_0) = \textstyle\bigoplus_{\mathbb{Z}^{d+1}} \mathbb{Z},$$

and

$$\pi_2(\mathcal{N}_0) = \cdots = \pi_{d-1}(\mathcal{N}_0) = \{0\}.$$

In addition, $\pi_d(\mathcal{N}_0)$ is finitely generated over the action of $\pi_1(\mathcal{N}_0)$. Finally, the higher order homotopy groups of $\mathcal{N}_0$, $\pi_j(\mathcal{N}_0)$ with $j > d$, may be computed as the corresponding homotopy groups of a bouquet of infinitely many j-spheres, one for each element of $\mathbb{Z}^{d+1}$.

Proof. First of all, it is straightforward from the natural cell complex structure of $\widetilde{\mathcal{N}_0}$ that its integer homology groups of are given by

$$H_j(\widetilde{\mathscr{N}_0}) = \begin{cases} \{0\} & \text{if } j \neq d, \\ \bigoplus_{\mathbb{Z}^{d+1}} \mathbb{Z} & \text{if } j = d. \end{cases}$$

Actually, there is one copy of $\mathbb{Z}$ in $H_d(\widetilde{\mathscr{N}_0})$ for every boundary of a $(d+1)$-cube in $\widetilde{\mathscr{N}_0}$.

Therefore, Hurewicz's theorem or a direct argument implies that $\pi_1(\widetilde{\mathscr{N}_0}) = \cdots = \pi_{d-1}(\widetilde{\mathscr{N}_0}) = \{0\}$, while $\pi_d(\widetilde{\mathscr{N}_0}) = \bigoplus_{\mathbb{Z}^{d+1}} \mathbb{Z}$. Since $\pi\colon \widetilde{\mathscr{N}_0} \to \mathscr{N}_0$ is the covering associated to the action of $\mathbb{Z}^{d+1}$ on $\widetilde{\mathscr{N}_0}$, we deduce that $\pi_1(\mathscr{N}_0) = \mathbb{Z}^{d+1}$, $\pi_d(\mathscr{N}_0) = \bigoplus_{\mathbb{Z}^{d+1}} \mathbb{Z}$, and $\pi_2(\mathscr{N}_0) = \cdots = \pi_{d-1}(\mathscr{N}_0) = \{0\}$. Moreover, we know that $\pi_1(\mathscr{N}_0)$ acts as $\mathbb{Z}^{d+1}$ by translation on $\widetilde{\mathscr{N}_0}$; see e.g. [Hat02, Proposition 1.40]. Let us investigate the effect of this action on $\pi_d(\mathscr{N}_0) = \pi_d(\widetilde{\mathscr{N}_0})$. Assume without loss of generality that the origin 0 has been chosen as the basepoint for the homotopy groups of $\widetilde{\mathscr{N}_0}$. Then, any loop in $\gamma \in \pi_1(\mathscr{N}_0) = \mathbb{Z}^{d+1}$ can be lifted to a path in $\widetilde{\mathscr{N}_0}$ connecting the basepoint 0 to the associated endpoint $z_\gamma \in \mathbb{Z}^{d+1}$. Let a be the element of $\pi_d(\mathscr{N}_0)$ obtained by projecting the element of $\pi_d(\widetilde{\mathscr{N}_0})$ that covers once the cycle $\partial[0,1]^{d+1}$. By construction, the element $\gamma a \in \pi_d(\mathscr{N}_0)$ lifts to the element of $\pi_d(\widetilde{\mathscr{N}_0})$ that covers once the cycle $z_\gamma + \partial[0,1]^{d+1}$; see also [Hat02, §4.1, Exercise 4]. (The action can also be described as a particular case of a general bijection between pointed homotopy classes and free homotopy classes in the covering space [Spa66, Corollary 7.3.7], [Whi78, III (1.15)].) When γ runs over $\pi_1(\mathscr{N}_0)$, it is readily checked that we obtain the generating family of $\pi_d(\widetilde{\mathscr{N}_0})$ associated with the homology basis shown above, one element for each boundary of a unit $(d+1)$-cube in $\widetilde{\mathscr{N}_0}$. Therefore, $\pi_d(\mathscr{N}_0)$ is generated by one element over the action of $\pi_1(\mathscr{N}_0)$.

Finally, the affirmation concerning the higher order homotopy groups of $\mathscr{N}_0$ comes from the fact that $\widetilde{\mathscr{N}_0}$ is homotopically equivalent to a bouquet of infinitely many d-spheres, one for each cube in $\widetilde{\mathscr{N}_0}$. This can be proved by a hand-made construction. However, for the sake of completeness, we sketch a proof, relying on a more general argument. We start with the following exact sequence, which is a part of the long exact sequence relating the homology of a space, a subspace, and the associated quotient space, see e.g. [Hat02, Theorem 2.13]:

$$\begin{tikzcd} \{0\} = H_d(\widetilde{\mathscr{M}}_0^{d-1}) \arrow[r] & H_d(\widetilde{\mathscr{M}_0}) \arrow[r, "j_\sharp"] & H_d(\widetilde{\mathscr{M}_0}/\widetilde{\mathscr{M}}_0^{d-1}) \arrow[dll, "\partial"] \\ H_{d-1}(\widetilde{\mathscr{M}}_0^{d-1}) \arrow[r] & H_{d-1}(\widetilde{\mathscr{M}_0}) = \{0\}, & \end{tikzcd} \tag{5.3.16}$$

where $\widetilde{\mathscr{M}}_0^{d-1}$ is the $(d-1)$-skeleton of $\widetilde{\mathscr{M}_0}$. The quotient space $\widetilde{\mathscr{M}_0}/\widetilde{\mathscr{M}}_0^{d-1}$ is a bouquet of d-spheres, one for each d-cell in $\widetilde{\mathscr{M}_0}$, the map j is the quotient map that sends each cell to the corresponding sphere, and the map ∂ sends every sphere to the boundary of the associated cell, which is a $(d-1)$-cycle. The exact sequence (5.3.16) implies that the map ∂ is an epimorphism. Since $H_{d-1}(\widetilde{\mathscr{M}}_0^{d-1})$ is free, this map admits a section σ, and moreover, it can constructed so that its image $\operatorname{Im}\sigma$ is the submodule generated by a subset of the spheres constituting $\widetilde{\mathscr{M}_0}/\widetilde{\mathscr{M}}_0^{d-1}$. This section induces the direct sum decomposition

$$H_d(\widetilde{\mathscr{M}_0}/\widetilde{\mathscr{M}}_0^{d-1}) = H_d(\widetilde{\mathscr{M}_0}) \oplus \operatorname{Im}\sigma.$$

Collapsing the spheres which generate $\operatorname{Im}\sigma$, the map j induces a map h from $\widetilde{\mathscr{M}_0}$ to a (possibly smaller) bouquet of spheres, and our argument shows that this map induces an isomorphism in homology in degree d. It also induces an isomorphism in homology in all the other degrees, since all the other homology groups of both spaces involved are trivial (except for the H_0, which are both $\mathbb{Z}$, and on which this map also induces an isomorphism). A corollary of Whitehead's and Hurewicz's theorems, see e.g. [Hat02, Corollary 4.33], allows to conclude that h is a homotopy equivalence. □

We note importantly that, although there is a correspondence between the cubes in $\widetilde{\mathscr{N}_0}$ and the spheres of the bouquet to which it is homotopic, the homotopy equivalence between the bouquet of spheres and $\widetilde{\mathscr{N}_0}$ *does not* map each generator associated to a sphere to the generator associated to the corresponding cube. Instead, a generator associated to a sphere may be mapped to a generator associated to a cycle containing an arbitrary large number of cubes in $\widetilde{\mathscr{N}_0}$. This can be intuitively seen from the proof: $\widetilde{\mathscr{M}_0}$ is sent to a bouquet of spheres by collapsing its $(d-1)$-skeleton, sending all d-faces to spheres, and then collapsing a certain number of those spheres. Therefore, the spheres of the bouquet are in correspondence with a family of cycles that form a basis of $H_d(\widetilde{\mathscr{M}_0})$ chosen such that the faces that have not been collapsed are part of one and only one cycle. However, it is readily seen that it is impossible to select some faces of $\widetilde{\mathscr{M}_0}$ in such a way that the corresponding family of cycles can be chosen to be all boundaries of unit cubes.

This can be put into the framework of the third strategy for proving weak approximation explained in the introduction, based on the construction and elimination of connections [BCL86, ABL88, Bet90, ABO03, CO19, PR03]. Indeed, by composing an $\mathscr{N}_0$-valued $W^{1,d}$ mapping u with the homotopy equivalence $h\colon \widetilde{\mathscr{M}_0} \to \bigvee_{\mathbb{Z}^{d+1}} \mathbb{S}^d$ constructed above — which can be taken to be Lipschitz, as it merely amounts to collapse some cells of $\widetilde{\mathscr{M}_0}$ and map the other ones to spheres — one can proceed with the first part of the strategy, namely construct and analyze the topological singular set of the map u

and obtain a minimal connection for this set, whose length is controlled by the Sobolev energy of u, identifying separately the contributions of the different spheres of the bouquet. However, one cannot proceed with the second step of the strategy, as eliminating the singularities corresponding to the different spheres of the bouquet cannot be done with uniformly controlled maps for creating the dipoles, due to the fact that some of these spheres are associated with arbitrarily large cycles in $\widetilde{\mathscr{M}_0}$. This explains why this strategy cannot be applied to obtain weak approximation in our context.

We now provide an instance of a sequence of Sobolev mappings with values into the *compact* skeleton $\mathscr{N}_0$ whose relaxed energy grows up superlinearly with respect to the Sobolev energy. We define $v = \pi \circ u$, where the u is the map from Proposition 5.3.6. In particular, we note that the map v is $\mathbb{Z}^n$-periodic.

Proposition 5.3.8. *For every $d \geq 2$,*

$$\liminf_{\ell \to +\infty} \frac{\mathcal{E}^{1,d}_{\mathrm{rel}}(v, Q_\ell)}{\ell^{d+1} \ln \ell} > 0.$$

The map v being $\mathbb{Z}^{d+1}$-periodic, it satisfies in particular $\mathcal{E}^{1,d}(v, Q_\ell) \lesssim \ell^{d+1}$, so that the above theorem indeed provides a superlinear growth of the relaxed energy with respect to the Sobolev energy.

Proof of Proposition 5.3.8. Assume that we are given a sequence $(w_n)_{n \in \mathbb{N}}$ of mappings in $W^{1,d}(Q_\ell; \mathscr{N}_0) \cap C(Q_\ell; \mathscr{N}_0)$ that converges weakly to v in $W^{1,d}$ on Q_ℓ. We perform a lifting in the spirit of [BBM00, BC07, BZ88]. Since Q_ℓ is simply connected, the maps w_n may be lifted as maps $\widetilde{w}_n \colon Q_\ell \to \widetilde{\mathscr{N}_0}$. In addition, since π is a local isometry, it holds that

$$\int_{Q_\ell} |\mathrm{D} w_n|^d = \int_{Q_\ell} |\mathrm{D} \widetilde{w}_n|^d.$$

Moreover, taking profit of the fact that $\mathbb{Z}^{d+1}$ acts on $\widetilde{\mathscr{N}_0}$ and that those deck transformations are translations, we may choose the liftings $\widetilde{w}_n$ so that

$$\left(\fint_{Q_\ell} \widetilde{w}_n \right)_{n \in \mathbb{N}} \quad \text{is bounded.}$$

Therefore, the Poincaré–Wirtinger inequality ensures that the sequence $(\widetilde{w}_n)_{n \in \mathbb{N}}$ is bounded in L^d, and hence, up to extraction of a subsequence, it converges weakly to some map $w \in W^{1,d}(Q_\ell, \widetilde{\mathscr{N}_0})$. Here we use the fact that $d > 1$. By the continuity of the covering map π, we have almost everywhere

$$\pi \circ w = \lim_{n \to +\infty} \pi \circ \widetilde{w}_n = \lim_{n \to +\infty} w_n = v = \pi \circ u. \tag{5.3.17}$$

Since the action of $\mathbb{Z}^{d+1}$ by translation is transitive, by the uniqueness of liftings in Sobolev spaces (see e.g. [BBM00, BC07, BZ88], and also [MVS21a, Proposition 4.2]), it follows that $w = u + a$ for some $a \in \mathbb{Z}^{d+1}$. Hence, we deduce that

$$\liminf_{n\to+\infty} \int_{Q_\ell} |\mathrm{D}w_n|^d = \liminf_{n\to+\infty} \int_{Q_\ell} |\mathrm{D}\widetilde{w}_n|^d \geq \mathcal{E}^{1,d}_{\mathrm{rel}}(u, Q_\ell).$$

Taking the infimum over all sequences weakly converging to v, we obtain

$$\mathcal{E}^{1,d}_{\mathrm{rel}}(v, Q_\ell) \geq \mathcal{E}^{1,d}_{\mathrm{rel}}(u, Q_\ell).$$

The conclusion follows from Proposition 5.3.6. □

We point out that all our analysis before Proposition 5.3.8 did not rely on the fact that $d \geq 2$. Since weak density always holds in $W^{1,1}$, a uniform linear bound on the relaxed energy must hold and thus Proposition 5.3.8 cannot be extended to that case. Let us examine why our construction does not contradict the weak density in $W^{1,1}$. Proposition 5.3.6 yields

$$\liminf_{\ell\to+\infty} \frac{\mathcal{E}^{1,1}_{\mathrm{rel}}(u, Q_\ell)}{\ell^2 \ln \ell} > 0.$$

When $d + 1 = 2$, $\widetilde{\mathscr{N}_0}$ is the 1-skeleton of $\mathbb{R}^2$, and $\mathscr{N}_0$ is a bouquet of two circles. In particular, $\pi\colon \widetilde{\mathscr{N}_0} \to \mathscr{N}_0$ is no longer the universal cover, but it is nevertheless a covering. Actually, the effect of passing from $\mathscr{N}_0$ to $\widetilde{\mathscr{N}_0}$ is to abelianize the π_1: the covering map $\pi\colon \widetilde{\mathscr{M}_0} \to \mathscr{N}_0$ is normal, maps $\pi_1(\widetilde{\mathscr{M}_0})$ to the commutator group of $\pi_1(\mathscr{N}_0)$, and has the abelianization of $\pi_1(\mathscr{N}_0)$ as a group of deck transformations. This is however no obstruction to construct the map v, and to proceed to the above reasoning up to (5.3.17) included. However, the main difference is that here, $d = 1$, and hence, boundedness in $W^{1,1}$ only implies weak convergence to a BV map, not to a $W^{1,1}$ map. Since there is no uniqueness of lifting in BV, one cannot conclude that $w = u + a$ and hence transfer the energy gap estimate for u to v. Actually, it is not difficult to construct by hand a BV lifting of v such that there indeed exists a sequence of continuous maps, bounded in $W^{1,1}$, that converges to this lifting almost everywhere. Of course, it is therefore not $W^{1,1}$ itself.

5.3.5 From the skeleton to a manifold

We now transfer our construction to a manifold.

Given $l \in \mathbb{N}$ and $\lambda \in [0, +\infty)$, we define the set

$$\mathscr{N}_\lambda = V^{-1}(\{\lambda\}),$$

where the function $V\colon \mathbb{T}^{d+1} \times \mathbb{R}^l \to \mathbb{R}$ is given for $x \in \mathbb{T}^{d+1} \times \mathbb{R}^l \subseteq \mathbb{R}^{2(d+1)} \times \mathbb{R}^l$ by

$$V(x) = \prod_{j=1}^{d+1} \frac{1 + x_{2j-1}}{2} + \sum_{j=1}^{l} |x_{2(d+1)+j}|^2.$$

We observe that

$$\mathscr{N}_0 = V^{-1}(\{0\}) = \{x \in \mathbb{T}^{d+1}\colon \min_{1 \le j \le d+1} x_{2j-1} = -1\} \times \{0\}$$

is the set that was used in the previous section, along with its universal covering $\widetilde{\mathscr{M}_0}$.

Proposition 5.3.9. *For every $\lambda \in (0,1)$, $\mathscr{N}_\lambda$ is a smooth manifold and there exists a Lipschitz map from $\mathscr{N}_\lambda$ to $\mathscr{N}_0$. If moreover $l > 0$,*

$$\mathscr{N}_\lambda \supseteq \mathscr{N}_0 \times \mathbb{S}^{l-1}_{\sqrt{\lambda}},$$

and the restriction of the Lipschitz map $\mathscr{N}_\lambda \to \mathscr{N}_0$ above to $\mathscr{N}_0 \times \mathbb{S}^{l-1}_{\sqrt{\lambda}}$ coincides with $\mathrm{id}_{\mathscr{N}_0} \times 0$. If $i < l$, then

$$\pi_i(\mathscr{N}_\lambda) = \pi_i(\mathscr{N}_0) = \begin{cases} \mathbb{Z}^{d+1} & \text{if } i = 1, \\ \{0\} & \text{if } 2 \le i \le d-1, \\ \pi_i(\bigwedge_{\mathbb{Z}^{d+1}} \mathbb{S}^d) & \text{if } i \ge d. \end{cases}$$

Moreover, if $d < l$, then $\pi_d(\mathscr{N}_\lambda)$ is finitely generated over the action of $\pi_1(\mathscr{N}_\lambda)$.

Proof. In angular coordinates $x = (\cos\theta_1, \sin\theta_1, \dots, \cos\theta_{d+1}, \sin\theta_{d+1}, z_1, \dots, z_l)$, we have

$$V(x) = \prod_{j=1}^{d+1} \frac{1 + \cos\theta_j}{2} + |z|^2 = \prod_{j=1}^{d+1} \bigl(\cos\tfrac{\theta_j}{2}\bigr)^2 + |z|^2.$$

In particular, if $V(x) < 1$, we have $\theta \neq 0$. Taking the gradient, we have, if $0 < V(x) < 1$,

$$|\mathrm{D}V(x)|^2 = \left(\sum_{j=1}^{d+1} \tan\bigl(\tfrac{\theta_j}{2}\bigr)^2\right) \prod_{j=1}^{d+1} \bigl(\cos\tfrac{\theta_j}{2}\bigr)^4 + 4|z|^2 > 0.$$

Hence, for every $\lambda \in (0,1)$, the set $\mathscr{N}_\lambda = V^{-1}(\{\lambda\})$ is a smooth submanifold of $\mathbb{T}^{d+1}$.

Next, we define for $(\theta, z) \in V^{-1}([0,1)) \subseteq ([-\pi,\pi]^{d+1} \setminus \{0\}) \times \mathbb{R}^l$,

$$\Theta(t,\theta,z) = \Big((1 + t(\tfrac{\pi}{|\theta|_\infty} - 1))\theta, (1-t)z\Big) = (1-t)(\theta,z) + t\Theta(1,\theta,z).$$

We observe that if $t_0, t_1 \in [0,1]$ satisfy $t_0 < t_1$ and if $x \notin \mathscr{N}_0$, then

$$V(\Theta(t_0,x)) > V(\Theta(t_1,x)).$$

It follows from the facts that (i) Θ is continuous, (ii) $\Theta([0,1] \times V^{-1}([0,\lambda])) \subseteq V^{-1}([0,\lambda])$, (iii) $\Theta(\{1\} \times V^{-1}([0,\lambda])) \subseteq V^{-1}(\{0\})$, and (iv) for every $x \in V^{-1}(\{0\})$, $\Theta(t,x) = (t,x)$, that the set $\mathscr{N}_0 = V^{-1}(\{0\})$ is a (strong) deformation rectract of $V^{-1}([0,\lambda])$. In particular, for every $j \in \mathbb{N}$, we have

$$\pi_j(V^{-1}([0,\lambda])) = \pi_j(\mathscr{N}_0).$$

Moreover, it is readily checked that the map $\Theta(1,\cdot)_{|\mathscr{N}_\lambda}$ is the required Lipschitz map $\mathscr{N}_\lambda \to \mathscr{N}_0$.

The map $\Theta_{|[0,1)\times\mathscr{N}_\lambda} \colon [0,1) \times \mathscr{N}_\lambda \to V^{-1}((0,\lambda])$ is a bijection. An inspection of its definition shows its injectivity. For the surjectivity, defining for $(\theta,z) \in V^{-1}([0,1)) \subseteq ([-\pi,\pi]^{d+1} \setminus \{0\}) \times \mathbb{R}^l$,

$$\Lambda(t,\theta,z) = \left(\frac{1 - t\pi/|\theta|_\infty}{1-t}\theta, \frac{z}{1-t}\right),$$

we observe that if $t < |\theta|_\infty/\pi$, $\Lambda(t,\theta,z) \in V^{-1}([0,1)) \subseteq ([-\pi,\pi]^{d+1} \setminus \{0\}) \times \mathbb{R}^l$ and $\Theta(t, \Lambda(t,\theta,z)) = (\theta,z)$. Since $t \in (0, |\theta|_\infty/\pi) \mapsto V(\Lambda(t,\theta,z))$ is increasing and continuous and since $\lim_{t \to |\theta|_\infty/\pi} V(\Lambda(t,\theta,z)) > 1$, Θ has the required surjectivity property.

For every $\varepsilon \in (0,\lambda)$, the set $\Theta^{-1}(V^{-1}([\varepsilon,\lambda])) \subseteq [0,1) \times \mathscr{N}_\lambda \subseteq [0,1] \times \mathscr{N}_\lambda$ is closed and thus $(\Theta^{-1}_{|[0,1)\times\mathscr{N}_\lambda})_{|V^{-1}([\varepsilon,\lambda])}$ is continuous. Since V is continuous, $\Theta^{-1}_{|[0,1)\times\mathscr{N}_\lambda}$ is continuous on $V^{-1}((0,\lambda])$. Writing $(T,R) = \Theta^{-1}_{|[0,1)\times\mathscr{N}_\lambda}$ with $T \in C(V^{-1}((0,\lambda]); [0,1))$ and $R \in C(V^{-1}((0,\lambda]); \mathscr{N}_\lambda)$, we note that the map

$$\Xi \colon (t,x) \in [0,1] \times V^{-1}((0,\lambda]) \mapsto \Theta((1-t)T(x), R(x))$$

is continuous, that $\Xi(0,\cdot) = \mathrm{id}$, and that $\Xi(t,\cdot)_{|\mathscr{N}_\lambda} = \mathrm{id}_{\mathscr{N}_\lambda}$. Therefore, $V^{-1}(\{\lambda\})$ is a (strong) deformation retract of $V^{-1}((0,\lambda])$, and thus for every $j \in \mathbb{N}$,

$$\pi_j(V^{-1}((0,\lambda])) = \pi_j(\mathscr{N}_\lambda).$$

By a cellular or simplicial approximation (see [Hat02, Theorem 2C.1 or 4.8]), since $\mathcal{N}_0 = V^{-1}(\{0\})$ and $\dim \mathcal{N}_0 = d$, we have if $j \in \{1, \dots, l-1\}$,

$$\pi_j(V^{-1}((0,\lambda])) = \pi_j(V^{-1}([0,\lambda])).$$

It remains to prove that, if $d < l$, then $\pi_d(\mathcal{N}_\lambda)$ is finitely generated over the action of $\pi_1(\mathcal{N}_\lambda)$. Since $\mathcal{N}_0 \simeq V^{-1}([0,\lambda])$ and $V^{-1}((0,\lambda]) \simeq \mathcal{N}_\lambda$, it suffices to prove that this property is preserved when passing from $V^{-1}([0,\lambda])$ to $V^{-1}((0,\lambda])$. But this follows from the same general position argument. Indeed, let g be a generator of $\pi_d(V^{-1}([0,\lambda])) = \pi_d(V^{-1}((0,\lambda]))$ over the action of $\pi_1(V^{-1}([0,\lambda]))$, where the equality has been proved above. Let also a be any element of $\pi_d(V^{-1}((0,\lambda])) = \pi_d(V^{-1}([0,\lambda]))$. By assumption, there exists $\gamma \in \pi_1(V^{-1}([0,\lambda])) = \pi_1(V^{-1}((0,\lambda]))$ such that $\gamma g = a$ in $\pi_d(V^{-1}([0,\lambda]))$. But an additional application of the general position argument shows that this equality then also holds in $\pi_d(V^{-1}((0,\lambda]))$, which establishes our claim and concludes the proof. □

Remark 5.3.10. Let us write

$$\mathcal{N}_\lambda = \mathcal{N}_\lambda^0 \cup \mathcal{N}_\lambda^1,$$

where

$$\mathcal{N}_\lambda^0 = \mathcal{N}_\lambda \setminus (\mathbb{R}^{2(d+1)} \times B^l_{\sqrt{\lambda}/2}(0)) \qquad \text{and} \qquad \mathcal{N}_\lambda^0 = \mathcal{N}_\lambda \cap (\mathbb{R}^{2(d+1)} \times \overline{B^l_{\sqrt{\lambda}/2}}(0)).$$

We observe that

$$\mathcal{N}_\lambda^0 \simeq (\mathbb{T}^{d+1} \setminus \mathbb{B}^{d+1}) \times \mathbb{S}^{l-1}, \qquad \mathcal{N}_\lambda^1 \simeq \mathbb{S}^d \times \mathbb{B}^l, \qquad \text{and} \qquad \mathcal{N}_\lambda^0 \cap \mathcal{N}_\lambda^1 \simeq \mathbb{S}^d \times \mathbb{S}^{l-1}.$$

Without giving detailed arguments, these claims can be justified as follows. In the sequel, we denote an element of $\mathcal{N}_\lambda \subset \mathbb{R}^{2(d+1)} \times \mathbb{R}^l$ as (x,z). For the description of $\mathcal{N}_\lambda^0$, we first note that, for a fixed $x \in \mathbb{R}^{2(d+1)}$, the set of those $z \in \mathbb{R}^l$ such that $(x,z) \in \mathcal{N}_\lambda$ forms a sphere centered at the origin, unless it is empty. In order to enforce the additional constraint that this sphere should have a radius at least $\sqrt{\lambda}/2$, we have to remove those x such that $\prod_{j=1}^{d+1} \frac{1+x_{2j-1}}{2} \geq \sqrt{\lambda}/2$, which corresponds to removing from $\mathbb{T}^{d+1}$ a set homeomorphic to a ball, centered at the point $(1,\dots,1)$, hence yielding $\mathcal{N}_\lambda^0 \simeq (\mathbb{T}^{d+1} \setminus \mathbb{B}^{d+1}) \times \mathbb{S}^{l-1}$.

For the description of $\mathcal{N}_\lambda^1$, we argue in the other direction, by first observing from the definition of V that, for a fixed $z \in \mathbb{R}^l$, the set of those $x \in \mathbb{R}^{2(d+1)}$ such that $(x,z) \in \mathcal{N}_\lambda$ is homeomorphic to the sphere $\mathbb{S}^d$, unless it is empty. The fact that we restrict to $z \in \overline{B^l_{\sqrt{\lambda}/2}}(0)$ yields the homeomorphism $\mathcal{N}_\lambda^1 \simeq \mathbb{S}^d \times \mathbb{B}^l$.

The description of $\mathcal{N}_\lambda^0 \cap \mathcal{N}_\lambda^1$ follows from a careful combination of the considerations above for both $\mathcal{N}_\lambda^0$ and $\mathcal{N}_\lambda^1$.

This shows that

$$\mathcal{N}_\lambda \simeq (\mathbb{T}^{d+1} \setminus \mathbb{B}^{d+1}) \times \mathbb{S}^{l-1} \cup_\partial \mathbb{S}^d \times \mathbb{B}^l,$$

where $\cup_\partial$ denotes the attachment along the boundary (see e.g. [Lee13, Theorem 9.29]).

In the special case where $l = 1$, then $(\mathbb{T}^{d+1} \setminus \mathbb{B}^{d+1}) \times \mathbb{S}^{l-1} = (\mathbb{T}^{d+1} \setminus \mathbb{B}^{d+1}) \times \{0, 1\}$ consists of two disjoint copies of $(\mathbb{T}^{d+1} \setminus \mathbb{B}^{d+1})$, and $\mathbb{S}^d \times \mathbb{B}^l = \mathbb{S}^d \times (-1, 1)$ is a cylinder that collapses after attaching, so that

$$\mathcal{N}_\lambda \simeq \mathbb{T}^{d+1} \# \mathbb{T}^{d+1}. \qquad \square$$

We extend now to Proposition 5.3.8 to a map in a manifold.

Proposition 5.3.11. *For every $d \geq 2$ and $\lambda \in (0, 1)$,*

$$\liminf_{\ell \to +\infty} \frac{\mathcal{E}^{1,d}_{\mathrm{rel}}(v, Q_\ell)}{\ell^{d+1} \ln \ell} > 0.$$

The difference between Propositions 5.3.8 and 5.3.11 is that the relaxed energy is taken with respect to sequences respectively in $\mathcal{N}_0$ and $\mathcal{N}_\lambda$.

Proof. We view the map v from Proposition 5.3.8 as a Sobolev mapping into $\mathcal{N}_\lambda$, taking advantage of the inclusion $\mathcal{N}_0 \subset \mathcal{N}_\lambda$ exhibited in Proposition 5.3.9. The conclusion hence follows from Proposition 5.3.8 and the fact that any weakly converging sequence of continuous Sobolev maps into $\mathcal{N}_\lambda$ can be projected to a weakly converging sequence of continuous Sobolev maps into $\mathcal{N}_0$ through the Lipschitz map $\Phi\colon \mathcal{N}_\lambda \to \mathcal{N}_0$ provided by Proposition 5.3.9. We note that this relies on the fact that Φ is a left-inverse of the embedding of $\mathcal{N}_0$ into $\mathcal{N}_\lambda$. □

We are now in position to prove our first main result as stated in the introduction.

Proof of Theorem 5.1.1. It suffices to apply the nonlinear uniform boundedness principle for the weak approximation [HL03b, Th. 9.6] (see also [MVS19]). □

We observe that this proof yields a slightly stronger conclusion than required: it shows that the obstruction to weak approximation already arises if one considers continuous Sobolev maps rather than smooth maps.

5.4 Whitehead products and spheres

5.4.1 Construction of mappings

We now move towards our second main result, Theorem 5.1.2. We first use the Whitehead product construction [Whi41] to construct a periodic map $\mathbb{R}^{4d} \to \mathbb{S}^{2d}$ with singularities located at every point in $\mathbb{Z}^{4d}$ and Hopf degree 2 around each singularity.

Proposition 5.4.1. *For every $d \geq 1$, there exists a mapping $u\colon \mathbb{R}^{4d} \to \mathbb{S}^{2d}$ such that*

(*i*) *for every $h \in \mathbb{Z}^{4d}$,*

$$u(x+h) = u(x),$$

(*ii*) $\deg_H u_{|\partial[-1/2,1/2]^{4d}} = 2$,

(*iii*) *$u \in W^{1,4d-1}_{\mathrm{loc}}(\mathbb{R}^{4d};\mathbb{S}^{2d}) \cap C(\mathbb{R}^{4d} \setminus \mathbb{Z}^{4d};\mathbb{S}^{2d})$ and $|Du(x)| \operatorname{dist}(x,\mathbb{Z}^{4d}) \leq C$ for a.e. $x \in \mathbb{R}^{4d} \setminus \mathbb{Z}^{4d}$.*

Proof. We consider a mapping $f \in C^\infty(\mathbb{R}^{2d};\mathbb{S}^{2d})$ such that $f = b$ in $\mathbb{R}^{2d} \setminus [-1/2,1/2]^{2d}$ and $\deg f = 1$. We note that, since f is constant outside a compact set, it can be associated thanks to a stereographic projection to a map from $\mathbb{S}^{2d}$ to $\mathbb{S}^{2d}$. Therefore, $\deg f$ is well-defined. In fact, such a map f can actually be defined as a truncated inverse stereographic projection.

We define then $v\colon \partial([-1/2,1/2]^{4d}) \to \mathbb{S}^{2d}$ as the Whitehead product of f with itself [Whi41]. More precisely we set, for $x = (x',x'') \in \partial([-1/2,1/2]^{4d}) \subseteq \mathbb{R}^{4d} = \mathbb{R}^{2d}\times\mathbb{R}^{2d}$,

$$v(x',x'') = \begin{cases} f(x') & \text{if } |x'|_\infty < 1/2, \\ f(x'') & \text{if } |x''|_\infty < 1/2, \\ b & \text{otherwise.} \end{cases} \tag{5.4.1}$$

The Hopf invariant of v can be computed by classical properties of the Whitehead product as $\deg_H(v) = 2(\deg(f))^2 = 2$; see e.g. [Whi78, XI (2.5)]. Here we use the fact that we work with a sphere of even dimension. Indeed, the exact same construction could have been performed to obtain a map $v\colon \partial[-1/2,1/2]^{4d+2} \to \mathbb{S}^{2d+1}$, but one would then have had $\deg_H(v) = 0$. More generally, the Hopf degree of *any* continuous $\mathbb{S}^{4d+1} \to \mathbb{S}^{2d+1}$ map is zero; see e.g. [Whi78, XI (2.4)].

We extend then v homogeneously to $[-1/2,1/2]^{4d} \setminus \{0\}$ by setting $v(x) = v(x/(2|x|_\infty))$. It follows from (5.4.1) that, if $x, y \in [-1/2,1/2]^{4d} \setminus \mathbb{Z}^{4d}$ and $x - y \in \mathbb{Z}^{4d}$, then $v(x) = v(y)$, so that v can be extended periodically to $\mathbb{R}^{4d}$. □

5.4.2 Lower estimate on the relaxed energy

Below, the target $\mathscr{N}$ is assumed to be a compact manifold without boundary. In the application we have in mind, $\mathscr{N}$ will be a sphere. We start by describing a basic cylinder-type construction. Given $m \geq 1$, we let Q^m be the unit m-dimensional cube. If $u,\ v \in W^{1,p}(Q^{m-1};\mathscr{N}) \cap C(Q^{m-1};\mathscr{N})$, then there exists $\delta \in (0,+\infty)$ such that, if $\|u-v\|_{L^\infty(\partial Q^{m-1})} \leq \delta$, then the map $w\colon \partial Q^m \to \mathscr{N}$ given by

$$w(x', x_m) = \begin{cases} u(x') & \text{if } x_m = -1, \\ \Pi_{\mathscr{N}}\left(\frac{1-x_m}{2}u(x') + \frac{1+x_m}{2}v(x')\right) & \text{if } x' \in \partial Q^{m-1}, \\ v(x') & \text{if } x_m = 1 \end{cases}$$

is well-defined and belongs to $W^{1,p}(\partial Q^m;\mathscr{N}) \cap C(\partial Q^m;\mathscr{N})$. Moreover, for every such δ, a straightforward computation shows that

$$\int_{\partial Q^m} |\mathrm{D}w|^p \leq \int_{Q^{m-1}} |\mathrm{D}u|^p + \int_{Q^{m-1}} |\mathrm{D}v|^p + C\int_{\partial Q^{m-1}} |\mathrm{D}u|^p + C\int_{\partial Q^{m-1}} |\mathrm{D}v|^p + C\delta^p. \quad (5.4.2)$$

With this construction at hand, we prove the following lower estimate on the relaxed energy. In order to state the following proposition, we recall that any (subset of a) hyperplane can be endowed with two different orientations. For a face σ of $Q^m \subset \mathbb{R}^m$, we consider the natural orientation induced by the outward normal vector with respect to Q^m. Of course, if σ is now viewed as a face of a skeleton made of several cubes, then it may receive both orientations, depending on the cube with respect to which it is considered. We will view the two copies of the same face with reverse orientations as two distinct oriented faces. Moreover, we will denote by $|\sigma|$ the unoriented face σ.

Proposition 5.4.2. *Let L^j_ℓ be the j-dimensional skeleton of the cube $Q_\ell = [0,\ell]^{4d}$. Assume that $(u_n)_{n\in\mathbb{N}}$ is a sequence in $C(Q_\ell;\mathbb{S}^{2d}) \cap W^{1,4d-1}(Q_\ell;\mathbb{S}^{2d})$ converging weakly to some $u \in C(L^{4d-1}_\ell;\mathbb{S}^{2d}) \cap W^{1,4d-1}(L^{4d-1}_\ell;\mathbb{S}^{2d})$ and such that the traces on L^{4d-2}_ℓ also converge weakly in $W^{1,4d-1}$. Then, for every oriented face σ of L^{4d-1}_ℓ, there exists some $d_\sigma \in \mathbb{Z}$ such that $d_{-\sigma} = -d_\sigma$, where $-\sigma$ denotes the face σ with reverse orientation, and there exists some $n_* \in \mathbb{N}$ such that for every cube Q of L^{4d}_ℓ, up to a subsequence,*

$$\sum_{\sigma \text{ face of } Q} d_\sigma = \deg_H(u_{|\partial Q}) - \deg_H(u_{n_*|\partial Q}),$$

and

$$\sum_{\sigma \text{ face of } L^{4d-1}} |d_\sigma|^{1-\frac{1}{4d}} \leq \liminf_{n\to+\infty} C\left(\int_{L_\ell^{4d-1}} |\mathrm{D}u_n|^{4d-1} + \int_{L_\ell^{4d-2}} |\mathrm{D}u_n|^{4d-1}\right).$$

The apparent notational complexity of the statement should not obscure the intuition behind it: the difference of degrees between the maps u_n for large n and the limiting map u can be attributed in a coherent way to the different faces of L_ℓ^{4d-1}, so that the difference of degrees on one cube is the sum of the contributions of all its oriented faces. Moreover, we have an estimate available for these partial degrees on faces by the Sobolev energy of the approximating maps, involving a power $1 - \frac{1}{4d}$, whose importance will become apparent later on, in relation with branched optimal transportation considerations.

Proposition 5.4.2 leaves some freedom to the d_σ on the boundary. Indeed, whereas interior faces all belong to two cubes and are involved, since $d_\sigma = -d_{-\sigma}$, in two degree conservation constraints, the interior faces are merely involved in a single one, so that they are much less constrained.

Proof of Proposition 5.4.2. By compactness of the embedding $W^{1,4d-1} \hookrightarrow L^\infty$ in dimension $4d-2$, we assume without loss of generality that

$$\|u_n - u\|_{L^\infty(L_\ell^{4d-2})} \leq \delta.$$

For every face σ of L_ℓ^{4d-1}, we consider u_σ and $u_{\sigma,n}$ the restrictions of u and u_n to σ, and we let $w_{\sigma,n}$ be the map obtained using the above cylinder construction with respect to $u_{\sigma,n}$ and u_σ, with the following orientations: σ is placed in $\mathbb{R}^{4d-1} \times \{0\}$ so that the orienting normal vector points upwards, the map $u_{\sigma,n}$ is placed at the bottom of the cylinder, and the map u_σ is placed at the top of the cylinder. We define $d_{\sigma,n}$ to be the Hopf degree of the map $w_{\sigma,n}$. With our orientation convention, we have $d_{-\sigma,n} = -d_{\sigma,n}$: Indeed, reversing the orientation in our construction amounts to reflect the domain with respect to the hyperplane $\mathbb{R}^{4d-1} \times \{1/2\}$, which, in terms of homotopy, corresponds to taking the inverse.

We first prove that

$$\sum_{\sigma \text{ face of } Q} d_{\sigma,n} = \deg_H(u_{|\partial Q}) - \deg_H(u_{n|\partial Q}).$$

For this purpose, we consider the set $\widetilde{Q}$ obtained by taking the union of Q and all the cubes congruent to Q and sharing exactly one face with it. We observe that $\partial\widetilde{Q}$ is homeomorphic to the boundary of a cube. We define a map w on the $(4d-1)$-skeleton of $\widetilde{Q}$ by letting w be equal to u_n on ∂Q, and to $w_{\sigma,n}$ on the boundary of the cube that has been glued to Q along the face $|\sigma|$ (after a suitable rotation so that both maps coincide on $|\sigma|$). It is straightforward to see that $w_{|\partial\widetilde{Q}}$ is homotopic to $u_{|\partial Q}$, since the maps $w_{\sigma,n}$ are constructed on the vertical faces as the projection of a linear interpolation between $u_{\sigma,n}$ and u_{σ} on $\partial|\sigma|$. On the other hand, we have the identity

$$\deg_H(w_{|\partial\widetilde{Q}}) = \sum_{\sigma \text{ face of } Q} d_{\sigma,n} + \deg_H(u_{n|\partial Q}).$$

This follows directly from the construction of w, since gluing two maps defined on the boundary of two neighboring cubes corresponds exactly to taking the sum of the associated homotopy classes. Combining both these pieces of information, we deduce that

$$\deg_H(u_{|\partial Q}) = \deg_H(w_{|\partial\widetilde{Q}}) = \sum_{\sigma \text{ face of } Q} d_{\sigma,n} + \deg_H(u_{n|\partial Q}),$$

which proves the required formula.

It remains to prove the integral estimate. For this purpose, we recall Rivière's estimate on the Hopf invariant [Riv98]

$$|\deg_H(f)|^{1-\frac{1}{4d}} \lesssim \int_{\mathbb{S}^{4d-1}} |\mathrm{D}f|^{4d-1}.$$

Applied to the maps $w_{\sigma,n}$, this yields

$$|d_{\sigma,n}|^{1-\frac{1}{4d}} \lesssim \int_{\partial Q^{4d}} |\mathrm{D}w_{\sigma,n}|^{4d-1}.$$

Invoking (5.4.2), the compactness of the embedding $W^{1,4d-1} \hookrightarrow L^{\infty}$ in dimension $4d-2$, and the weak lower semicontinuity of the norm of the gradient, we obtain

$$\limsup_{n\to+\infty} \sum_{\sigma \text{ face of } L^{4d-1}} |d_{\sigma,n}|^{1-\frac{1}{4d}} \lesssim \liminf_{n\to+\infty}\left(\int_{L^{4d-1}_{\ell}} |\mathrm{D}u_n|^{4d-1} + \int_{L^{4d-2}_{\ell}} |\mathrm{D}u_n|^{4d-1}\right).$$

The conclusion then holds provided $n_* \in \mathbb{N}$ is taken sufficiently large. □

Corollary 5.4.3. *Under the assumptions of Proposition 5.4.2, if for every cube Q of L^{4d}_{ℓ},*

$$\deg_H(u_{|Q}) = 2,$$

then

$$\liminf_{n\to+\infty} \int_{L_\ell^{4d-1}} |\mathrm{D}u_n|^{4d-1} + \int_{L_\ell^{4d-2}} |\mathrm{D}u_n|^{4d-1} \geq C\ell^{4d} \ln \ell.$$

Proof. This follows from the lower estimate on branched transportation [Bet20, Theorem A.1], and is linked to the fact that the exponent $\alpha = 1 - 1/N$ is the critical exponent for the irrigability of the Lebesgue measure in dimension N; see [BCM09] and the references therein for a detailed discussion about branched optimal transportation problems. Indeed, since the maps u_n are assumed to be continuous on the whole Q_ℓ, it follows that $\deg_H u_{n|\partial Q} = 0$ for every cube Q of L_ℓ^{4d}, which implies that

$$\sum_{\sigma \text{ face of } Q} d_{\sigma,n} = 2. \tag{5.4.3}$$

We define a transport plan to the boundary for the measure that consists of one Dirac mass with weight equal to 2 at the center of each cube of L_ℓ^{4d} by connecting the center of each cube to the centers of all the neighboring cubes by a segment, where each segment is weighted by the number $d_{\sigma,n}$ corresponding to the only face σ that it crosses. The orientation of the segments is determined by the sign of $d_{\sigma,n}$. By the fact that $d_{-\sigma,n} = -d_{\sigma,n}$ and (5.4.3), this indeed defines a transport plan from the above described measure to the boundary, and the mass of this transport plan with respect to the branched optimal transportation with power $\alpha = 1 - \frac{1}{4d}$ is exactly given by

$$\sum_{|\sigma|} |d_\sigma|^{1-\frac{1}{4d}}.$$

We conclude by combining the lower estimate on branched transportation and Proposition 5.4.2. □

Remark 5.4.4. Strictly speaking, our lower bound just uses sublinear transport (without branching), which means that a self-contained analysis would not require to take into account the appearance of additional points. □

5.4.3 The counterexample

We take u given by Proposition 5.4.1. Thanks to Fubini–Tonelli and Fatou, we can apply Proposition 5.4.2, and we get the conclusion by the uniform boundedness principle for the weak approximation.

5.5 Obstruction for higher order Sobolev energies

In this final section, we briefly explain how a higher order counterpart can be deduced from our main results by adapting our previous constructions so that the higher order nonlinear uniform boundedness principle [MVS19, Theorem 6.1] can be applied. In order to do this, we need maps with improved regularity and with a control on the energy on an enlarged domain; the explicit construction that we made in Sections 5.3 and 5.4 can be adapted with reasonable effort.

We start with the construction in Section 5.3 of the map into the skeleton of the torus. Let $d \geq 2$, let $s \geq 1$, and let $1 \leq p < +\infty$ be such that $sp = d$. We keep denoting by L^j the j-dimensional skeleton of $\mathbb{R}^{d+1}$. In Section 5.3.5, our counterexample was built up from $v = \pi \circ u$ defined in Proposition 5.3.8, where $\pi \colon \widetilde{\mathscr{M}_0} \to \mathscr{N}_0$ was the universal covering and u was the homogeneous extension of the inclusion $i \colon L^d \to \widetilde{\mathscr{M}_0}$. This construction can be adapted to higher order regularity.

Lemma 5.5.1. *Given $\varepsilon > 0$, there exists a map $v^{\mathrm{sm}} \in C^\infty(\mathbb{R}^{d+1} \setminus \Sigma; \mathscr{N}_\lambda)$, with $\Sigma = (\mathbb{Z}+1/2)^{d+1}$, such that*

(i) v^{sm} *is periodic under the action of* $\mathbb{Z}^{d+1}$,

(ii) *for every* $j \in \mathbb{N} \setminus \{0\}$, $\sup_{x \in \mathbb{R}^{d+1} \setminus \Sigma} |\mathrm{D}^j v^{\mathrm{sm}}(x)| \operatorname{dist}(x, \Sigma)^j < +\infty$,

(iii) $\|v - v^{\mathrm{sm}}\|_{L^\infty(\mathbb{R}^{d+1})} \leq \varepsilon$.

Proof. By convolution of the map v with a suitable mollifier and projection on $\mathscr{N}_\lambda$ via the smooth nearest point projection, we obtain a smooth periodic map $v^{\mathrm{ex}} \colon L^d + B_\delta \to \mathscr{N}_\lambda$.

Then, we rely on a thickening procedure to extend v^{ex} smoothly to $\mathbb{R}^{d+1} \setminus \Sigma$. More precisely, on each cube of L^{d+1}, that we identify with the cube $[-1,1]^{d+1}$, instead of precomposing v^{ex} with $x/|x|_\infty$ as we would do for homogeneous extension, we precompose it with a map of the form $\lambda(|x|_{2q})x$, where $\lambda \colon (0,+\infty] \to [1,+\infty)$ is a suitable smooth nonincreasing map which is constantly equal to 1 in a neighborhood of $[1,+\infty)$ and $q \in \mathbb{N}$ is chosen sufficiently large so that $\{x \in \mathbb{R}^{d+1} \colon |x|_{2q} \geq 1\} \subseteq \partial([-1,1]^n) + B_\delta^{d+1}$; see [BPVS15, Section 4] or Chapter 2. Let us call v^{sm} the map obtained through this process.

Since the thickening procedure was applied to the periodic map v^{ex}, the map v^{sm} is periodic. In addition, a suitable choice of the number δ, the convolution parameter, and the function λ, ensures that (iii) holds. Property (ii) follows from the estimates for thickening. □

We may now compose v^{sm} with the Lipschitz map $\Phi \colon \mathscr{N}_\lambda \to \mathscr{N}_0$ provided by Proposition 5.3.9, which yields a map $v^{\mathrm{lip}} = \Phi \circ v^{\mathrm{sm}} \colon \mathbb{R}^{d+1} \to \mathscr{N}_0$ with the following properties:

(i) $v^{\mathrm{lip}} \in W^{1,d}_{\mathrm{loc}}(\mathbb{R}^{d+1}; \mathscr{N}_0) \cap C(\mathbb{R}^{d+1} \setminus \Sigma; \mathscr{N}_0)$;

(ii) v^{lip} is periodic under the action of $\mathbb{Z}^{d+1}$;

(iii) $\|v^{\mathrm{lip}} - v\|_{L^\infty(\mathbb{R}^{d+1})} \leq C_1 \varepsilon$.

Assertion (iii) follows from the fact that $\Phi \colon \mathscr{N}_\lambda \to \mathscr{N}_0$ is a left-inverse of the embedding of $\mathscr{N}_0$ into $\mathscr{N}_\lambda$.

If $Q_\ell = [0, \ell]^{d+1}$, the map $v^{\mathrm{lip}}_{|Q_\ell}$ can then be lifted to a map $u \in W^{1,d}(Q_\ell; \widetilde{\mathscr{N}_0}) \cap C(Q_\ell \setminus \Sigma; \widetilde{\mathscr{N}_0})$, which can easily be checked to satisfy the assumptions of Proposition 5.3.5, with the exception that (5.3.13) should be replaced by $u(Q_{\ell,\alpha}) \subset Q_{\ell,\alpha} + B^{d+1}_{C_1\varepsilon}$. However, it is readily verified that the conclusion of Proposition 5.3.5 remains valid under this slightly weaker assumption, provided that $\varepsilon > 0$ is chosen sufficiently small. Therefore, as the reasoning carried out in Section 5.3.4 does not depend on the specific form of the map, we deduce that the map v^{lip} also satisfies the estimate

$$\liminf_{\ell \to +\infty} \frac{\mathcal{E}^{1,d}_{\mathrm{rel}}(v^{\mathrm{lip}}, Q_\ell)}{\ell^{d+1} \ln \ell} > 0.$$

This implies that also

$$\liminf_{\ell \to +\infty} \frac{\mathcal{E}^{1,d}_{\mathrm{rel}}(v^{\mathrm{sm}}, Q_\ell)}{\ell^{d+1} \ln \ell} > 0, \tag{5.5.1}$$

since any sequence of smooth maps that weakly converges to v^{sm} can be projected to a sequence of Lipschitz maps weakly converging to v^{lip} through the Lipschitz map $\Phi \colon \mathscr{N}_\lambda \to \mathscr{N}_0$ provided by Proposition 5.3.9.

By compactness of $\mathscr{N}_\lambda$, we have $W^{s,p}(\mathscr{M}, \mathscr{N}_\lambda) \subseteq L^\infty$ and thus, by the Gagliardo–Nirenberg interpolation inequality $W^{s,p} \cap L^\infty \subset W^{1,sp}$, see e.g. [Run86, Lemma 3.1 p. 329] or [BM01, Corollary 3.2], for every $w \in W^{s,p}(Q_\ell, \mathscr{N}_\lambda)$, we have

$$\mathcal{E}^{1,sp}(w, Q_\ell) \lesssim \|w\|^{p(s-1)}_{L^\infty} (\mathcal{E}^{s,p}(w, Q_\ell) + \ell^{-sp} \|w\|^p_{L^p}) \lesssim (\mathcal{E}^{s,p}(w, Q_\ell) + \ell), \tag{5.5.2}$$

so that by definition of relaxed energy and (5.5.1),

$$\liminf_{\ell \to +\infty} \frac{\mathcal{E}^{s,p}_{\mathrm{rel}}(v^{\mathrm{sm}}, Q_\ell)}{\ell^{d+1} \ln \ell} > 0.$$

On the other hand, by the periodic character of the map v^{sm}, it is straightforward to check that, if $2Q_\ell$ denotes the cube with the same center as Q_ℓ and double edge length, we have

$$\mathcal{E}^{s,p}(v^{\mathrm{sm}}, 2Q_\ell) \lesssim \ell^{d+1} \qquad \text{and} \qquad \mathcal{E}^{1,p}(v^{\mathrm{sm}}, 2Q_\ell) \lesssim \ell^{d+1}.$$

The case where $s \notin \mathbb{N}$ might be somehow more subtle, since the Gagliardo seminorm is not additive. Working with mixed integrals, one has

$$\iint_{Q^m \times \mathbb{R}^m} \frac{|\mathrm{D}^{\lfloor s \rfloor} v(x) - \mathrm{D}^{\lfloor s \rfloor} v(y)|^p}{|x-y|^{m+(s-\lfloor s \rfloor)p}} \,\mathrm{d}x\,\mathrm{d}y < +\infty,$$

and one can conclude then by additivity and periodicity; see e.g. Lemma 2.2.1.

We are thus in position to apply the higher order nonlinear uniform boundedness principle [MVS19, Theorem 6.1], which proves the existence of a map $u \in W^{s,p}(\mathscr{M}; \mathscr{N}_\lambda)$ which cannot be weakly approximated by smooth maps in $W^{s,p}(\mathscr{M}; \mathscr{N}_\lambda)$, hence completing the proof of Theorem 5.1.3.

A similar process applied to our construction involving the Whitehead product yields a higher order counterpart to Theorem 5.1.2. Since the procedure is actually simpler than for Theorem 5.1.1, as there is no lifting or passing from a cell complex to a manifold involved, the target being a sphere, we omit the details. However, we importantly mention that there is indeed a smoothing procedure to be performed. Indeed, even though the map f in the proof of Proposition 5.4.1 is taken to be smooth, the resulting map u does not need to be smooth, not only because of the homogeneous extension procedure, but already in the definition of v, which need not be constant near the boundaries of the faces of $\partial([-1/2, 1/2]^{4d})$. (The image of the construction on $\partial([-1/2, 1/2]^2)$ is misleading as it is the only dimension where v is constant near the boundary of the faces.) Therefore, we need to argue as for the higher order counterpart of Theorem 5.1.1: build a smooth variant of the original construction by first extending continuously on a neighborhood of the faces, then use regularization by convolution and thickening to replace homogeneous extension, and finally study the relaxed energy thanks to the Gagliardo–Nirenberg interpolation inequality to get Theorem 5.1.4.

Let us close this section with some thoughts about the case $0 < s < 1$. In this case, there is no smoothing needed for our constructions to be in $W^{s,p}$, but the Gagliardo–Nirenberg interpolation inequality cannot be used to estimate the relaxed energy. Although it should be possible to adapt the essential part of the slicing and bubbling arguments, the crucial lower estimates on energies would require more work. In order to extend Theorem 5.1.3 to $0 < s < 1$, one would need a fractional version of the localized degree estimate of Proposition 5.3.2; for Theorem 5.1.4, Proposition 5.4.2 relies on Rivière's Hopf invariant estimate [Riv98], which has only been partially extended to the fractional case for $s \geq 1 - \frac{1}{4d}$ [SVS20].

Chapter 6

Pullback of closed forms by low regularity maps to manifolds, and applications

Résumé

Dans ce chapitre, on traite du problème de caractériser la clôture des applications lisses dans le cas où la propriété de densité forte échoue, ce qui correspond à la question (Q3). Comme expliqué dans l'introduction, une contribution pionnière dans cette direction est le travail de F. Bethuel, J.-M. Coron, F. Demengel, et F. Hélein, qui ont démontré que, lorsque la cible $\mathscr{N}$ a une topologie suffisamment simple pour que *sa cohomologie détecte son homotopie*, alors une application $W^{1,p}$ à valeurs dans $\mathscr{N}$ peut être fortement approchée par des applications lisses à valeurs dans $\mathscr{N}$ si et seulement si tous les tirés en arrière au sens des distributions par u des $\lfloor p \rfloor$-formes fermées sur $\mathscr{N}$ s'annulent. Dans ce chapitre, on étend ce résultat aux applications $W^{s,p}$ avec $0 < s < 1$. Chemin faisant, nous adaptons la théorie des invariants homotopiques de H. Brezis et L. Nirenberg aux applications VMO sur des espaces métriques mesurés, établissons l'existence et quelques propriétés clés des invariants intégraux pour des applications VMO sur des variétés Lipschitz, démontrons l'existence de tirés en arrière au sens des distributions par des applications de Sobolev fractionnaires et obtenons certaines de leurs propriétés, incluant plusieurs formules de désintégration, et caractérisons la fermeture des applications lisses en termes de restrictions à des squelettes génériques. Ceci correspond à un travail en collaboration avec P. Mironescu et Xiao K.

Abstract

In this chapter, we deal with the question of characterizing the closure of smooth maps in case the strong density property fails, which corresponds to question (Q3). As we explained in the introduction, a seminal contribution in this direction is the work of F. Bethuel, J.-M. Coron, F. Demengel, and F. Hélein, who proved that, when the target $\mathscr{N}$ has a sufficiently simple topology so that its cohomology sees its homotopy, then a $W^{1,p}$ map u to $\mathscr{N}$ can be strongly approximated with smooth maps to $\mathscr{N}$ if and only if all the distributional pullbacks by u of closed $\lfloor p \rfloor$-forms

on $\mathscr{N}$ vanish. In this chapter, we extend this result to $W^{s,p}$ maps with $0 < s < 1$. In the process, we adapt the theory of homotopical invariants by H. Brezis and L. Nirenberg to VMO maps on metric measure spaces, establish the existence and some main properties of integral invariants for VMO maps on Lipschitz manifolds, prove the existence of distributional pullbacks by fractional Sobolev maps and obtain some of their properties, including various slicing formulas, and characterize the closure of smooth maps in terms of restrictions on generic skeletons. This corresponds to a joint work with P. Mironescu and Xiao K.

6.1 Introduction

In this chapter, we investigate the question of topological and analytical characterizations of the space $H^{s,p}_S(\mathscr{M};\mathscr{N})$ in the case where it is strictly included in $W^{s,p}(\mathscr{M};\mathscr{N})$. We focus on the case where $0 < s < 1$, where, as far as analytical characterization are concerned, a first challenge is to even suitably *define* the relevant quantities to work with. When $s = 1$, after a significant contribution by F. Bethuel [Bet90] when $\mathscr{N} = \mathbb{S}^2$, relying on the distributional Jacobian (in the sense of J. Ball [Bal76]), a far-reaching generalization was announced by F. Bethuel, J.-M. Coron, F. Demengel, and F. Hélein [BCDH91]. A fruitful contribution of [BCDH91] is to highlight the important role played by the pullback of forms by Sobolev maps. The relevant object here is $u^\sharp\omega$, where ω is a smooth d-form on $\mathscr{N}$ and $u \in W^{1,d}(\mathbb{B}^m;\mathscr{N})$; clearly, this is a d-form with L^1 coefficients. A second significant contribution was to coin the importance of the topological assumption on $\mathscr{N}$ that *its cohomology sees its homotopy*, that we already mentioned in the main introduction. Let us now be more specific: the crucial assumption on $\mathscr{N}$ in order for this approach to be implemented is that

$$\left[\int_{\mathbb{S}^k} f^\sharp\omega = 0, \text{for every smooth closed } d\text{-form } \omega \text{ on } \mathscr{N}\right] \implies f \text{ is nullhomotopic.} \tag{A}$$

The main result in [BCDH91] asserts that, if (i) $1 \leq d \leq p < d+1 \leq N$; and (ii) the closed manifold $\mathscr{N}$ satisfies (A), a map $u \in W^{1,p}(\mathbb{B}^m;\mathscr{N})$ can be strongly approximated with smooth $\mathscr{N}$-valued maps if and only if, for each smooth *closed* d-form on $\mathscr{N}$, we have $\mathrm{d}[u^\sharp\omega] = 0$ in the sense of distributions. (More precisely, in the sense of currents.)

In [BCDH91], the authors present the main lines of proof of the above result. One of its main ingredients is the characterization of strongly approximable maps via the homotopy type of their restrictions to generic d-dimensional skeletons. This type of characterization has been subsequently formalized by Hang F. and Lin F. [HL03a]. However, a rigorous proof of the validity of such characterizations (in the $W^{1,p}$ setting) has only been achieved very recently by P. Bousquet, A. Ponce, and J. Van Schaftingen [BPVS25]. This leads to a full proof of the results announced in [BCDH91].

In this chapter, we obtain, in fractional Sobolev spaces $W^{s,p}$ with $0 < s < 1$, full counterparts of the above described results. In addition to the aforementioned difficulties, as we already mentioned, we have to cope with the fact that the pullback $u^\sharp\omega$ has no obvious meaning when $s < 1$.

We next describe our contributions and how they fit together to prove our main result.

VMO *and homotopy.* H. Brezis and L. Nirenberg [BN95] carried out a systematic study of the homotopy classes naturally associated with $\mathrm{VMO}(\mathscr{M};\mathscr{N})$, where $\mathscr{M}$, respectively $\mathscr{N}$, is a smooth compact manifold, respectively closed manifold. In our setting, a relevant $\mathscr{M}$ is the boundary of a cube. In Section 6.2, we establish the counterparts of the results in [BN95] in the rather general case where $\mathscr{M}$ is a compact metric measure space with a doubling measure. This seems a natural generalization and we hope that it is of independent interest. In particular, the mollifiers that we construct may prove useful in other contexts.

Integral invariants. In Section 6.3, we consider non-smooth versions of the integral invariants of the form $\mathscr{I}(f) = \int_{\mathscr{M}} f^\sharp\omega$, where $\mathscr{M}$ and $\mathscr{N}$ are smooth closed manifolds, $f : \mathscr{M} \to \mathscr{N}$ is smooth, $\mathscr{M}$ is d-dimensional, and ω is a smooth closed d-form on $\mathscr{N}$. In the smooth case, it is well-known that this is a homotopical invariant acting on de Rham cohomology classes. We extend this result to Lipschitz closed manifolds $\mathscr{M}$. Here, we opted for a completely elementary approach, avoiding geometric measure theory language and tools. We hope that making this part of the text low tech and essentially self-contained was worth a few extra pages.

Estimates for $\mathscr{I}(f)$. A first major difficulty in the proof of the main theorem arises in the estimate of $\mathscr{I}(f)$. When $\mathscr{M} = \mathscr{N} = \mathbb{S}^d$ and $f \in W^{s,p}$, with $sp = d$, this has been obtained in [BBM05]. In Section 6.4, we extend the result in [BBM05] to general $\mathscr{M}$ and $\mathscr{N}$.

The distribution $f^\sharp\omega$. In Section 6.5, we investigate whether one can naturally associate a distribution with $f^\sharp\omega$. This topic has been originally addressed by H. Brezis and Nguyen H.-M. [BN11] when $\mathscr{M} = \mathscr{N} = \mathbb{S}^d$ and ω is the standard volume form. We obtain counterparts of their results in the general case. We hope that this provides a muggle's approach to some *magical* identities in [BN11]. This route will be further pursued in [DX].

A higher dimensional version of $\mathscr{I}(f)$. A second major difficulty in the proof of the main theorem is related to the definition of the exterior differential $\mathrm{d}[u^\sharp\omega]$ when $\dim\mathscr{M} > d$. (In our case, $\mathscr{M}$ is typically a ball of dimension greater than d.)

Unlike the analysis in Sections 6.2 and 6.3, which naturally involves VMO maps, in this setting the right regularity of maps is Sobolev. The first main result in Section 6.6 provides, roughly speaking, a *robust* definition for $\mathrm{d}[u^\sharp\omega]$ when $u \in W^{s,p}(\mathscr{M};\mathscr{N})$, $\dim\mathscr{M} > d$, and $sp = d$. This generalizes a result in [BBM05], which corresponds to $\mathscr{M} = \mathbb{S}^{d+1}$, $\mathscr{N} = \mathbb{S}^d$, and ω the standard volume form on $\mathbb{S}^d$. (See also [HL00, BM14].) A similar direction of research was also investigated, using the language of geometric measure theory, by M. Giaquinta, G. Modica, and J. Souček [GMS04] for $W^{1/2,2}$ maps with values into $\mathbb{S}^1$, by M. Giaquinta and D. Mucci [GM05a] for $W^{1/2,2}$ maps into more general targets, and by D. Mucci [Muc12] for $W^{1/p,p}$ maps with $p > 1$. These latter contributions are in line with the theory of *Cartesian currents*, developed by M. Giaquinta, G. Modica, and J. Souček in $W^{1,p}$, and culminating with the monograph [GMS98a, GMS98b]. Our approach is purely analytical and avoids geometric measure theory.

We next show that, at least when u is sufficiently nice, $\mathrm{d}[u^\sharp\omega]$ encodes the singular set of u and the topology carried by the singularities. The first result of this type is due to H. Brezis, J.-M. Coron, and E. Lieb [BCL86]. For other similar results, see R. Jerrard and H. M. Soner [JS02, Theorem 1.2], G. Alberti, S. Baldo, and G. Orlandi [ABO03, Theorem 3.8], and P. Bousquet [Bou07, Proposition 1]. The result we prove was obtained by M. Giaquinta, G. Modica, and J. Souček [GMS98b, Section 4.2, Theorem 1]. However, the reader may find instructive our proof, relying only on an iterated use of the Stokes formula.

Slicing. A third major difficulty arises from the disintegration of $\mathrm{d}[u^\sharp\omega]$. When $u \in W^{1,d}$, a simple application of the Fubini theorem allows to recover $\mathrm{d}[u^\sharp\omega]$ from its $(d+1)$-dimensional slices. In the fractional Sobolev setting, a similar disintegration formula was obtained P. Mironescu, E. Russ, and Y. Sire [MRS20, Lemma 3.12] when $\mathscr{N} = \mathbb{S}^1$ and ω is the standard volume form. In Section 6.6, we prove such a formula in the general case.

A first answer to (Q3). In this context, it is more convenient to work with maps defined on $\mathbb{R}^m$ (with $0 < s < 1$ and $1 \leq d \leq sp < d+1 \leq m$). A main result in Section 6.6.5, Theorem 6.6.15, asserts that a map $u \in W^{s,p}(\mathbb{R}^m;\mathscr{N})$ is approximable with smooth $\mathscr{N}$-valued maps if and only if, on *sufficiently many* grids, its restriction to the boundaries of $(d+1)$-dimensional cubes is nullhomotopic. This relies on approximation techniques devised in [BM15]. A specific feature of the case $0 < s < 1$ (as opposed to the case where s is an integer, investigated in [BPVS25]), is the conceptually simpler approach for strong density proposed in [BM15], which substantially simplifies our task, especially when we have to quantify the notion of *genericity*.

Providing a rigorous proof of Theorem 6.6.15 is one of the main contributions of this chapter.

A second answer to (Q3). In Section 6.6.6, we prove the fractional counterpart of the main result in [BCDH91]. More specifically, we prove that, if (i) $1 \leq d \leq sp < d+1 \leq N$; and (ii) the closed manifold $\mathscr{N}$ satisfies (A), a map $u \in W^{s,p}(\mathbb{B}^N; \mathscr{N})$ can be strongly approximated with smooth $\mathscr{N}$-valued maps if and only if, for each smooth *closed* d-form on $\mathscr{N}$, we have $\mathrm{d}[u^\sharp \omega] = 0$ in the sense of distributions.

The proof follows the strategy in [BCDH91] and relies on all the above analytical tools and results. Its three main steps are the following. *Step 1.* Starting from *higher-dimensional integral invariants* and using a dimensional reduction relying on *slicing*, we determine the *integral invariants* on the boundaries of $(d+1)$-dimensional cubes. *Step 2.* Using assumption (A) and the value of the integral invariants computed in the first step, we obtain a *homotopical information* on the restrictions of u to the boundaries of $(d+1)$-dimensional cubes. *Step 3.* We conclude using the homotopical information obtained in Step 2 and the *first answer to* (Q2).

When $\mathscr{N} = \mathbb{S}^d$, the above result takes the following simpler form: a map $u \in W^{s,p}(\mathscr{M}; \mathbb{S}^d)$ is approximable with smooth $\mathbb{S}^d$-valued maps if and only if $Ju = 0$, where Ju is the distributional Jacobian introduced in [BBM05] and [BM14]. This result was announced in D. Mucci [Muc24]. As in our approach, the proof in [Muc24] follows the main lines in [BCDH91], with a sketch of the slicing argument.

About assumption (A). Assumptions in the spirit of (A) are crucial in various contributions subsequent to [BCDH91], including, but not only, M. Giaquinta, G. Modica, and J. Souček [GMS98a, GMS98b], R. Pakzad and T. Rivière [PR03], M. Giaquinta and D. Mucci [GM05b], G. Canevari and G. Orlandi [CO19], and P. Bousquet, A. Ponce, and J. Van Schaftingen [BPVS25]. In Section 6.7, we clarify how assumption (A) compares with the ones in the aforementioned references.

6.2 Homotopy classes of VMO maps on doubling metric measure spaces

In this section, with no claim of originality: (a) $\mathscr{M}$ is a compact doubling metric measure space (see below); (b) $\mathscr{N}$ is a closed manifold. We carefully adapt to this setting the results of H. Brezis and L. Nirenberg [BN95] concerning the existence and some basic properties of the homotopy classes of the space $\mathrm{VMO}(\mathscr{M}; \mathscr{N})$. (In [BN95, BN96], $\mathscr{M}$ is a compact manifold.)

More specifically, we assume that $\mathscr{M}$ is a compact metric space endowed with a non-trivial (finite) Borel measure μ satisfying the doubling property

$$\text{there exists } C_{\mathscr{M}} > 0 \text{ such that } 0 < \frac{\mu(B_{2r}(x))}{\mu(B_r(x))} \leq C_{\mathscr{M}}, \text{ for every } x \in \mathscr{M} \text{ and } r > 0. \tag{6.2.1}$$

(The balls we consider are open, but we could have also considered closed balls.)

The prototypical example of $\mathscr{M}$ we have in mind is $\mathscr{M} = \partial C^m$, where C^m is a cube in $\mathbb{R}^m$, with dist the geodesic or Euclidean distance and μ the $(m-1)$-dimensional Hausdorff measure.

Throughout this section, we assume that the doubling condition (6.2.1) *holds.* This is a crucial condition. In contrast, $\mathscr{M}$ is assumed to be compact mainly in order to stay on the safe side for all the statements in this section; in many of them, we could have assumed that $\mathscr{M}$ is merely bounded or totally bounded.

We note that the boundedness of $\mathscr{M}$ and the doubling condition (6.2.1) imply that there exists some $C_r > 0$ such that

$$\mu(B_r(x)) \geq C_r, \quad \text{for every } x \in \mathscr{M}. \tag{6.2.2}$$

We also note also that, since $\mathscr{M}$ is bounded, we have the following straightforward property:

$$\text{if (6.2.1) holds for any } 0 < r \leq r_0 \text{ and any } x, \text{ then (6.2.1) holds for any } r \text{ and } x. \tag{6.2.3}$$

6.2.1 BMO and VMO on doubling metric measure spaces

We first define BMO. For $f \in L^1(\mathscr{M})$, we define the seminorm

$$|f|_{\mathrm{BMO}} = \sup_{x \in \mathscr{M}, 0 < \varepsilon \leq \varepsilon_0} \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |f(y) - f(z)| \, \mathrm{d}\mu(y) \, \mathrm{d}\mu(z), \tag{6.2.4}$$

and let

$$\mathrm{BMO} = \mathrm{BMO}(\mathscr{M}) = \mathrm{BMO}(\mathscr{M}; \mathbb{R}) = \{f \in L^1(\mathscr{M}) \colon |f|_{\mathrm{BMO}} < +\infty\}.$$

Similarly for maps in $L^1(\mathscr{M}; \mathbb{R}^\nu)$.

In the above definition, $\varepsilon_0 > 0$ is a fixed constant. *By default, we let* $\varepsilon_0 = \operatorname{diam}(\mathscr{M})$ (if $\mathscr{M}$ contains at least two points), but, under the mild assumption that $\mathscr{M}$ is connected, ε_0 could be any positive number (see below).

We first establish a variant of [BN95, Lemma A.1].

Lemma 6.2.1. (1) *Assume that* $\varepsilon_0 \geq \operatorname{diam}(\mathscr{M})$. *Then, there exists a constant* $C > 0$

depending on $\mathscr{M}$ *and* μ *such that*

$$\|f\|_{L^1} \le C|f|_{\mathrm{BMO}} + \left|\int_{\mathscr{M}} f\right|, \quad \textit{for every } f \in \mathrm{BMO}. \tag{6.2.5}$$

(2) *Assume that* $\mathscr{M}$ *is connected. Then,* (6.2.5) *holds for some constant* $C > 0$ *depending on* $\mathscr{M}$, μ, *and* ε_0.

Proof of item (2). We will use the following straightforward property: (P) if $\mathscr{M}$ is connected, then any measurable function locally constant a.e. is actually constant a.e.

With no loss of generality, we may consider only functions with zero integral. We argue by contradiction. Assume that there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in BMO such that

$$\int_{\mathscr{M}} f_n = 0, \quad |f_n|_{\mathrm{BMO}} \to 0, \quad \text{and } \|f_n\|_{L^1} = 1.$$

Since $\mathscr{M}$ is compact, we can cover $\mathscr{M}$ with a finite number of balls $B_{\varepsilon_0}(x_i)$, $1 \le i \le N$. For fixed i, we have

$$\fint_{B_{\varepsilon_0}(x_i)} \left| f_n(y) - \fint_{B_{\varepsilon_0}(x_i)} f_n \right| \mathrm{d}\mu(y) \le |f_j|_{\mathrm{BMO}} \to 0. \tag{6.2.6}$$

Using (6.2.6) and $\|f_n\|_{L^1} = 1$, we find that $\left(\fint_{B_{\varepsilon_0}(x_i)} f_n\right)_{n\in\mathbb{N}}$ is bounded (for every fixed i). From the above, we deduce that, up to a subsequence, (j) $\left(\fint_{B_{\varepsilon_0}(x_i)} f_n\right)_{n\in\mathbb{N}}$ converges to some constant a_i; and (jj) on each $B_{\varepsilon_0}(x_i)$, f_n converges to a_i a.e. and in L^1. Since $(B_{\varepsilon_0}(x_i))_{1\le i\le N}$ is an open cover of $\mathscr{M}$, all the constants a_i are equal (by the property (P)), so that $f_n \to a_1$ in $L^1(\mathscr{M})$. Since $\int_{\mathscr{M}} f_n = 0$, we find that $a_1 = 0$, and thus $f_n \to 0$ in $L^1(\mathscr{M})$. This contradicts the assumption $\|f_n\|_{L^1} = 1$. ◻

Proof of item (1). The proof is essentially the same as above. Property (P) is not needed in this setting since, for any $x_i \in \mathscr{M}$, we have $\mathscr{M} = B_{\varepsilon_0}(x_i)$. ◻

Corollary 6.2.2. *Assume that* $\mathscr{M}$ *is connected. Then, two different values of* ε_0 *yield equivalent seminorms on* BMO.

Proof. In view of (6.2.5), it suffices to prove that, if $r_0 < \varepsilon_0$, then we have, for some constant $C > 0$ depending on r_0 and ε_0,

$$\fint_{B_\rho(x)} \fint_{B_\rho(x)} |f(y) - f(z)| \, d\mu(y) \, d\mu(z) \le C\|f\|_{L^1},$$

for every $f \in L^1$ such that $\displaystyle\int_{\mathscr{M}} f = 0$, for every $x \in \mathscr{M}$, for every $r_0 < \rho \le \varepsilon_0$.

With ρ as above and C_r as in (6.2.2), this follows from

$$\begin{aligned}
&\fint_{B_\rho(x)} \fint_{B_\rho(x)} |f(y) - f(z)| \, d\mu(y) \, d\mu(z) \\
&\quad = \frac{1}{[\mu(B_\rho(x))]^2} \int_{B_\rho(x)} \int_{B_\rho(x)} |f(y) - f(z)| \, d\mu(y) \, d\mu(z) \\
&\quad \le \frac{1}{[\mu(B_\rho(x))]^2} \int_{B_\rho(x)} \int_{B_\rho(x)} [|f(y)| + |f(z)|] \, d\mu(y) \, d\mu(z) \\
&\quad = \frac{2}{\mu(B_\rho(x))} \|f\|_{L^1} \le \frac{2}{C_{r_0}} \|f\|_{L^1}.
\end{aligned}$$ □

We now turn to VMO and its basic characterizations and properties, in the spirit of Sarason [Sar75]. For $f \in \mathrm{BMO}$ and $r > 0$, define

$$M_r(f) = \sup_{x \in \mathscr{M},\, 0 < s \le r} \fint_{B_s(x)} \fint_{B_s(x)} |f(y) - f(z)| \, d\mu(y) \, d\mu(z) \le |f|_{\mathrm{BMO}},$$

and

$$M_0(f) = \lim_{r \searrow 0} M_r(f).$$

We denote $\mathrm{VMO} = \mathrm{VMO}(\mathscr{M}) = \mathrm{VMO}(\mathscr{M}; \mathbb{R})$ the closure of continuous functions with respect to the BMO seminorm, i.e.,

$$\mathrm{VMO} = \overline{C(\mathscr{M})/\mathbb{R}}^{|\cdot|_{\mathrm{BMO}}}. \tag{6.2.7}$$

Similarly for $\mathrm{VMO}(\mathscr{M}; \mathbb{R}^\nu)$. We also denote

$$\operatorname{dist}(f, \mathrm{VMO}) = \inf_{g \in \mathrm{VMO}} |f - g|_{\mathrm{BMO}}.$$

We next introduce an approximation procedure adapted to the study of VMO under the doubling condition (6.2.1). For $x, y \in \mathscr{M}$ and $\varepsilon > 0$, let

$$\rho(x,\varepsilon,y) = [\varepsilon - \operatorname{dist}(x,y)]_+, \quad K(x,\varepsilon) = \left(\int_{\mathscr{M}} \rho(x,\varepsilon,y)\,\mathrm{d}\mu(y)\right)^{-1}, \tag{6.2.8}$$

and set

$$f_\varepsilon(x) = K(x,\varepsilon)\int_{\mathscr{M}} \rho(x,\varepsilon,y) f(y)\,\mathrm{d}\mu(y) = \fint_{\mathscr{M}} f\,\mathrm{d}[\rho(x,\varepsilon,\cdot)\mu]. \tag{6.2.9}$$

For further use, let us note the following straightforward inequalities.

Lemma 6.2.3. *We have, for every $x \in \mathscr{M}$ and $\varepsilon > 0$,*

$$\frac{\varepsilon}{2}\chi_{B_{\varepsilon/2}(x)} \le \rho(x,\varepsilon,\cdot) \le \varepsilon\chi_{B_\varepsilon(x)}, \tag{6.2.10}$$

$$\frac{\varepsilon}{2}\mu(B_{\varepsilon/2}(x)) \le \int_{\mathscr{M}} \rho(x,\varepsilon,y)\,\mathrm{d}\mu(y) \le \varepsilon\mu(B_\varepsilon(x)), \tag{6.2.11}$$

$$\frac{1}{\varepsilon\mu(B_\varepsilon(x))} \le K(x,\varepsilon) \le \frac{2C_{\mathscr{M}}}{\varepsilon\mu(B_\varepsilon(x))}, \tag{6.2.12}$$

$$\frac{1}{2\mu(B_\varepsilon(x))}\chi_{B_{\varepsilon/2}(x)} \le K(x,\varepsilon)\rho(x,\varepsilon,\cdot) \le \frac{2C_{\mathscr{M}}}{\mu(B_\varepsilon(x))}\chi_{B_\varepsilon(x)}. \tag{6.2.13}$$

Proof. The inequality (6.2.10) is clear. Integrating (6.2.10) yields (6.2.11). Then, (6.2.12) follows from (6.2.11) and the doubling assumption (6.2.1). Finally, (6.2.13) is a consequence of (6.2.10) and (6.2.12). □

The next result is crucial for the existence of well-behaved homotopy classes.

Lemma 6.2.4. *The map $\mathscr{M} \times (0,+\infty) \ni (x,\varepsilon) \mapsto f_\varepsilon(x)$ is continuous.*

Proof. Since f is integrable and $\mathscr{M}$ is compact, it suffices to prove that $K(x,\varepsilon)\rho(x,\varepsilon,y)$ is continuous with respect to (x,ε,y). Clearly, $\rho(x,\varepsilon,y)$ is continuous with respect to (x,ε,y). On the other hand, we have

$$+\infty > \varepsilon\mu(\mathscr{M}) \ge \int_{\mathscr{M}} \rho(x,\varepsilon,y)\,\mathrm{d}\mu(y) \ge \frac{\varepsilon}{2}\mu(B_{\varepsilon/2}(x)) > 0,$$

so that $K(x,\varepsilon)$ is well-defined and continuous with respect to (x,ε). □

We have the following versions of [BN95, Lemma A.5, Corollary 1].

Lemma 6.2.5. *There exists a finite constant A depending on $\mathscr{M}$ and μ such that*

$$M_0(f) \le \operatorname{dist}(f,\mathrm{VMO}) \le AM_0(f), \quad \textit{for every } f \in \mathrm{BMO}, \tag{6.2.14}$$

and

$$|f - f_\varepsilon|_{\mathrm{BMO}} \le AM_{2\varepsilon}(f), \quad \text{for every } f \in \mathrm{BMO}, \text{for every } 0 < \varepsilon \le \varepsilon_0/2. \tag{6.2.15}$$

In particular, we have

$$\mathrm{VMO} = \{f \in \mathrm{BMO}\colon M_0(f) = 0\}. \tag{6.2.16}$$

Corollary 6.2.6. *For $f \in \mathrm{VMO}$, we have $f_\varepsilon \in \mathrm{VMO}$ and $f_\varepsilon \to f$ in* BMO *as $\varepsilon \to 0$.*

We will often use Corollary 6.2.6 in conjunction with the following observation.

Lemma 6.2.7. *For $f \in L^1(\mathscr{M})$, we have $f_\varepsilon \to f$ in L^1 as $\varepsilon \to 0$.*

The proof of Lemma 6.2.5 relies on the following straightforward variant of [BN95, Lemma A.6].

Lemma 6.2.8. *For any given numbers $0 < r \le \rho$, any ball $B_\rho(x) \subset \mathscr{M}$ can be covered with a finite number K of balls $B_r(x_i)$ with $x_i \in B_\rho(x)$, $i = 1, \dots, K$, such that $\operatorname{dist}(x_i, x_j) \ge r$ for $i \ne j$ and*

$$\sum_{i=1}^{K} \mu(B_r(x_i)) \le (C_{\mathscr{M}})^2 \mu(B_\rho(x)).$$

(The number K may depend on r, ρ, and x.)

Proof of Lemma 6.2.8. Since $\mathscr{M}$ is compact, there exists a (finite) maximal collection of disjoint balls $B_{r/2}(x_i)$, $1 \le i \le K$, with centers x_i in $B_\rho(x)$. For any point $x' \in B_\rho(x) \setminus \bigcup_{i=1}^{K} B_{r/2}(x_i)$, there exists some i_0 such that $\operatorname{dist}(x', x_{i_0}) < r$ (for otherwise we can add $B_{r/2}(x')$ to the collection, which contradicts its maximality). Therefore, we have

$$B_\rho(x) \subset \bigcup_{i=1}^{K} B_r(x_i).$$

Since

$$B_{r/2}(x_i) \subset B_{\rho+r/2}(x) \subset B_{2\rho}(x)$$

and thus

$$\sum_{i=1}^{K} \mu(B_{r/2}(x_i)) \le \mu(B_{2\rho}(x)),$$

the doubling assumption (6.2.1) yields

$$\sum_{i=1}^{K} \mu(B_r(x_i)) \leq C_{\mathscr{M}} \sum_{i=1}^{K} \mu(B_{r/2}(x_i)) \leq C_{\mathscr{M}} \mu(B_{2\rho}(x)) \leq (C_{\mathscr{M}})^2 \mu(B_\rho(x)). \qquad \square$$

Proof of Lemma 6.2.5. We first prove that

$$M_0(f) \leq \operatorname{dist}(f, \mathrm{VMO}), \quad \text{for every } f \in \mathrm{BMO}. \tag{6.2.17}$$

Clearly, if $f, g \in \mathrm{BMO}$, then, for any $r \geq 0$,

$$M_r(f) \leq M_r(f-g) + M_r(g). \tag{6.2.18}$$

On the other hand, if $g \in C(\mathscr{M})$, then g is uniformly continuous, and therefore $M_0(g) = 0$. Letting $r \to 0$ in (6.2.18), we find that

$$M_0(f) \leq M_0(f-g) \leq |f-g|_{\mathrm{BMO}}, \quad \text{for every } f \in \mathrm{BMO} \text{ and } g \in C(\mathscr{M}). \tag{6.2.19}$$

Inequality (6.2.17) follows from (6.2.19) and the definition of VMO.

We next assume that (6.2.15) holds. Then, combined with Lemma 6.2.4, it implies that

$$\operatorname{dist}(f, \mathrm{VMO}) \leq AM_{2\varepsilon}(f), \quad \text{for every } f \in \mathrm{BMO}, \text{ for every } \varepsilon > 0. \tag{6.2.20}$$

Letting $\varepsilon \to 0$ in (6.2.20) yields the second inequality in (6.2.14).

Therefore, it suffices to establish (6.2.15), which amounts to the existence of some finite A, independent of f and of ε and r as below, such that

$$\begin{gathered} \fint_{B_r(x)} \fint_{B_r(x)} |(f-f_\varepsilon)(y) - (f-f_\varepsilon)(z)| \,\mathrm{d}\mu(y)\,\mathrm{d}\mu(z) \leq AM_{2\varepsilon}(f), \\ \text{for every } 0 < \varepsilon \leq \varepsilon_0/2, \text{ for every } 0 < r \leq \varepsilon_0. \end{gathered} \tag{6.2.21}$$

Proof of (6.2.21) when $r \leq \varepsilon$. We first note that

$$\begin{aligned} &\fint_{B_r(x)} \fint_{B_r(x)} |f(y) - f_\varepsilon(y) - f(z) + f_\varepsilon(z)| \,\mathrm{d}\mu(y)\,\mathrm{d}\mu(z) \\ &\quad \leq \fint_{B_r(x)} \fint_{B_r(x)} |f(y) - f(z)| \,\mathrm{d}\mu(y)\,\mathrm{d}\mu(z) + \sup_{y,z \in B_r(x)} |f_\varepsilon(y) - f_\varepsilon(z)| \\ &\quad \leq M_r(f) + \sup_{y,z \in B_r(x)} |f_\varepsilon(y) - f_\varepsilon(z)|. \end{aligned} \tag{6.2.22}$$

In order to estimate the latter quantity in (6.2.22), we start from the identity

$$\begin{aligned} & f_\varepsilon(y) - f_\varepsilon(z) \\ & \quad = f_\varepsilon(y) \int_{\mathscr{M}} K(z,\varepsilon)\rho(z,\varepsilon,\eta)\,\mathrm{d}\mu(\eta) - f_\varepsilon(z) \int_{\mathscr{M}} K(y,\varepsilon)\rho(y,\varepsilon,\xi)\,\mathrm{d}\mu(\xi) \\ & \quad = \int_{\mathscr{M}} \int_{\mathscr{M}} K(y,\varepsilon)K(z,\varepsilon)\rho(y,\varepsilon,\xi)\rho(z,\varepsilon,\eta)[f(\xi) - f(\eta)]\,\mathrm{d}\mu(\xi)\,\mathrm{d}\mu(\eta). \end{aligned} \tag{6.2.23}$$

Combining (6.2.23) and (6.2.13) we obtain, for $y, z \in B_r(x)$,

$$\begin{aligned} & |f_\varepsilon(y) - f_\varepsilon(z)| \\ & \quad \leq \frac{4(C_{\mathscr{M}})^2}{\mu(B_\varepsilon(y))\mu(B_\varepsilon(z))} \int_{B_\varepsilon(z)} \int_{B_\varepsilon(y)} |f(\xi) - f(\eta)|\,\mathrm{d}\mu(\xi)\,\mathrm{d}\mu(\eta) \\ & \quad \leq 4(C_{\mathscr{M}})^6 \fint_{B_{2\varepsilon}(x)} \fint_{B_{2\varepsilon}(x)} |f(\xi) - f(\eta)|\,\mathrm{d}\mu(\xi)\,\mathrm{d}\mu(\eta) \leq 4(C_{\mathscr{M}})^6 M_{2\varepsilon}(f), \end{aligned} \tag{6.2.24}$$

where, in the last line, we use the fact that $B_{2\varepsilon}(x) \subset B_{4\varepsilon}(y)$ and thus, thanks to (6.2.1),

$$\frac{\mu(B_{2\varepsilon}(x))}{\mu(B_\varepsilon(y))} = \frac{\mu(B_{2\varepsilon}(x))}{\mu(B_{4\varepsilon}(y))} \frac{\mu(B_{4\varepsilon}(y))}{\mu(B_\varepsilon(y))} \leq (C_{\mathscr{M}})^2. \tag{6.2.25}$$

Combining (6.2.22) and (6.2.24), we obtain, for $0 < r \leq \varepsilon$,

$$\fint_{B_r(x)} \fint_{B_r(x)} |f(y) - f_\varepsilon(y) - f(z) - f_\varepsilon(z)|\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(z) \leq (4(C_{\mathscr{M}})^6 + 1)M_{2\varepsilon}(f). \tag{6.2.26}$$

Proof of (6.2.21) *when* $r \geq \varepsilon$. By Lemma 6.2.8 and the doubling assumption (6.2.1), $B_r(x)$ can be covered by a finite number of balls $B_\varepsilon(x_i)$ such that

$$\sum_i \mu(B_{2\varepsilon}(x_i)) \leq (C_{\mathscr{M}})^3 \mu(B_r(x)). \tag{6.2.27}$$

Using successively (6.2.13), (6.2.25), and (6.2.27) , we have

$$
\begin{aligned}
&\fint_{B_r(x)} \fint_{B_r(x)} |f(y) - f_\varepsilon(y) - f(z) + f_\varepsilon(z)| \,\mathrm{d}\mu(y)\,\mathrm{d}\mu(z) \\
&\quad \le 2 \fint_{B_r(x)} |f(y) - f_\varepsilon(y)| \,\mathrm{d}\mu(y) \le \frac{2}{\mu(B_r(x))} \sum_i \int_{B_\varepsilon(x_i)} |f(y) - f_\varepsilon(y)| \,\mathrm{d}\mu(y) \\
&\quad = \frac{2}{\mu(B_r(x))} \sum_i \int_{B_\varepsilon(x_i)} \left| \int_{B_\varepsilon(y)} K(y,\varepsilon)\rho(y,\varepsilon,z)[f(y) - f(z)] \,\mathrm{d}\mu(z) \right| \mathrm{d}\mu(y) \\
&\quad \le \frac{2}{\mu(B_r(x))} \sum_i \int_{B_\varepsilon(x_i)} \int_{B_\varepsilon(y)} K(y,\varepsilon)\rho(y,\varepsilon,z)|f(y) - f(z)| \,\mathrm{d}\mu(z)\,\mathrm{d}\mu(y) \qquad (6.2.28) \\
&\quad \le \frac{4C_{\mathscr{M}}}{\mu(B_r(x))} \sum_i \int_{B_\varepsilon(x_i)} \int_{B_\varepsilon(y)} \frac{1}{\mu(B_\varepsilon(y))} |f(y) - f(z)| \,\mathrm{d}\mu(z)\,\mathrm{d}\mu(y) \\
&\quad \le \frac{4(C_{\mathscr{M}})^3}{\mu(B_r(x))} \sum_i \int_{B_{2\varepsilon}(x_i)} \int_{B_{2\varepsilon}(x_i)} \frac{1}{\mu(B_{2\varepsilon}(x_i))} |f(y) - f(z)| \,\mathrm{d}\mu(z)\,\mathrm{d}\mu(y) \\
&\quad \le \frac{4(C_{\mathscr{M}})^3}{\mu(B_r(x))} \sum_i \mu(B_{2\varepsilon}(x_i)) M_{2\varepsilon}(f) \le 4(C_{\mathscr{M}})^6 M_{2\varepsilon}(f).
\end{aligned}
$$

Combining (6.2.26) and (6.2.28), we obtain

$$
|f - f_\varepsilon|_{\mathrm{BMO}} \le (4(C_{\mathscr{M}})^6 + 1) M_{2\varepsilon}(f), \quad \text{for every } f \in \mathrm{BMO}, \text{ for every } 0 < \varepsilon \le \varepsilon_0/2,
$$

so that (6.2.21) and (6.2.15) hold with $A = 4(C_{\mathscr{M}})^6 + 1$. □

Proof of Corollary 6.2.6. Combine (6.2.15) and Lemma 6.2.4. □

Proof of Lemma 6.2.7. Set

$$
T_\varepsilon(f) = f_\varepsilon, \quad \text{for every } f \in L^1(\mathscr{M}), \text{ for every } \varepsilon > 0.
$$

Clearly, (j) T_ε is linear; and (jj) if $f \in C(\mathscr{M})$, then, as $\varepsilon \to 0$, $T_\varepsilon(f) \to f$ uniformly, and thus in $L^1(\mathscr{M})$. In order to conclude (via (j), (jj), and density), it suffices to find some constant $C > 0$ such that

$$
\|T_\varepsilon\|_{\mathscr{L}(L^1(\mathscr{M});L^1(\mathscr{M}))} \le C, \quad \text{for every } \varepsilon > 0. \tag{6.2.29}
$$

Estimate (6.2.29) follows from (6.2.13), which yields

$$\begin{aligned}
\|T_\varepsilon(f)\|_{L^1} &\leq \int_{\mathscr{M}} \int_{\mathscr{M}} K(x,\varepsilon)\rho(x,\varepsilon,y)|f(y)|\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x) \\
&\leq 2C_{\mathscr{M}} \int_{\mathscr{M}} \frac{1}{\mu(B_\varepsilon(x))} \int_{B_\varepsilon(x)} |f(y)|\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x) \\
&= 2C_{\mathscr{M}} \int_{\mathscr{M}} |f(y)| \int_{B_\varepsilon(y)} \frac{1}{\mu(B_\varepsilon(x))}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \\
&\leq 2C_{\mathscr{M}} \int_{\mathscr{M}} |f(y)| \int_{B_\varepsilon(y)} \frac{1}{\mu(B_\varepsilon(x))} \frac{\mu(B_{2\varepsilon}(x))}{\mu(B_\varepsilon(y))}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \\
&\leq 2(C_{\mathscr{M}})^2 \int_{\mathscr{M}} |f(y)| \int_{B_\varepsilon(y)} \frac{1}{\mu(B_\varepsilon(y))}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) = 2(C_{\mathscr{M}})^2 \|f\|_{L^1},
\end{aligned}$$

where we have used the obvious inclusion $B_\varepsilon(y) \subset B_{2\varepsilon}(x)$ and the assumption (6.2.1). □

6.2.2 Homotopy classes of VMO$(\mathscr{M};\mathscr{N})$

If $\mathscr{N} \subset \mathbb{R}^\nu$, we naturally define

$$\mathrm{VMO}(\mathscr{M};\mathscr{N}) = \{f \in \mathrm{VMO}(\mathscr{M};\mathbb{R}^\nu) : f(x) \in \mathscr{N}, \text{ for a.e. } x \in \mathscr{M}\},$$

and similarly for BMO$(\mathscr{M};\mathscr{N})$.

We note that these definitions do not depend on the choice of the embedding $\mathscr{N} \subset \mathbb{R}^\nu$. Indeed, since $\mathscr{N}$ is compact, if $\Phi_j : \mathscr{N} \to \mathscr{N}_j \subset \mathbb{R}^{\nu_j}$, $j = 1, 2$, are isometric embeddings, then the geodesic and Euclidean distance on each $\mathscr{N}_j$ are equivalent, and thus the transition map $\Phi = \Phi_2 \circ \Phi_1^{-1} : \mathscr{N}_1 \to \mathscr{N}_2$ is bi-Lipschitz. The independence of VMO$(\mathscr{M};\mathscr{N})$ or BMO$(\mathscr{M};\mathscr{N})$ on the choice of the embedding is then clear, from (6.2.4).

In view of the above, *from now on, we assume that $\mathscr{N}$ is a smooth closed manifold embedded in $\mathbb{R}^\nu$. We also recall that we assume that $\mathscr{M}$ is compact and satisfies the doubling condition* (6.2.1).

We first note the following simple result.

Lemma 6.2.9. *For every integrable map $f : \mathscr{M} \to \mathscr{N}$, we have*

$$\operatorname{dist}(f_\varepsilon(x), \mathscr{N}) \leq 2C_{\mathscr{M}} M_\varepsilon(f), \quad \textit{for every } x \in \mathscr{M}, \textit{for every } \varepsilon > 0. \tag{6.2.30}$$

Proof. For every $y \in \mathscr{M}$, we have $\operatorname{dist}(f_\varepsilon(x), \mathscr{N}) \le |f_\varepsilon(x) - f(y)|$, so that (using (6.2.13))

$$
\begin{aligned}
\operatorname{dist}(f_\varepsilon(x), \mathscr{N}) &\le \fint_{B_\varepsilon(x)} |f(y) - f_\varepsilon(x)| \, \mathrm{d}\mu(y) \\
&\le \fint_{B_\varepsilon(x)} \int_{B_\varepsilon(x)} K(x, \varepsilon)\rho(x, \varepsilon, z)|f(y) - f(z)| \, \mathrm{d}\mu(z) \, \mathrm{d}\mu(y) \\
&\le 2C_{\mathscr{M}} \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |f(y) - f(z)| \, \mathrm{d}\mu(z) \, \mathrm{d}\mu(y) \le 2C_{\mathscr{M}} M_\varepsilon(f). \qquad \square
\end{aligned}
$$

We next recall the existence, if $\iota > 0$ is chosen sufficiently small, of a smooth retraction Π onto $\mathscr{N}$, defined on the tubular neighborhood $\mathscr{N}_\iota = \mathscr{N} + B^\nu_\iota$; see Proposition 1.4.2 in the introduction. In what follows, ι is implicitly assumed to be sufficiently small such that Π is well-defined and has bounded derivatives on $\mathscr{N}_\iota$.

By (6.2.30) and (6.2.16), for each $f \in \mathrm{VMO}(\mathscr{M}; \mathscr{N})$, there exists some $\varepsilon_1 = \varepsilon_1(f)$ such that

$$
f_\varepsilon(x) \in \mathscr{N}_\iota, \quad \text{for every } x \in \mathscr{M}, \text{ for every } 0 < \varepsilon \le \varepsilon_1. \tag{6.2.31}
$$

Therefore, if we set

$$
f^\varepsilon = \Pi \circ f_\varepsilon \colon \mathscr{M} \to \mathscr{N}, \tag{6.2.32}
$$

then f^ε is well-defined, for every $0 < \varepsilon \le \varepsilon_1$. By Lemma 6.2.4, if (6.2.1) holds, then the mapping

$$
(0, \varepsilon_1] \ni \varepsilon \mapsto f^\varepsilon \in C(\mathscr{M}; \mathscr{N}) \tag{6.2.33}
$$

is continuous, and therefore the following definition makes sense.

Definition 6.2.10. *For $f \in \mathrm{VMO}(\mathscr{M}; \mathscr{N})$, we define the homotopy class $[f]$ of f by $[f] = [f^\varepsilon]$ for small ε, i.e.,*

$$
[f] = \{h \in C(\mathscr{M}; \mathscr{N}) \colon h \sim f^\varepsilon \text{ for some } \ \varepsilon \le \varepsilon_1\}. \tag{6.2.34}
$$

Two maps $f, g \in \mathrm{VMO}(\mathscr{M}; \mathscr{N})$ are homotopic (and this is denoted $f \sim g$) if $[f] = [g]$.

We first note that (by (6.2.33)), in (6.2.34), it is equivalent to ask that $h \sim f^\varepsilon$ for *some* ε or *each* ε. We next note that, when f is continuous, $[f]$ is the classical homotopy class of f. Indeed, in this case $f_\varepsilon \to f$, and therefore $f^\varepsilon \to f$, uniformly as $\varepsilon \to 0$, so that the claim follows from the stability of the homotopy classes.

We next prove the fundamental fact that the homotopy class is stable under $\mathrm{BMO} \cap L^1$

convergence (analogue of [BN95, Theorem 1]).

Proposition 6.2.11. *Let $f \in \mathrm{VMO}(\mathscr{M};\mathscr{N})$. Let $\varepsilon_2 = \varepsilon_2(f) \leq \varepsilon_0$ be such that $M_{\varepsilon_2}(f) \leq \iota/(8C_{\mathscr{M}})$. Then, with C_r as in (6.2.2), we have*

$$[g \in \mathrm{VMO}(\mathscr{M};\mathscr{N}), |g-f|_{\mathrm{BMO}} \leq \iota/(8C_{\mathscr{M}}), \|f-g\|_1 \leq \iota C_{\varepsilon_2}/(4C_{\mathscr{M}})] \implies [g^\varepsilon \sim f^\varepsilon, 0 < \varepsilon \leq \varepsilon_2]. \tag{6.2.35}$$

In particular, under the assumptions of (6.2.35), we have $g \sim f$.

Corollary 6.2.12. *If $(f_n)_{n\in\mathbb{N}} \subset \mathrm{VMO}(\mathscr{M};\mathscr{N})$, $f \in \mathrm{VMO}(\mathscr{M};\mathscr{N})$, and $f_n \to f$ in $\mathrm{BMO} \cap L^1$, then, for large n, $f_n \sim f$.*

Proof of Proposition 6.2.11. Let g satisfy the assumptions of (6.2.35). By (6.2.18), we have $M_{\varepsilon_2}(g) \leq \iota/(4C_{\mathscr{M}})$, for every $0 < \varepsilon \leq \varepsilon_2$, and thus (by (6.2.30))

$$f_\varepsilon(x),\ g_\varepsilon(x) \in \mathscr{N}_{\iota/2}, \quad \text{for every } x \in \mathscr{M}, \text{ for every } 0 < \varepsilon \leq \varepsilon_2. \tag{6.2.36}$$

In order to complete the proof, we prove that $g^{\varepsilon_2} \sim f^{\varepsilon_2}$. For this purpose, it suffices to establish the estimate

$$\|g_{\varepsilon_2} - f_{\varepsilon_2}\|_{L^\infty} \leq \iota/2. \tag{6.2.37}$$

Indeed, granted (6.2.37), we have, thanks to (6.2.36),

$$(1-t)f_{\varepsilon_2}(x) + t g_{\varepsilon_2}(x) = f_{\varepsilon_2}(x) + t(g_{\varepsilon_2}(x) - f_{\varepsilon_2}(x)) \in \mathscr{N}_\iota, \quad \text{for every } x \in \mathscr{M}, \text{ for every } 0 \leq t \leq 1,$$

and thus $[0,1] \ni t \mapsto \Pi((1-t)f_{\varepsilon_2} + t g_{\varepsilon_2})$ is a homotopy between f^{ε_2} and g^{ε_2}.

But, we note that (6.2.37) follows, under the assumptions of (6.2.35), from

$$|g_{\varepsilon_2}(x) - f_{\varepsilon_2}(x)| \leq \frac{2C_{\mathscr{M}}}{\mu(B_{\varepsilon_2}(x))}\|g-f\|_{L^1} \leq \frac{2C_{\mathscr{M}}}{C_{\varepsilon_2}}\|g-f\|_{L^1} \leq \iota/2,$$

where we have used (6.2.13). □

Although we will not use the next result in what follows, we state it since it gives some insight concerning Definition 6.2.10.

Lemma 6.2.13. *For f, $g \in \mathrm{VMO}(\mathscr{M};\mathscr{N})$, we have $f \sim g$ if and only if there exists $F \in C([0,1];(\mathrm{VMO}\cap L^1)(\mathscr{M};\mathscr{N}))$ such that $F_0 = f$ and $F_1 = g$.*

Here, we recall that we use the standard notation $F_t = F(t,\cdot)$.

Proof. If $f \sim g$, by definition, there exists some sufficiently small $\overline{\varepsilon}$ such that $f^{\varepsilon} \sim g^{\varepsilon}$ for every $\varepsilon \leq \overline{\varepsilon}$, which implies that there exists a continuous map $H\colon [0,1] \to C(\mathscr{M};\mathscr{N})$ such that $H_0 = f^{\overline{\varepsilon}}$ and $H_1 = g^{\overline{\varepsilon}}$. Then we define F by

$$F_t = \begin{cases} f, & \text{if } t = 0, \\ f^t, & \text{if } 0 < t \leq \overline{\varepsilon}, \\ H_{(t-\overline{\varepsilon})/(1-2\overline{\varepsilon})}, & \text{if } \overline{\varepsilon} \leq t \leq 1 - \overline{\varepsilon}, \\ g^{1-t}, & \text{if } 1 - \overline{\varepsilon} \leq t < 1, \\ g, & \text{if } t = 1. \end{cases}$$

Since $C(\mathscr{M};\mathscr{N}) \hookrightarrow (\mathrm{VMO} \cap L^1)(\mathscr{M};\mathscr{N})$, $t \mapsto F_t$ belongs to $F \in C((0,1);(\mathrm{VMO} \cap L^1)(\mathscr{M};\mathscr{N}))$. In order to prove the continuity of F on $[0,1]$ and complete the proof of the direct implication, it therefore suffices to check the continuity at $t = 0$ and $t = 1$. For this purpose, we rely on Corollary 6.2.6, Lemma 6.2.7, and the fact that the superposition with Lipschitz functions is continuous in VMO (see [BN95, Lemma A.8]).

For the reverse implication, by Proposition 6.2.11, the map $t \mapsto [F_t]$ is locally constant. By a standard argument, it is constant, whence the conclusion. □

A final result in this section concerns maps such that $|f|_{\mathrm{BMO}}$ is sufficiently small.

Proposition 6.2.14. *There exists some constant $C > 0$ depending on $\mathscr{M}$ and $\mathscr{N}$ such that*

$$[f \in \mathrm{VMO}(\mathscr{M};\mathscr{N}), |f|_{\mathrm{BMO}} \leq C] \implies f \sim \xi \text{ for some point } \xi \in \mathscr{N}. \tag{6.2.38}$$

If, in addition, $\mathscr{N}$ is connected, then (6.2.38) holds for any $\xi \in \mathscr{N}$.

Proof. We may assume that $\varepsilon_0 = \operatorname{diam} \mathscr{M}$ (see Corollary 6.2.2). Let $f \in \mathrm{VMO}(\mathscr{M};\mathscr{N})$. Since $\mathscr{M} = B_{\varepsilon_0}(x)$, for every $x \in \mathscr{M}$, there exists some $z = z(f) \in \mathscr{M}$ such that

$$\fint_{\mathscr{M}} |f(y) - f(z)| \,\mathrm{d}\mu(y) \leq |f|_{\mathrm{BMO}}. \tag{6.2.39}$$

Set $\xi = f(z) \in \mathscr{N}$. From Proposition 6.2.11 (with the constant map ξ playing the role of f and $\varepsilon_2 = \varepsilon_0 = \operatorname{diam} \mathscr{M}$) and (6.2.39), we find that (6.2.38) holds, provided that $C \leq \min\left\{\frac{\iota}{8C_{\mathscr{M}}}, \frac{C_{\varepsilon_0}\iota\mu(\mathscr{M})}{4C_{\mathscr{M}}}\right\}$. □

6.3 Integral invariants for VMO maps to manifolds

In this section, again with no claim of originality, we assume that: (a) $\mathscr{M}$ is a Lipschitz d-dimensional manifold embedded into some $\mathbb{R}^N$, endowed with a finite bi-Lipschitz chart structure, considered as a metric subspace of $\mathbb{R}^N$ and endowed with the natural measure, i.e., the d-dimensional Hausdorff measure $\mathscr{H}^d$; (b) $\mathscr{N}$ is a closed smooth manifold embedded into some $\mathbb{R}^\nu$; (c) ω is a smooth *closed* d-form on $\mathscr{N}$. (For $\mathscr{M}$, the prototypical example we have in mind is $\mathscr{M} = \partial C^{d+1}$, with C^{d+1} a cube in $\mathbb{R}^{d+1}$.) The main objective here is to give, when $\mathscr{M}$ is compact, a robust meaning to $\int_{\mathscr{M}} f^\sharp \omega$ when $f \in \mathrm{VMO}(\mathscr{M}; \mathscr{N})$.

To be more specific, the instrumental definition of Lipschitz manifolds we adopt here is the following.

Definition 6.3.1. *A d-dimensional finite chart structure on $\mathscr{M} \subset \mathbb{R}^N$ is a finite family $\{(U_i, V_i, \varphi_i)\}_{i \in I}$ such that:*

(i) U_i is open in $\mathscr{M} \subset \mathbb{R}^N$, for each $i \in I$, and $\bigcup_{i \in I} U_i$ is a cover of $\mathscr{M}$;

(ii) V_i is an open subset of $\mathbb{R}^d$, for each $i \in I$;

(iii) $\varphi_i \colon V_i \to U_i$ is bi-Lipschitz, for each $i \in I$.

A *(d-dimensional) Lipschitz manifold* is a set $\mathscr{M}$ embedded into some $\mathbb{R}^N$ and endowed with a d-dimensional finite chart structure (in the sense of Definition 6.3.1).

Considering a *finite* chart structure is a matter of convenience. As we will see, working with Lipschitz maps requires excluding exceptional null sets, and we wanted to avoid working with infinite unions of null sets. In practice, $\mathscr{M}$ will most of the time be compact, so that considering a finite chart structure is not a real limitation. Another not so common feature is the *bi-Lipschitz character of the φ_i's* (this condition is clearly satisfied, at least locally, in the smooth case). This is also a matter of convenience, for avoiding using the decomposition of rectifiable sets as images of bi-Lipschitz maps (see, e.g., [Fed69, Lemma 3.2.18]).

Smooth closed manifolds are examples of such $\mathscr{M}$'s. More generally, if $\mathscr{M}$ is bi-Lipschitz homeomorphic with some smooth closed manifold $\mathscr{M}'$, then $\mathscr{M}'$ naturally induces a chart structure on $\mathscr{M}$. This includes, as special cases, ∂C^{d+1}, and more generally, ∂B, where B is a ball for some norm in $\mathbb{R}^{d+1}$, and even more generally, boundaries of convex bodies in $\mathbb{R}^{d+1}$. Indeed, such boundaries are bi-Lipschitz homeomorphic with the Euclidean unit sphere $\mathbb{S}^d$. (See, e.g., Section 6.3.4 below for more details.)

This section is organized as follows. First, we prove, in Section 6.3.1, that $\mathscr{M}$ as above, when compact and endowed with the natural distance and measure, fits into the framework developed in Section 6.2.1. Next, in Sections 6.3.2–6.3.6, we carefully adapt notions as the tangent space, the differential, and the calculus with forms (exterior calculus, pullback, integration on oriented manifolds) to the context of Lipschitz manifolds. Since our final purpose is to establish integral estimates associated with such forms, we adopt an analytic point of view, working mainly in local coordinates. While consistent with the smooth case, this approach has the advantage of making obvious the main properties of the calculus with forms. Finally, in Section 6.3.7, which is at the heart of this part, we define $\int_{\mathscr{M}} f^\sharp\omega$ when $\mathscr{M}$ is compact, $f \in \mathrm{VMO}(\mathscr{M};\mathscr{N})$, and ω is a closed smooth d-form on $\mathscr{N}$, and prove that this quantity is a homotopical invariant. In Section 6.3.8, we consider the special case of $W^{1,d}$ maps and prove that, as expected, in this case $\int_{\mathscr{M}} f^\sharp\omega$ is a genuine integral.

Although most of the results we establish in Sections 6.3.2–6.3.6 can be derived from more general advanced assertions from geometric measure theory (we have in mind in particular the analysis on rectifiable sets and on finite perimeter sets, as in H. Federer [Fed69, Section 3.2], and the homological integration in [Fed69, Chapter 4]), we have opted for a low tech and essentially self-contained exposition that does not require any knowledge of geometric measure theory.

6.3.1 Compact Lipschitz manifolds are doubling metric measure spaces

In this short section, $\mathscr{M}$ is compact and is endowed with a finite chart structure in the sense of Definition 6.3.1. We establish the following result.

Lemma 6.3.2. *We endow $\mathscr{M}$ with the Euclidean distance* (*or any distance induced by a norm on $\mathbb{R}^m$*) *and with the Hausdorff measure $\mathscr{H}^d$. Then $\mathscr{M}$ satisfies the doubling condition* (6.2.1)*.*

If $\mathscr{M}$ is connected, then the same holds for the geodesic distance.

Proof. Since all the above distances are equivalent to the Euclidean distance on $\mathscr{M}$ (for the geodesic distance, this follows from Definition 6.3.1 (iii)), it suffices to consider the Euclidean distance $|\cdot|$. Let $0 < K_1 \le K_2 < +\infty$ be such that

$$K_1|v-w| \le |\varphi_i(v)-\varphi_i(w)| \le K_2|v-w|, \quad \text{for each } i, \text{ for every } v, w \in V_i. \tag{6.3.1}$$

We claim that, if $B \subset U_i$ is a Borel set, then

$$(K_1)^d\mathscr{H}^d(\varphi_i^{-1}(B)) \le \mathscr{H}^d(B) \le (K_2)^d\mathscr{H}^d(\varphi_i^{-1}(B)). \tag{6.3.2}$$

Indeed, (6.3.2) clearly follows from: (i) the fact that a K-Lipschitz map $\varphi\colon A \subset \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ can be extended to a K-Lipschitz map to the whole $\mathbb{R}^{n_1}$ (Kirszbraun's theorem); (ii) the fact that, if $\varphi\colon \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ is K-Lipschitz, then

$$\mathscr{H}^l(\varphi(B)) \le K^l \mathscr{H}^l(B), \quad \text{for every } l > 0, \text{ for every Borel set } B \subset \mathbb{R}^{n_1}.$$

Let r_0 be such that for every $x \in \mathscr{M}$, there exists some i such that $B_{r_0}(x) \subset U_i$. Let $x \in \mathscr{M}$. If $r \le r_0$ and i are such that $B_r(x) \subset U_i$, we write $x = \varphi_i(v)$ for some $v \in V_i$. By (6.3.1), we have

$$B_{r/K_2}(v) \subset \varphi_i^{-1}(B_r(x)) \subset B_{r/K_1}(v). \tag{6.3.3}$$

Combining (6.3.2) and (6.3.3), we find that

$$\mathscr{H}^d(B_r(x)) \sim r^d, \quad \text{for every } 0 < r \le r_0, \text{ for every } x \in \mathscr{M}. \tag{6.3.4}$$

We conclude via (6.3.4) and (6.2.3). □

From now on, any compact Lipschitz d-manifold is implicitly assumed to be endowed with the Euclidean distance and the d-dimensional Hausdorff measure.

6.3.2 Tangent spaces on Lipschitz manifolds

We are here in the setting of Definition 6.3.1 and $\mathscr{M}$ need not be compact. Coordinates of points in $\mathbb{R}^d$ and $\mathbb{R}^N$ appear as superscripts, e.g., $v = (v^1, \dots, v^d)$. The differential of a map φ at v is denoted $\mathrm{D}\varphi(v)$. The canonical basis in $\mathbb{R}^d$ is denoted $\{e_1, \dots, e_d\}$.

For every $i \in I$ and almost every $v \in V_i$, φ_i is differentiable at v (by Rademacher's theorem). If $x = \varphi_i(v)$ for such v, then we set

$$\left.\frac{\partial}{\partial v^\ell}\right|_x = \frac{\partial \varphi_i}{\partial v^\ell}(v) = \mathrm{D}\varphi_i(v)[e_\ell] = \text{ (in short) } \frac{\partial}{\partial v^\ell} \text{ or } \frac{\partial}{\partial v_i^\ell}, \ \ell = 1, \dots, d, \tag{6.3.5}$$

$$T_x\mathscr{M} = \mathrm{D}\varphi_i(v)[\mathbb{R}^d] = \operatorname{span}\left\{ \left.\frac{\partial}{\partial v^\ell}\right|_x : \ell = 1, \dots, d \right\}. \tag{6.3.6}$$

We first note that the above definitions are consistent with the ones for differentiable manifolds. We next check that $T_x\mathscr{M}$ enjoys two basic expected properties.

Lemma 6.3.3. *We have* $\dim T_x\mathscr{M} = d$.

Lemma 6.3.4. *The definition of $T_x\mathscr{M}$ does not depend on i.*

Proof of Lemma 6.3.3. We have to prove that $\mathrm{D}\varphi_i(v)$ is one-to-one. Let $K_1 > 0$ be such

that

$$|\varphi_i(v) - \varphi_i(w)| \geq K_1|v - w|, \quad \text{for every } w \in V_i. \tag{6.3.7}$$

By (6.3.7), we have

$$|D\varphi_i(v)[\xi]| = \lim_{t\to 0}\left|\frac{\varphi_i(v+t\xi) - \varphi_i(v)}{t}\right| \geq K_1|\xi|, \quad \text{for every } \xi \in \mathbb{R}^d,$$

whence the conclusion. □

Proof of Lemma 6.3.4. Assume that $x = \varphi_i(v_i) = \varphi_j(v_j)$, with φ_i, respectively φ_j, differentiable at v_i, respectively v_j. It suffices to prove that $D\varphi_i(v_i)[\mathbb{R}^d]$ and $D\varphi_j(v_j)[\mathbb{R}^d]$ have the same unit sphere. By the proof of Lemma 6.3.3, $D\varphi_i(v_i)$ and $D\varphi_j(v_j)$ are one-to-one. In view of Definition 6.3.1, the conclusion of the lemma follows from the following claim.

Claim. Let $V \subset \mathbb{R}^d$ be an open set. Let $\varphi\colon V \to \varphi(V) \subset \mathbb{R}^N$ be such that (i) $0 \in V$ and $\varphi(0) = 0$; (ii) φ is differentiable at the origin; (iii) $D\varphi(0)$ is one-to-one; and (iv) φ is a homeomorphism. Then, for w a unit vector of $\mathbb{R}^N$, we have

$$\begin{aligned} &w \in D\varphi(0)[\mathbb{R}^d] \quad \text{if and only if} \\ &\text{there exists } (x_n)_{n\in\mathbb{N}} \text{ in } \varphi(V)\setminus\{0\} \text{ such that } x_n \to 0 \text{ and } \frac{x_n}{|x_n|} \to w. \end{aligned} \tag{6.3.8}$$

To establish the claim, let first w be a unit vector in $D\varphi(0)[\mathbb{R}^d]$. Let $\xi \in \mathbb{R}^d$ be such that $D\varphi(0)[\mathbb{R}^d] = w$. Then, for large n, $x_n = \varphi(n^{-1}\xi)$ belongs to $\varphi(V)\setminus\{0\}$ and satisfies $x_n \to 0$ and $\dfrac{x_n}{|x_n|} \to w$. (Here, we do not use the assumptions (iii) and (iv).)

For the reverse implication, let $(x_n)_{n\in\mathbb{N}}$ be as in (6.3.8). Write $x_n = \varphi(v_n)$, with $v_n \in V\setminus\{0\}$. By the assumption (iv), we have $v_n \to 0$. Write $v_n = t_n\xi_n$, with $t_n > 0$, $\xi_n \in \mathbb{R}^d$, $|\xi_n| = 1$, $t_n \to 0$. Up to a subsequence, we may assume that $\xi_n \to \xi$. By the assumption (iii), we have $D\varphi(0)[\xi] \neq 0$, and then one easily sees that

$$w = \lim_{n\to+\infty}\frac{\varphi(t_n\xi_n)}{|\varphi(t_n\xi_n)|} = \frac{D\varphi(0)[\xi]}{|D\varphi(0)[\xi]|} = D\varphi(0)[\xi/|D\varphi(0)[\xi]|]. \qquad \square$$

In what follows, we implicitly consider only *regular* points $x \in \mathscr{M}$, i.e., points x such that, if $x = \varphi_i(v_i)$ for some i, then φ_i is differentiable at v_i. By the above, the complement of the regular points is an $\mathscr{H}^d$-null set, and the tangent space at any regular point $x \in U_i$ is expressed via (6.3.5)–(6.3.6).

Remark 6.3.5. A digression about measurability issues. Given a locally Lipschitz function $g\colon V \to \mathbb{R}$, where V is an open set in $\mathbb{R}^d$, the exceptional set A of points where g is not differentiable is a Borel set. Moreover, the gradient (and thus the differential) $V \setminus A \ni x \mapsto \nabla g(x) \in \mathbb{R}^d$ is a Borel function. Both these properties are well-known to experts, but we could not find a reference. They may be derived, for example, by following the proof of Rademacher's theorem (see, e.g., L. C. Evans and R. F. Gariepy [EG15, Section 3.1]), which implicitly contains explicit formulas for A and for ∇g allowing to check their Borel measurability.

In what follows, we do not discuss anymore measurability issues but, following this remark, it is easy to prove that all the forms and functions we construct below are Borel measurable and defined up to an $\mathscr{H}^d$-null Borel set. □

6.3.3 Lipschitz maps on $\mathscr{M}$: differential and pullback of forms

Here, we are again in the setting of Definition 6.3.1 and $\mathscr{M}$ need not be compact. We consider (locally) Lipschitz maps defined on $\mathscr{M}$, since this setting is sufficient for most of the applications we have in mind (see, however, Section 6.3.8 for $W^{1,d}$ maps), but with more effort some of the results below can be extended to approximately differentiable maps.

Given a locally Lipschitz function $f\colon \mathscr{M} \to \mathbb{R}$, we define, for $\mathscr{H}^d$-a.e. $x \in \mathscr{M}$, $\mathrm{d}_x f$ as follows.

Definition 6.3.6. *Let $x = \varphi_i(v) \in \mathscr{M}$ be a regular point such that $f \circ \varphi_i$ is differentiable at $v \in V_i$. We define*

$$\mathrm{d}_x f\colon T_x\mathscr{M} \to \mathbb{R},\ \mathrm{d}_x f(\mathrm{D}\varphi_i(v)[\xi]) = \mathrm{D}(f \circ \varphi_i)(v)[\xi], \quad \textit{for every } \xi \in \mathbb{R}^d. \tag{6.3.9}$$

Similarly when $f\colon \mathscr{M} \to \mathbb{R}^\nu$.

We note that the above definition is consistent with the one for smooth manifolds and, by Rademacher's theorem, $\mathrm{d}_x f$ is defined except on an $\mathscr{H}^d$-null set. (This null set depends on f.)

We first check that the definition is correct, in the sense that it is independent of the chart. This is a straightforward consequence of the chain rule combined with the following result.

Lemma 6.3.7. *Let $x = \varphi_i(v_i) = \varphi_j(v_j) \in \mathscr{M}$ be a regular point. Let $W_j = \varphi_j^{-1}(U_i \cap U_j)$ and $W_i = \varphi_i^{-1}(U_i \cap U_j)$. Then,*

$$\varphi = \varphi_i^{-1} \circ \varphi_j\colon W_j \to W_i$$

is differentiable at v_j.

Proof. With no loss of generality, we may assume that $U_i = U_j$, $v_i = v_j = 0$, and $\varphi_i(0) = \varphi_j(0) = 0$. By Lemmas 6.3.3 and 6.3.4, there exists a unique (linear, bijective) map $A\colon \mathbb{R}^d \to \mathbb{R}^d$ such that

$$D\varphi_j(0)[a] = D\varphi_i(0)[Aa], \quad \text{for every } a \in \mathbb{R}^d. \tag{6.3.10}$$

For $w \in V_j$, let $y = \varphi_i^{-1}(\varphi_j(w)) \in V_i$. The conclusion of the lemma follows from the equality

$$y = Aw + o(|w|) \text{ as } w \to 0, \tag{6.3.11}$$

that we next prove.

By the assumption (iii) in Definition 6.3.1, we have

$$|y| \sim |w| \text{ as } w \to 0. \tag{6.3.12}$$

Next, using: (i) (6.3.10); (ii) the equation $\varphi_j(w) = \varphi_i(y)$ under the self-explaining form $D\varphi_j(0)[w] + o(|w|) = D\varphi_i(0)[y] + o(|y|)$; and (iii) the equivalence (6.3.12), we find that

$$\begin{aligned} D\varphi_i(0)[Aa] + o(|w|) = D\varphi_j(0)[w] + o(|w|) &= D\varphi_i(0)[y] + o(|y|) \\ &= D\varphi_i(0)[y] + o(|w|). \end{aligned} \tag{6.3.13}$$

We obtain (6.3.11) from (6.3.13), (6.3.12), and the fact that $D\varphi_i(0)$ is one-to-one. □

Remark 6.3.8. Let $\mathscr{N}$ be a C^1-submanifold of $\mathbb{R}^\nu$ and V be an open subset of $\mathbb{R}^d$. Assume that $g\colon V \to \mathbb{R}^\nu$ is differentiable at some point $v \in V$ and that $g(V) \subset \mathscr{N}$. It is straightforward that g, seen as a map from V to $\mathscr{N}$, is differentiable at v, and that $Dg(v)[\mathbb{R}^d] \subset T_{g(v)}\mathscr{N}$.

This consideration leads to the following. Let $f\colon \mathscr{M} \to \mathscr{N}$ be locally Lipschitz. Then, at each regular point $x = \varphi_i(v) \in \mathscr{M}$ such that $f \circ \varphi_i$ is differentiable at v, we have $d_x f\colon T_x\mathscr{M} \to T_{f(x)}\mathscr{N}$. □

As in the smooth case, we associate with $x = \varphi_i(v) \in U_i$ its *coordinates*

$$x^\ell = x_i^\ell = x^\ell(x) = v^\ell, \ \ell = 1, \dots, d.$$

The maps $U_i \ni x \mapsto x^\ell \in \mathbb{R}$ are Lipschitz. Moreover, one sees (from (6.3.9)) that, at each regular point,

$$\mathrm{d}_x x_i^{\ell}\left(\frac{\partial}{\partial v_i^{\ell'}}\bigg|_x\right) = \delta_{\ell\ell'},\ 1 \le \ell, \ell' \le k. \tag{6.3.14}$$

Therefore, when $x \in \mathscr{M}$ is a regular point and $1 \le p \le d$, an alternate form $\eta = \eta(x)$ of order l (in short, an l-form) on $T_x\mathscr{M}$ can be uniquely written as

$$\begin{aligned}\eta(x) &= \sum_{1\le \ell_1<\ell_2<\cdots<\ell_l\le d} \eta^i_{\ell_1,\dots,\ell_l}(x)\,\mathrm{d}_x x_i^{\ell_1} \wedge \cdots \wedge \mathrm{d}_x x_i^{\ell_l} \\ &= \text{(in short)} \sum_{1\le \ell_1<\ell_2<\cdots<\ell_l\le d} \eta_{\ell_1,\dots,\ell_l}(x)\,\mathrm{d}_x x^{\ell_1} \wedge \cdots \wedge \mathrm{d}_x x^{\ell_l}.\end{aligned} \tag{6.3.15}$$

More specifically, for every $f_1, \dots, f_l \in \mathrm{Lip}(\mathscr{M}; \mathscr{N})$ and $\xi_1, \dots, \xi_l \in T_x\mathscr{M}$, we have

$$\mathrm{d}_x f_1 \wedge \cdots \wedge \mathrm{d}_x f_l(\xi_1, \cdots, \xi_l) = \det(\mathrm{d}_x f_i(\xi_j)). \tag{6.3.16}$$

Combining (6.3.16) with (6.3.14) and (6.3.15), this implies that

$$\eta_{\ell_1,\dots,\ell_l}(x) = \eta(x)\left(\frac{\partial}{\partial v^{\ell_1}}\bigg|_x, \dots, \frac{\partial}{\partial v^{\ell_l}}\bigg|_x\right). \tag{6.3.17}$$

Definition 6.3.9. *If $A \subset \mathscr{M}$ is an $\mathscr{H}^d$-null Borel subset such that $\mathscr{M} \setminus A$ consists of regular points, and if, for each $x \in \mathscr{M} \setminus A$, we are given an l-form $\eta(x)$ as in (6.3.15), we say that η is Borel measurable, respectively bounded, if the (locally defined) coefficients $\eta_{\ell_1,\dots,\ell_l}$ are Borel measurable, respectively bounded.*

As in the smooth case, one checks, using (i) Lemma 6.3.7; (ii) the bi-Lipschitz character of the chart system; and (iii) the chain rule, that the Borel measurable or bounded character of η does not depend on the chart.

In what follows, we consider only forms that are implicitly defined up to an $\mathscr{H}^d$-null Borel set $A \subset \mathscr{M}$ as in Definition 6.3.9.

We next define the pullback of forms in the two special cases we are interested in.

Definition 6.3.10. *If η is an l-form on $\mathscr{M}$ defined at a regular point $x = \varphi_i(v) \in \mathscr{M}$, we set*

$$(\varphi_i)^\sharp\eta(v)(\xi_1, \dots, \xi_l) = \eta(x)(\mathrm{D}\varphi_i(v)[\xi_1], \dots, \mathrm{D}\varphi_i(v)[\xi_l], \quad \textit{for every } \xi_1, \dots, \xi_l \in \mathbb{R}^d. \tag{6.3.18}$$

Definition 6.3.11. *Let $\mathscr{N}$ be a C^1-submanifold of $\mathbb{R}^\nu$. Let ω be an l-form on $\mathscr{N}$ (defined everywhere) and $f\colon \mathscr{M} \to \mathscr{N}$ be locally Lipschitz. If $x = \varphi_i(v) \in \mathscr{M}$ is a regular point such* *that $f \circ \varphi_i$ is differentiable at v, we set*

$$f^\sharp\omega(x)(y_1,\dots,y_l) = \omega(f(x))(\mathrm{d}_x f(y_1),\dots,\mathrm{d}_x f(y_l)), \quad \textit{for every } y_1,\dots,y_l \in T_x\mathscr{M}. \tag{6.3.19}$$

We note that (6.3.19) does not depend on i.

Clearly, $(\varphi_i)^\sharp\eta$ is an l-form on V_i, while $f^\sharp\omega$ is an l-form on $\mathscr{M}$. Moreover, assuming f Lipschitz and $\mathscr{N}$ compact, if η (respectively ω) is Borel measurable or bounded, then so is $(\varphi_i)^\sharp\eta$ (respectively $f^\sharp\omega$).

On the other hand, with f and ω as above, one can classically define, at each regular point $x = \varphi_i(v) \in \mathscr{M}$ such that $f \circ \varphi_i$ is differentiable at v,

$$\begin{aligned}(f\circ\varphi_i)^\sharp\omega(v)(\xi_1,\dots,\xi_l) = \omega(f(x))(\mathrm{D}(f\circ\varphi_i)(v)[\xi_1],\dots,\mathrm{D}(f\circ\varphi_i)(v)[\xi_l]),&\\ \text{for every } \xi_1,\dots,\xi_l \in \mathbb{R}^d.&\end{aligned} \tag{6.3.20}$$

Using successively (6.3.18), (6.3.9), and (6.3.20), we find that

$$(\varphi_i)^\sharp(f^\sharp\omega) = (f\circ\varphi_i)^\sharp\omega \quad \mathscr{H}^d\text{-a.e. on } V_i. \tag{6.3.21}$$

Let us note the following obvious consequence of the discussions in this section.

Lemma 6.3.12. *Assume $\mathscr{N}$ compact and $f\colon \mathscr{M} \to \mathscr{N}$ Lipschitz. Let ω be a (everywhere defined) bounded Borel l-form on $\mathscr{N}$. Then $f^\sharp\omega$ is a bounded Borel l-form on $\mathscr{M}$.*

6.3.4 Orientation

Definition 6.3.13. *The finite chart structure in Definition 6.3.1 defines an orientation on $\mathscr{M}$ if, for each i and j, $\det \mathrm{D}(\varphi_j^{-1}\circ\varphi_i)(v) > 0$ for a.e. $v \in \varphi_i^{-1}(U_i \cap U_j)$.*

We say that $\mathscr{M}$ is oriented whenever we are given a chart structure as above.

As in the case of differentiable manifolds, an orientation allows to define, for $\mathscr{H}^d$-a.e. $x \in \mathscr{M}$, the notion of direct basis of $T_x\mathscr{M}$. On the other hand, if $\varphi_i^{-1}(U_i \cap U_j)$ is connected (which is equivalent to requiring that $U_i \cap U_j$ itself is connected, since φ_i is bi-Lipschitz), then the sign of $v \mapsto \det \mathrm{D}(\varphi_j^{-1}\circ\varphi_i)(v)$ is constant almost everywhere on $\varphi_i^{-1}(U_i \cap U_j)$. For this (not so obvious) property, the reader may refer to [Fed69, Corollary 4.1.26].

A basic class of oriented Lipschitz manifolds is given by the bi-Lipschitz images of smooth oriented manifolds.

Example 6.3.14. Assume that $\mathscr{M}'$ is a smooth closed oriented manifold, and that $\mathscr{M} = g(\mathscr{M}')$ for some bi-Lipschitz map $g\colon \mathscr{M}' \to \mathscr{M}$. Then g naturally induces a structure of oriented manifold on $\mathscr{M}$. Indeed, let the orientation of $\mathscr{M}'$ be given by a finite atlas

$\{(U_i', V_i', \varphi_i')\}_{i\in I}$. Then, clearly, $\{(g(U_i'), V_i', g\circ\varphi_i')\}_{i\in I}$ endows $\mathscr{M}$ with a finite chart structure. This structure defines an orientation. Indeed, for every i and j, we find, using the fact that the atlas on $\mathscr{M}'$ defines an orientation, that

$$\det \mathrm{D}((g\circ\varphi_j')^{-1}\circ(g\circ\varphi_i'))(v) = \det \mathrm{D}((\varphi_j')^{-1}\circ\varphi_i')(v) > 0$$

for each $v \in (g\circ\varphi_i')^{-1}(g(U_i')\cap g(U_j')) = (\varphi_i')^{-1}(U_i'\cap U_j')$. □

We now give more insight about the orientation induced in Example 6.3.14. Motivated by the applications we have in mind, we focus on the particular case where $\mathscr{M}'$ is a sphere and $\mathscr{M}$ is the boundary of a convex body (though the same study could be performed, at the cost of more technicality, for the boundary of a Lipschitz open set). This class of examples is sufficiently large to include as a particular instance the case where $\mathscr{M}$ is the boundary of a cube, which will be of crucial importance for us in the sequel.

Example 6.3.15. Let us recall that a convex body in $\mathbb{R}^N$ is a compact convex subset C of $\mathbb{R}^N$ with nonempty interior. Without loss of generality, we may assume that $0 \in \operatorname{int} C$. Consider the *Minkowsky gauge* λ_C associated with C,

$$\lambda_C\colon \mathbb{R}^N \to \mathbb{R}_+, \lambda_C(y) = \inf\left\{t > 0\colon \frac{1}{t}y \in C\right\}, \quad \text{for every } y \in \mathbb{R}^N. \tag{6.3.22}$$

The following properties are well-known (and straightforward):

$$\lambda_C \text{ is positively 1-homogeneous,} \tag{6.3.23}$$

$$\text{when } y \neq 0, \text{ the inf in (6.3.22) is actually a min, and } \frac{1}{\lambda_C(y)}y \in \partial C, \tag{6.3.24}$$

$$\lambda_C \text{ is convex (and thus locally Lipschitz).} \tag{6.3.25}$$

Set

$$\Phi_C\colon \mathbb{R}^N \to \mathbb{R}^N, \Phi_C(y) = \begin{cases} \dfrac{|y|}{\lambda_C(y)}y = \dfrac{1}{\lambda_C(y/|y|)}y, & \text{if } y \neq 0,\\ 0, & \text{if } y = 0,\end{cases} \tag{6.3.26}$$

$$\Psi_C\colon \mathbb{R}^N \to \mathbb{R}^N, \Psi_C(x) = \begin{cases} \lambda_C(x/|x|)x, & \text{if } x \neq 0,\\ 0, & \text{if } x = 0.\end{cases} \tag{6.3.27}$$

It is straightforward (using (6.3.23)–(6.3.24)) that (j) $\Phi_C(\overline{\mathbb{B}^N}) = C$; (jj) $\Phi_C(\mathbb{S}^{N-1}) = \partial C$; and (jjj) Ψ_C is the reciprocal of Φ_C. Moreover, using (i) the definitions (6.3.26)–(6.3.27); (ii) (6.3.25); and (iii) standard properties of products and superpositions of locally Lipschitz maps, we find that Φ_C and Ψ_C are locally Lipschitz. Combining the above, we find that Φ_C is a bi-Lipschitz homeomorphism between $\overline{\mathbb{B}^N}$ and C, whose restriction g to $\mathbb{S}^{N-1}$ is a bi-Lipschitz homeomorphism between $\mathbb{S}^{N-1}$ and ∂C. Thus, ∂C fits Example 6.3.14, with $d = N-1$, $\mathscr{M}' = \mathbb{S}^{N-1}$, and $g = \Phi_{C|\mathbb{S}^{N-1}}$.

Assume that $\mathbb{S}^{N-1}$ is oriented consistently with Stokes' formula on $\mathbb{B}^N$, that is, for every $y = \varphi(v) \in \mathbb{S}^{N-1}$ in the codomain of a chart φ,

$$\text{the } N\text{-tuple } \left(y, \frac{\partial \varphi}{\partial v^1}(v), \ldots, \frac{\partial \varphi}{\partial v^{N-1}}(v)\right) \text{ is a direct basis of } \mathbb{R}^N. \tag{6.3.28}$$

Consider on ∂C the induced parametrization $\psi = g \circ \varphi$. We claim that ψ is consistent with Stokes' formula on C: if $x = \psi(v)$ and ψ is differentiable at v, then

$$\text{the } N\text{-tuple } \left(x, \frac{\partial \psi}{\partial v^1}(v), \ldots, \frac{\partial \psi}{\partial v^{N-1}}(v)\right) \text{ is a direct basis of } \mathbb{R}^N. \tag{6.3.29}$$

Indeed, if we set $t(v) = \dfrac{1}{\lambda_C(\varphi(v))} > 0$ then (j) (by (6.3.26)) $\psi = t\varphi$; (jj) ψ is differentiable at v if and only if t is differentiable at v; and (jjj) condition (6.3.29) is equivalent to

$$\det\left(t\varphi, \frac{\partial t}{\partial v^1}\varphi + t\frac{\partial \varphi}{\partial v^1}, \ldots, \frac{\partial t}{\partial v^{N-1}}\varphi + t\frac{\partial \varphi}{\partial v^{N-1}}\right) > 0 \tag{6.3.30}$$

(where the above determinant is evaluated at v). We complete the proof of claim (6.3.29) by combining (6.3.30) with the fact that (6.3.28) is equivalent to

$$\det\left(\varphi, \frac{\partial \varphi}{\partial v^1}, \ldots, \frac{\partial \varphi}{\partial v^{N-1}}\right) > 0. \qquad \square$$

Example 6.3.16. Let us now consider the special case where C is a cube aligned with the coordinate axes. For simplicity, we let $C = [-1,1]^N$, but the considerations below, in particular the description of the orientation of the faces, do not depend on this specific choice. Clearly, $\lambda_C(y) = |y|_\infty$ and g is smooth in a neighborhood of $y \in \mathbb{S}^{N-1}$ provided $|y^i| \neq |y^j|$ when $i \neq j$. Thus, the procedure described in Example 6.3.15 provides an orientation on ∂C, with parametrizations that are smooth in the interiors of the faces of ∂C. Consider, e.g., the open face

$$F = \{x = (x', 1) \colon x' \in \mathbb{R}^{N-1}, |x'|_\infty < 1\}.$$

Then, clearly, $T_x \partial C = \mathbb{R}^{N-1} \times \{0\}$, for every $x \in F$. Moreover, in view of claim (6.3.29), we have

$$((e_1,0),\dots,(e_{N-1},0)) \in (\mathbb{R}^{N-1}\times\{0\})^{N-1} \text{ is a direct basis of } T_x\partial C \\ \text{if and only if} \quad (-1)^{N-1}\det(e_1,\dots,e_{N-1}) > 0.$$

Similar considerations apply to the other faces (see also Example 6.3.18). □

6.3.5 Integral of forms

In this section, we briefly check that everything goes as expected for the integral of d-forms; this crucially relies on the area formula (instead of the standard change of variables formula). We assume that: (a) $\mathscr{M}$ is a compact d-dimensional Lipschitz manifold oriented by a finite chart structure $\{(U_i,V_i,\varphi_i)\}_{i\in I}$; (b) $\{\xi_i\}_{i\in I}$ is a Lipschitz partition of unity subordinated to the covering $\{U_i\}_{i\in I}$.

Let η be a Borel d-form defined in some Borel subset $\widetilde{U}_i$ of U_i. By (6.3.15), we may uniquely write, for $\mathscr{H}^d$-a.e. $x\in\widetilde{U}_i$,

$$\eta(x) = \alpha_i(x)\,\mathrm{d}_x x_i^1\wedge\cdots\wedge\mathrm{d}_x x_i^d, \tag{6.3.31}$$

with α_i a Borel function.

As in the smooth case, we have the following result.

Lemma 6.3.17. *If η is defined in $U_i\cap U_j$ and α_i is integrable on $U_i\cap U_j$, then α_j is integrable on $U_j\cap U_i$ and (with W_i and W_j as in Lemma 6.3.7)*

$$\int_{W_i}\alpha_i\circ\varphi_i = \int_{W_j}\alpha_j\circ\varphi_j. \tag{6.3.32}$$

Proof. If $\varphi\colon W_j\to W_i$ is the transition map in Lemma 6.3.7, using (i) Lemma 6.3.7; (ii) the chain rule; (iii) the exterior calculus rules; (iv) the fact that $\mathscr{M}$ is oriented; and (v) the area formula (for the last line), we find (as in the smooth case)

$$\alpha_j(\varphi_j(v)) = \alpha_i(\varphi_j(v))\det\mathrm{D}\varphi(v) \quad \text{for } \mathscr{H}^d\text{-a.e. } v\in W_j,$$

and finally

$$\begin{aligned}\int_{W_j}\alpha_j(\varphi_j(v))\,\mathrm{d}v &= \int_{W_j}\alpha_i(\varphi_j(v))\det\mathrm{D}\varphi(v)\,\mathrm{d}v = \int_{W_j}\alpha_i(\varphi_j(v))\,|\det\mathrm{D}\varphi(v)|\,\mathrm{d}v\\ &= \int_{W_j}\alpha_i(\varphi_i(\varphi(v)))\,|\det\mathrm{D}\varphi(v)|\,\mathrm{d}v = \int_{W_i}\alpha_i(\varphi_i(w))\,\mathrm{d}w.\end{aligned}$$ □

Assume next that η is an L^1 d-form on $\mathscr{M}$, in the sense that, for each i, $\alpha_i \circ \varphi_i$ is integrable on V_i. Using Lemma 6.3.17, we see that the definition

$$\int_{\mathscr{M}} \eta = \sum_i \int_{V_i} (\xi_i \alpha_i) \circ \varphi_i = \sum_i \int_{V_i} (\xi_i \circ \varphi_i)(\alpha_i \circ \varphi_i) \tag{6.3.33}$$

is correct, in the sense that it does not depend on the choice of the chart structure, and yields a finite real number. Moreover, this definition is consistent with the one in the classical setting.

Example 6.3.18. Consider the special case where $\mathscr{M}$ is the boundary of a cube C as in Example 6.3.16, say $C = [-1, 1]^{d+1}$. We will establish an explicit formula for the integral of a form on ∂C. Consider the open faces

$$F_{\ell,\pm} = \{(x^1, \dots, x^{\ell-1}, \pm 1, x^{\ell+1}, \dots, x^{d+1}) \in \partial C \colon x^j \in (-1, 1), \text{ for every } j \neq \ell\} \sim (-1, 1)^d,$$

and $F = \bigcup_{\ell,\pm} F_{\ell,\pm}$, so that $\partial C \setminus F$ is an $\mathscr{H}^d$-null set. Consider a sequence $(\zeta_n)_{n\in\mathbb{N}} \subset C^\infty_c(F; [0, 1])$ such that $\zeta_n(x) \to 1$ for $\mathscr{H}^d$-a.e. $x \in \partial C$. If η is an L^1 d-form on ∂C, then, by dominated convergence,

$$\int_{\partial C} \eta = \lim_{n\to+\infty} \int_{\partial C} \zeta_n \eta = \lim_{n\to+\infty} \sum_{\ell,\pm} \int_{F_{\ell,\pm}} \zeta_n \eta \chi_{F_{\ell,\pm}} = \sum_{\ell,\pm} \int_{F_{\ell,\pm}} \eta, \tag{6.3.34}$$

where $F_{\ell,\pm}$ is equipped with the orientation induced by ∂C.

Moreover, in $F_{\ell,\pm}$ (with $\pm$ fixed) we may write $\eta = \alpha_{\ell,\pm} \,\widehat{\mathrm{d}x^\ell}$, with the convention

$$\widehat{\mathrm{d}x^\ell} = \mathrm{d}x^1 \wedge \cdots \mathrm{d}x^{\ell-1} \wedge \mathrm{d}x^{\ell+1} \wedge \cdots \wedge \mathrm{d}x^{d+1} \tag{6.3.35}$$

and $\alpha_{\ell,\pm} \in L^1$.

As explained in Example 6.3.16, $(-e_1, \dots, -e_{\ell-1}, e_{\ell+1}, \dots, e_{d+1})$ is a direct basis of $T_x\partial C$, for every $x \in F_{\ell,+}$ (and a similar formula holds for $F_{\ell,-}$). Combining this with (6.3.34) and (6.3.35), we find that

$$\int_{F_{\ell,+}} \eta = \int_{F_{\ell,+}} \alpha_{\ell,+} \,\widehat{\mathrm{d}x^\ell} = (-1)^{\ell-1} \int_{F_{\ell,+}} \alpha_{\ell,+}.$$

Similarly for $\int_{F_{\ell,-}} \alpha$. Finally, we obtain

$$\int_{\partial C} \eta = \sum_\ell (-1)^{\ell-1} \Big(\int_{(-1,1)^d} \alpha_{\ell,+} \,\widehat{\mathrm{d}x^\ell} - \int_{(-1,1)^d} \alpha_{\ell,-} \,\widehat{\mathrm{d}x^\ell} \Big), \tag{6.3.36}$$

where we have identified $F_{\ell,\pm}$ with $(-1,1)^d$ with the standard orientation. □

Lemma 6.3.19. *Let $f\colon \mathscr{M} \to \mathscr{N}$ be a Lipschitz map. Let ω be a (everywhere defined) bounded Borel d-form on $\mathscr{N}$. Then*

$$\int_{\mathscr{M}} f^\sharp\omega = \sum_i \int_{V_i} (\xi_i \circ \varphi_i)\,(f\circ\varphi_i)^\sharp\omega.$$

Proof. Let $\eta = f^\sharp\omega$ and $(e_1,\dots,e_d)$ be the canonical basis of $\mathbb{R}^d$. In view of the definition (6.3.33), it suffices to check that, for every i, the function α_i associated with $f^\sharp\omega$ as in (6.3.31) satisfies

$$\alpha_i(\varphi_i(v)) = (f\circ\varphi_i)^\sharp\omega(v)(e_1,\dots,e_d) \quad \text{for } \mathscr{H}^d\text{-a.e. } v\in V_i. \tag{6.3.37}$$

At a regular point $x = \varphi_i(v) \in U_i$ such that $f\circ\varphi_i$ is differentiable at v, we have, via (i) (6.3.17); (ii) (6.3.19) and (6.3.5); and (iii) (6.3.20),

$$\begin{aligned}\alpha_i(x) &= (f^\sharp\omega)(x)\left(\frac{\partial}{\partial v_i^1}\Big|_x,\dots,\frac{\partial}{\partial v_i^d}\Big|_x\right)\\ &= \omega(f(x))(\mathrm{D}(f\circ\varphi_i)(v)[e_1],\dots,\mathrm{D}(f\circ\varphi_i)(v)[e_d])\\ &= (f\circ\varphi_i)^\sharp\omega(v)(e_1,\dots e_d),\end{aligned}$$

whence (6.3.37). □

Remark 6.3.20. Let us note a variant of the above considerations and definitions if, instead of $\mathscr{M}$, we consider the product $\widetilde{\mathscr{M}} = \mathscr{M}\times J$ with $J = (a,b) \subset \mathbb{R}$ a non-empty open interval. Clearly, if $\mathscr{M}$ has a (oriented) finite chart structure, then $\widetilde{\mathscr{M}}$ has a natural (oriented) finite chart structure, by setting

$$\widetilde{V}_i = V_i\times J, \widetilde{U}_i = U_i\times J, \widetilde{\varphi}_i(v,t) = (\varphi_i(v),t), \quad \text{for every } v\in V_i \text{ and } t\in J. \tag{6.3.38}$$

Assume that $\mathscr{M}$ is compact and I is bounded. Given a bounded Borel $(d+1)$-form η on $\widetilde{\mathscr{M}}$, we write, in $\widetilde{U}_i$, $\eta(x) = \alpha_i(x,t)\,\mathrm{d}_x x_i^1\wedge\cdots\wedge\mathrm{d}_x x_i^d\wedge\mathrm{d}t$, and naturally set

$$\int_{\widetilde{\mathscr{M}}} \eta = q\sum_i \int_{V_i\times J} (\xi_i\circ\varphi_i)\,(\alpha_i\circ\widetilde{\varphi}_i).$$

This definition is correct, consistent with the case of smooth manifolds, and the analogue of Lemma 6.3.19 holds, i.e., when $F\colon \widetilde{\mathscr{M}} \to \mathscr{N}$ is Lipschitz and λ is a (everywhere defined) bounded Borel $(d+1)$-form on $\mathscr{N}$, we have

$$\int_{\widetilde{\mathscr{M}}} F^\sharp \lambda = \sum_i \int_{V_i \times J} (\xi_i \circ \varphi_i)(F \circ \widetilde{\varphi}_i)^\sharp \lambda. \qquad \square \quad (6.3.39)$$

Remark 6.3.21. For further use, we note the following identity. Consider the setting in Remark 6.3.20 and assume that I is bounded. Let $\varphi\colon W_j \to W_i$ be as in Lemma 6.3.7, and set $\widetilde{\varphi}(v,t) = (\varphi(v),t)$, for every $v \in W_j$, for every $t \in \mathbb{R}$. Let ω be a (everywhere defined) bounded Borel d-form on $\mathscr{N}$. Let $f\colon V_i \times J \to \mathbb{R}$ be a bounded Borel function supported in $W_i \times J$. Let $g\colon V_i \times J \to \mathbb{R}$, respectively $G\colon V_i \times J \to \mathscr{N}$, be Lipschitz maps. Then,

$$\int_{V_i \times J} f \, \mathrm{d}g \wedge G^\sharp \omega = \int_{V_j \times J} f \circ \widetilde{\varphi}(\mathrm{d}(g \circ \widetilde{\varphi})) \wedge (G \circ \widetilde{\varphi})^\sharp \omega. \qquad (6.3.40)$$

Formula (6.3.40) is obtained by repeating the proof of (6.3.32) and using the exterior differential calculus rules for Lipschitz maps (see, e.g., the proof of (6.3.37)). $\square$

We next extend the definition of $\int_{\mathscr{M}} f^\sharp \omega$ to $W^{1,d}$ maps. Clearly, the definition of $W^{1,p}(\mathscr{M})$ adapted to our setting is the following: a map $f\colon \mathscr{M} \to \mathbb{R}$ belongs to $W^{1,p}(\mathscr{M})$ whenever $f \circ \varphi_i \in W^{1,p}(V_i)$ for every i. The next definition is also natural.

Definition 6.3.22. *For almost every regular point $x = \varphi_i(v) \in \mathscr{M}$, we let*

$$\mathrm{d}_x f\colon T_x\mathscr{M} \to \mathbb{R},\ \mathrm{d}_x f(D\varphi_i(v)[\xi]) = D(f \circ \varphi_i)(v)[\xi], \quad \textit{for every } \xi \in \mathbb{R}^d. \qquad (6.3.41)$$

Similarly when $f\colon \mathscr{M} \to \mathbb{R}^\nu$.

It is obvious, by the chain rule, that the above definitions do not depend on the choice of the chart.

Remark 6.3.23. We present a counterpart of Remark 6.3.8 adapted to Sobolev maps. Let $\mathscr{N}$ be a C^1-submanifold of $\mathbb{R}^\nu$ and V be an open subset of $\mathbb{R}^d$. Assume that $g \in W^{1,1}_{\mathrm{loc}}(V;\mathbb{R}^\nu)$ is such that $g(V) \subset \mathscr{N}$. We claim that, for a.e. $v \in V$, $Dg(v)[\mathbb{R}^d] \subset T_{g(v)}\mathscr{N}$. To prove this, one may rely, e.g., on the following argument. Let $(g_n)_{n\in\mathbb{N}} \subset C^\infty(V;\mathbb{R}^\nu)$ converge to g in $W^{1,1}_{\mathrm{loc}}(V)$. Up to extraction of a subsequence, we may further assume that $g_n \to g$ and $Dg_n \to Dg$ almost everywhere. Let Π be as in Proposition 1.4.2, and let $\widetilde{\Pi} \in C^\infty_c(\mathbb{R}^\nu;\mathbb{R}^\nu)$ be such that

$$\widetilde{\Pi}(z) = \Pi(z), \quad \text{for every } z \in \mathscr{N}_{\iota/2}.$$

By the chain rule, the map $\widetilde{\Pi} \circ g_n$ belongs to $W^{1,p}(V; \mathbb{R}^\nu)$, and satisfies

$$D(\widetilde{\Pi} \circ g_n)(v)[\xi] = D\widetilde{\Pi}(g_n(v))[Dg_n(v)[\xi]], \quad \text{for every } v \in V, \text{ for every } \xi \in \mathbb{R}^d.$$

When $n \to +\infty$, we have

$$D(\widetilde{\Pi} \circ g_n)(v)[\xi] \to D\Pi(g(v))[Dg(v)[\xi]] \in T_{g(v)}\mathscr{N}, \text{ for a.e. } v \in V \text{ and every } \xi \in \mathbb{R}^d,$$

where we have used the fact that $\widetilde{\Pi} = \Pi$ near $\mathscr{N}$.

On the other hand, by the continuity of the superposition operator, and up to a further extraction, we may assume that

$$D(\widetilde{\Pi} \circ g_n)(v)[\xi] \to D(\widetilde{\Pi} \circ g)(v)[\xi] = Dg(v)[\xi], \quad \text{for a.e. } v \in V, \text{ for every } \xi \in \mathbb{R}^d,$$

which shows our claim. □

We endow $W^{1,p}(\mathscr{M})$ with the natural norm $f \mapsto \sum_i \|f \circ \varphi_i\|_{W^{1,p}(V_i)}$. It is straightforward that two different chart structures yield equivalent norms.

We can extend the definition of $f^\sharp\omega$ (see Definition 6.3.11) and Lemma 6.3.19 to the case where $f \in W^{1,p}(\mathscr{M}; \mathscr{N})$.

Definition 6.3.24. *Let $\mathscr{N}$ be a C^1-submanifold of $\mathbb{R}^\nu$. Let ω be an l-form on $\mathscr{N}$ (defined everywhere) and $f \in W^{1,1}_{\mathrm{loc}}(\mathscr{M}; \mathscr{N})$. For almost every regular point $x = \varphi_i(v) \in \mathscr{M}$, we let*

$$f^\sharp\omega(x)(y_1, \dots, y_l) = \omega(f(x))(\mathrm{d}_x f(y_1), \dots, \mathrm{d}_x f(y_l)), \qquad \textit{for every } y_1, \dots, y_l \in T_x\mathscr{M}. \tag{6.3.42}$$

We note that the above definition is consistent with Definition 6.3.11 (which involves Lipschitz maps), and does not depend on i.

Similar to (6.3.20), one can define, using (6.3.42), for a.e. regular point $x = \varphi_i(v) \in \mathscr{M}$,

$$(f \circ \varphi_i)^\sharp\omega(v)(\xi_1, \dots, \xi_l) = \omega(f(x))(D(f \circ \varphi_i)(v)[\xi_1], \dots, D(f \circ \varphi_i)(v)[\xi_l]), \qquad \text{for every } \xi_1, \dots, \xi_l \in \mathbb{R}^d.$$

Then we have the analogue of (6.3.21),

$$(\varphi_i)^\sharp(f^\sharp\omega) = (f \circ \varphi_i)^\sharp\omega, \quad \mathscr{H}^d\text{-a.e. on } V_i.$$

Lemma 6.3.25. *For $f \in W^{1,d}(\mathscr{M}; \mathscr{N})$ and ω a (everywhere defined) bounded Borel d-form on*

$\mathscr{N}$, we have

$$\int_{\mathscr{M}} f^\sharp\omega = \sum_i \int_{V_i} (\xi_i \circ \varphi_i)\,(f \circ \varphi_i)^\sharp \omega.$$

Lemma 6.3.25 is obtained by repeating the proof of Lemma 6.3.19.

6.3.6 An adapted Stokes' formula

Throughout this section: (a) $\mathscr{M}$ is a compact d-dimensional Lipschitz manifold oriented by a finite chart structure $\{(U_i, V_i, \varphi_i)\}_{i\in I}$; (b) $J = (a, b)$ is a bounded interval; (c) $\mathscr{N}$ is a closed manifold; (d) ω is a smooth d-form on $\mathscr{N}$. We state and prove a formula in the spirit of the Stokes formula on $\mathscr{M} \times J$. (See Remark 6.3.20 for the integration on $\mathscr{M} \times J$.) For the sake of concision, given a map $F\colon X \times Y \to Z$ and $y \in Y$, we write $F_y = F(\cdot, y)$.

Proposition 6.3.26. *Let $F\colon \mathscr{M} \times [a, b] \to \mathscr{N}$ be a Lipschitz map. Then,*

$$\int_{\mathscr{M}\times(a,b)} F^\sharp(\mathrm{d}\omega) = \int_{\mathscr{M}} (F_b)^\sharp\omega - \int_{\mathscr{M}} (F_a)^\sharp\omega. \tag{6.3.43}$$

Similarly, if $F\colon \mathscr{M} \times [a, b] \to \mathbb{R}^\nu$ is a Lipschitz map and α is a smooth d-form on $\mathbb{R}^\nu$ with bounded coefficients, then

$$\int_{\mathscr{M}\times(a,b)} F^\sharp(\mathrm{d}\alpha) = \int_{\mathscr{M}} (F_b)^\sharp\alpha - \int_{\mathscr{M}} (F_a)^\sharp\alpha. \tag{6.3.44}$$

When both $\mathscr{M}$ and F are smooth, (6.3.43) is a special case of the Stokes formula on $\mathscr{M} \times (a, b)$.

Proof. We only prove (6.3.43), since (6.3.44) follows from a similar argument. In particular, one readily checks that all the ingredients involved in the proof of (6.3.43) below have a valid counterpart when F is $\mathbb{R}^\nu$ and α is a smooth d-form on $\mathbb{R}^\nu$ with bounded coefficients.

Let $\{\xi_i\}_{i\in I}$ be a partition of unity subordinated to the finite chart structure $\{(U_i, V_i, \varphi_i)\}_{i\in I}$ on $\mathscr{M}$. With $J = (a, b)$, we use the notation in (6.3.38).

By Lemma 6.3.19, we have

$$\begin{aligned}\int_{\mathcal{M}}(F_b)^\sharp\omega-\int_{\mathcal{M}}(F_a)^\sharp\omega&=\sum_i\int_{V_i}(\xi_i\circ\varphi_i)(F_b\circ\varphi_i)^\sharp\omega-\sum_i\int_{V_i}(\xi_i\circ\varphi_i)(F_a\circ\varphi_i)^\sharp\omega\\&=\sum_i\int_{V_i}(\xi_i\circ\varphi_i)[(F\circ\widetilde{\varphi}_i)_b]^\sharp\omega\\&\quad-\sum_i\int_{V_i}(\xi_i\circ\varphi_i)[(F\circ\widetilde{\varphi}_i)_a]^\sharp\omega.\end{aligned}\tag{6.3.45}$$

We extend $\xi_i\circ\varphi_i$ and $F\circ\widetilde{\varphi}_i$ to Lipschitz maps G_i and H_i defined on $V_i\times\mathbb{R}$ by letting

$$G_i(v,t)=\xi_i\circ\varphi_i(v),\text{ respectively }H_i(v,t)=\begin{cases}F\circ\widetilde{\varphi}_i(v,t), & \text{if } a\le t\le b,\\ F\circ\widetilde{\varphi}_i(v,a), & \text{if } t\le a,\\ F\circ\widetilde{\varphi}_i(v,b), & \text{if } t\ge b.\end{cases}$$

We next consider open neighborhoods W_i of the support of $\xi_i\circ\varphi_i$ such that $\overline{W_i}\subset V_i$. Finally, we set $G_{i,\varepsilon}=\rho_\varepsilon * G_i$ and $H_{i,\varepsilon}=\rho_\varepsilon * H_i$, where ρ is a standard mollifier in $\mathbb{R}^{d+1}$ and $\varepsilon>0$ is chosen sufficiently small so that (i) $G_{i,\varepsilon}$ and $H_{i,\varepsilon}$ are well-defined and smooth in $W_i\times\mathbb{R}$; and (ii) $G_{i,\varepsilon}$ are supported in $W_i\times\mathbb{R}$.

It is readily seen that (j) $G_{i,\varepsilon}\to G_i$ and $H_{i,\varepsilon}\to H_i$ uniformly as $\varepsilon\to 0$; (jj) $\mathrm{D}G_{i,\varepsilon}(v,t)\to \mathrm{D}G_i(v,t)$ and $\mathrm{D}H_{i,\varepsilon}(v,t)\to\mathrm{D}H_i(v,t)$ for almost every (v,t) as $\varepsilon\to 0$; (jjj) for any $t\notin[a,b]$, $\mathrm{D}H_{i,\varepsilon}(v,t)\to\mathrm{D}H_i(v,t)$ for almost every v as $\varepsilon\to 0$; and (jjjj) $G_{i,\varepsilon}$, $H_{i,\varepsilon}$, $\mathrm{D}G_{i,\varepsilon}$, and $\mathrm{D}H_{i,\varepsilon}$ are uniformly bounded, independently of ε, on $W_i\times\mathbb{R}$.

Fix $c<a<b<d$. Using (i) the fact that $\xi_i\circ\varphi_i$ is compactly supported in W_i; (ii) the fact that $G_{i,\varepsilon}$ does not depend on t; (iii) the divergence theorem for smooth forms; and (iv) the exterior product rules, we find that

$$\begin{aligned}\int_{W_i}G_{i,\varepsilon}\cdot[(H_{i,\varepsilon})_d]^\sharp\omega-\int_{W_i}G_{i,\varepsilon}\cdot[(H_{i,\varepsilon})_c]^\sharp\omega\qquad&\\=\int_{W_i\times(c,d)}&\mathrm{d}[G_{i,\varepsilon}\cdot(H_{i,\varepsilon})^\sharp\omega]\\=\int_{W_i\times(c,d)}&[(\mathrm{d}G_{i,\varepsilon})\wedge(H_{i,\varepsilon})^\sharp\omega+G_{i,\varepsilon}\cdot\mathrm{d}((H_{i,\varepsilon})^\sharp\omega)]\\=\int_{W_i\times(c,d)}&[(\mathrm{d}G_{i,\varepsilon})\wedge(H_{i,\varepsilon})^\sharp\omega+G_{i,\varepsilon}\cdot(H_{i,\varepsilon})^\sharp(\mathrm{d}\omega)].\end{aligned}\tag{6.3.46}$$

Letting $\varepsilon\to 0$ in (6.3.46) and using (j)–(jjjj) above to justify the use of the dominated convergence theorem, we find

$$\int_{W_i} G_i \cdot [(H_i)_d]^\sharp \omega - \int_{W_i} G_i \cdot [(H_i)_c]^\sharp \omega \\ = \int_{W_i \times (c,d)} [(\mathrm{d}G_i) \wedge (H_i)^\sharp \omega + G_i \cdot (H_i)^\sharp (\mathrm{d}\omega)]. \tag{6.3.47}$$

Then, we observe that, by construction of G_i and H_i, we have

$$\int_{W_i} G_i \cdot [(H_i)_d]^\sharp \omega - \int_{W_i} G_i \cdot [(H_i)_c]^\sharp \omega \\ = \int_{W_i} G_i \cdot [(H_i)_b]^\sharp \omega - \int_{W_i} G_i \cdot [(H_i)_a]^\sharp \omega. \tag{6.3.48}$$

Letting $c \to a$ and $d \to b$ in (6.3.47), summing over i, and using (i) (6.3.48); (ii) (6.3.45); and (iii) the fact that $\xi_i \circ \varphi_i$ is compactly supported in W_i, we deduce that

$$\int_{\mathscr{M}} (F_b)^\sharp \omega - \int_{\mathscr{M}} (F_a)^\sharp \omega = \sum_i \int_{V_i \times (a,b)} [(\mathrm{d}G_i) \wedge (H_i)^\sharp \omega + G_i \cdot (H_i)^\sharp (\mathrm{d}\omega)]. \tag{6.3.49}$$

From (6.3.49), (6.3.39) (with $\lambda = \mathrm{d}\omega$), and the fact that, on $V_i \times (a,b)$, we have $G_i = \xi_i \circ \varphi_i$ and $H_i = F \circ \widetilde{\varphi}_i$, we find that (6.3.43) holds provided that we have the identity

$$\sum_i \int_{V_i \times (a,b)} (\mathrm{d}(\xi_i \circ \varphi_i)) \wedge (F \circ \widetilde{\varphi}_i)^\sharp \omega = 0, \tag{6.3.50}$$

that we next prove. Let S denote the sum in (6.3.50). Since $\sum_j \xi_j = 1$ on $\mathscr{M}$, we have

$$S = \sum_{i,j} \int_{V_i \times (a,b)} \xi_j \circ \varphi_i (\mathrm{d}(\xi_i \circ \varphi_i)) \wedge (F \circ \widetilde{\varphi}_i)^\sharp \omega. \tag{6.3.51}$$

We next apply to the integrals in (6.3.51) the identity (6.3.40) (with $f(v,t) = \xi_j \circ \varphi_i(v)$, $g(v,t) = \xi_i \circ \varphi_i(v)$, $G = F \circ \widetilde{\varphi}_i$), and obtain

$$S = \sum_{i,j} \int_{V_j \times (a,b)} \xi_j \circ \varphi_j (\mathrm{d}(\xi_i \circ \varphi_j)) \wedge (F \circ \widetilde{\varphi}_j)^\sharp \omega,$$

where we have used the fact that $\varphi_i(\varphi(v)) = \varphi_j(v)$, for every $v \in W_j$. Finally, since $\sum_i \xi_i \circ \varphi_j = 1$ for every j, we have $\sum_i \mathrm{d}(\xi_i \circ \varphi_j) = 0$ a.e. in $V_j \times (a,b)$ for every j. Therefore, $S = 0$, and thus (6.3.50) holds, as claimed. □

6.3.7 Integral invariants for $\mathrm{VMO}(\mathscr{M};\mathscr{N})$ maps

Throughout this section: (a) $\mathscr{M}$ is a compact d-dimensional Lipschitz manifold oriented by a finite chart structure $\{(U_i, V_i, \varphi_i)\}_{i\in I}$; (b) $\mathscr{N}$ is a closed manifold; (c) ω is a smooth *closed* d-form on $\mathscr{N}$. We prove that ω induces a homotopical invariant $\int_{\mathscr{M}} f^\sharp\omega$ on $\mathrm{VMO}(\mathscr{M};\mathscr{N})$.

We first investigate the case of Lipschitz maps. Let $\iota = \iota(\mathscr{N})$ be as is Proposition 1.4.2.

Proposition 6.3.27. *Consider two Lipschitz maps $f, g\colon \mathscr{M} \to \mathscr{N}$ such that $\|f-g\|_{L^\infty} \le \iota/2$. Then*

$$\int_{\mathscr{M}} g^\sharp\omega = \int_{\mathscr{M}} f^\sharp\omega.$$

Proof. We have $tg(x) + (1-t)f(x) \in \mathscr{N}_\iota$, for every $x \in \mathscr{M}$ and $t \in [0,1]$. Therefore, the map $F(x,t) = \Pi(tg + (1-t)f)$, $x \in \mathscr{M}$, $t \in [0,1]$, with Π as in Proposition 1.4.2, is well-defined and Lipschitz. Moreover, we have $F_1 = g$ and $F_0 = f$. (Recall the notation $F_y = F(\cdot, y)$.) Hence, we are in position to apply Proposition 6.3.26, which yields

$$\int_{\mathscr{M}} g^\sharp\omega - \int_{\mathscr{M}} f^\sharp\omega = \int_{\mathscr{M}} (F_1)^\sharp\omega - \int_{\mathscr{M}} (F_0)^\sharp\omega = \int_{\mathscr{M}\times(0,1)} F^\sharp(\mathrm{d}\omega) = 0. \qquad \square$$

Corollary 6.3.28. *For $f \in C(\mathscr{M};\mathscr{N})$, set $\mathscr{I}(f) = \mathscr{I}_{\mathscr{M},\omega}(f) = \int_{\mathscr{M}} g^\sharp\omega$, where $g \in \mathrm{Lip}(\mathscr{M};\mathscr{N})$ is such that $\|f-g\|_{L^\infty} \le \iota/4$. Then, the definition is correct (i.e., it does not depend on g) and $\mathscr{I}(f)$ is a homotopical invariant.*

(The existence of such g is straightforward, since $\mathscr{M}$ is a compact subset of $\mathbb{R}^N$. On the other hand, if f happens to be Lipschitz, then $\mathscr{I}(f)$ coincides with $\int_{\mathscr{M}} f^\sharp\omega$.)

Corollary 6.3.29. *For $f \in \mathrm{VMO}(\mathscr{M};\mathscr{N})$, set $\mathscr{I}(f) = \mathscr{I}(f^\varepsilon)$ for $\varepsilon < \varepsilon_1$, with ε_1 as in (6.2.31). Then the definition is correct (i.e., it does not depend on ε) and $\mathscr{I}(f)$ is a homotopical invariant and locally constant. Moreover, the definition of $\mathscr{I}(f)$ is consistent with the one in Corollary 6.3.28.*

Proof. Combine the discussions preceding and following Definition 6.2.10 with the previous corollary. $\square$

Remark 6.3.30. If f is Lipschitz, then $\int_{\mathscr{M}} f^\sharp\omega$ makes sense, as an integral, for every smooth d-form ω, not necessarily closed. However, Proposition 6.3.26 suggests that the closedness assumption is necessary to make this quantity a homotopical invariant. On the other hand, if f is merely VMO, then the assumption that ω is closed is required even to *define* $\mathscr{I}(f)$. $\square$

The following corollary asserts that the integral invariant we have just defined is stable under composition with orientation preserving bi-Lipschitz transformations.

Corollary 6.3.31. *Let $\widetilde{\mathscr{M}}$ be a Lipschitz manifold and $\Psi\colon \widetilde{\mathscr{M}} \to \mathscr{M}$ be a bi-Lipschitz orientation preserving map. For $f \in \mathrm{VMO}(\mathscr{M};\mathscr{N})$, we have*

$$\mathscr{I}_{\mathscr{M},\omega}(f) = \mathscr{I}_{\widetilde{\mathscr{M}},\omega}(f \circ \Psi).$$

Proof. This is clear if f is Lipschitz (by the chain rule and Lemma 6.3.19). The case where f is continuous follows by approximation, via Corollary 6.3.28.

For a general map $f \in \mathrm{VMO}(\mathscr{M};\mathscr{N})$, by Corollary 6.3.29 we have, for small ε,

$$\mathscr{I}_{\mathscr{M},\omega}(f) = \mathscr{I}_{\mathscr{M},\omega}(f^{\varepsilon}) = \mathscr{I}_{\widetilde{\mathscr{M}},\omega}(f^{\varepsilon} \circ \Psi) = \mathscr{I}_{\widetilde{\mathscr{M}},\omega}(\Pi \circ f_{\varepsilon} \circ \Psi).$$

By Corollary 6.2.6 and Lemma 6.2.7, we have $f_\varepsilon \to f$ in $\mathrm{BMO} \cap L^1$. It is then straightforward that $f_\varepsilon \circ \Psi \to f \circ \Psi$ in $\mathrm{BMO} \cap L^1$ (since Ψ is bi-Lipschitz). In particular, we have $f \circ \Psi \in \mathrm{VMO}$ (see the definition (6.2.7) of VMO). Next, we use the fact that the superposition with Lipschitz functions is continuous in VMO (see [BN95, Lemma A.8]) to deduce that

$$\Pi \circ f_\varepsilon \circ \Psi \to \Pi \circ f \circ \Psi = f \circ \Psi \text{ in } \mathrm{BMO} \cap L^1. \tag{6.3.52}$$

We complete the proof by combining (6.3.52) with Corollary 6.3.29. □

Combining Corollary 6.3.29 with Proposition 6.2.14, we obtain the following result.

Corollary 6.3.32. *There exists some constant $C > 0$ depending on $\mathscr{M}$ and $\mathscr{N}$ such that*

$$[f \in \mathrm{VMO}(\mathscr{M};\mathscr{N}), |f|_{\mathrm{BMO}} \le C] \implies \mathscr{I}(f) = 0.$$

For pedagogical reasons, we postpone the study of further properties of $f^\sharp\omega$ and $\mathscr{I}(f)$ to Section 6.5; see, in particular, Sections 6.5.1 and 6.5.2.

6.3.8 The case of $W^{1,d}(\mathscr{M};\mathscr{N})$ maps

Let us recall the embedding

$$W^{1,d}(\mathscr{M}) \hookrightarrow (\mathrm{VMO} \cap L^1)(\mathscr{M}). \tag{6.3.53}$$

Indeed, this is well-known when $\mathscr{M}$ is smooth. In order to prove (6.3.53) in the Lipschitz case, it suffices to repeat the argument in [BN95, Example 1]. Consequently, when ω is closed, the invariant $\mathscr{I}(f)$ makes sense (see Corollary 6.3.29) and it is viewed as an *extension* of $\int_{\mathscr{M}} f^\sharp\omega$. However, we have at hand another natural definition of $\int_{\mathscr{M}} f^\sharp\omega$ as the integral of an L^1 function defined a.e. (see Lemma 6.3.25). Let us note that $\int_{\mathscr{M}} f^\sharp\omega$ makes sense even if ω is not closed. The following proposition shows that, for $f \in W^{1,d}(\mathscr{M};\mathscr{N})$ and ω a smooth closed d-form, the two definitions yield the same quantity.

Proposition 6.3.33. *Let $f \in W^{1,d}(\mathscr{M};\mathscr{N})$ and ω a smooth closed d-form. Then, we have*

$$\mathscr{I}(f) = \int_{\mathscr{M}} f^\sharp\omega. \tag{6.3.54}$$

Proof. Combining (i) the following density result (Lemma 6.3.34 below); (ii) the fact that $f_j \to f$ in $W^{1,d}(\mathscr{M};\mathscr{N})$ implies $\int_{\mathscr{M}} f_j^\sharp\omega \to \int_{\mathscr{M}} f^\sharp\omega$; (iii) the embedding (6.3.53); (iv) Corollary 6.2.12; and (v) Corollary 6.3.29, one gets (6.3.54). □

Lemma 6.3.34. *The space* $\mathrm{Lip}(\mathscr{M};\mathscr{N})$ *is dense in* $W^{1,d}(\mathscr{M};\mathscr{N})$.

Proof. We let (U_i, V_i, φ_i), K_2 be as in Sections 6.3.1 and 6.3.5. Let $\varepsilon_1 = \varepsilon_1(\mathscr{M}) > 0$ be such that the open sets

$$U_i' = \{x \in U_i \colon B^d_{\varepsilon_1}((\varphi_i)^{-1}(x)) \subset V_i\} = \varphi_i(\{v \in V_i \colon \mathrm{dist}\,(v, (V_i)^c) > \varepsilon_1\}) \tag{6.3.55}$$

cover $\mathscr{M}$.

For $f \in L^1(\mathscr{M};\mathbb{R}^\nu)$ and $v \in V_i$, set $\overline{f}_i(v) = f \circ \varphi_i(v)$. Consider a Lipschitz partition of unity $\{\xi_i\}_{i\in I}$ subordinated to the cover $\{U_i'\}_{i\in I}$ of $\mathscr{M}$. Let $\rho \in C^\infty_c(\mathbb{B}^d)$ be a mollifier. For $0 < \varepsilon \le \varepsilon_1$ and $x \in \mathscr{M}$, set

$$\overline{f}_{i,\varepsilon} = \xi_i\,[(\overline{\rho}_\varepsilon * f_i) \circ (\varphi_i)^{-1}] \quad \text{and} \quad \overline{f}_\varepsilon = \sum_i \overline{f}_{i,\varepsilon} \tag{6.3.56}$$

(with the natural convention that $\overline{f}_{i,\varepsilon}(x) = 0$ if $x \notin U_i'$). Clearly, $\overline{f}_{i,\varepsilon}$ is Lipschitz, and thus so is $\overline{f}_\varepsilon$.

Step 1. — We have $\overline{f}_\varepsilon \to f$ in $W^{1,d}(\mathscr{M})$ as $\varepsilon \to 0$. In order to see this, it suffices to prove that $\overline{f}_{i,\varepsilon} \to \xi_i f$ in $W^{1,d}(\mathscr{M})$. Clearly, we have

$$\overline{f}_{i,\varepsilon} \circ \varphi_i = (\xi_i \circ \varphi_i)(\rho_\varepsilon * \overline{f}_i) \to (\xi_i \circ \varphi_i)\overline{f}_i = (\xi_i \circ \varphi_i)(f \circ \varphi_i) \text{ in } W^{1,d}(V_i). \tag{6.3.57}$$

Combining (6.3.57) with the fact that $\varphi_i^{-1} \circ \varphi_j$ is bi-Lipschitz, we obtain that $\overline{f}_{i,\varepsilon} \circ \varphi_j \to (\xi_i \circ \varphi_j)(f \circ \varphi_j)$ in $W^{1,d}(V_j)$, and therefore $\overline{f}_{i,\varepsilon} \to \xi_i f$ in $W^{1,d}(\mathscr{M})$.

Step 2. — We have $\operatorname{dist}(\overline{f}_\varepsilon(x), \mathscr{N}) \to 0$ uniformly as $\varepsilon \to 0$. Indeed, starting from the identity $\sum_i \xi_i(x) f(y) = f(y)$ and using (6.3.53), we find that

$$\begin{aligned}
\operatorname{dist}(\overline{f}_\varepsilon(x), \mathscr{N}) &\leq \fint_{B_{K_2\varepsilon}(x)} |\overline{f}_\varepsilon(x) - f(y)| \, \mathrm{d}\mathscr{H}^d(y) \\
&\leq \sum_i \xi_i(x) \fint_{B_{K_2\varepsilon}(x)} |\rho_\varepsilon * [\overline{f}_i - f(y)]((\varphi_i)^{-1}(x))| \, \mathrm{d}\mathscr{H}^d(y) \\
&\leq C M_{K_2\varepsilon}(f).
\end{aligned} \tag{6.3.58}$$

Step 3. — We have $\Pi \circ \overline{f}_\varepsilon \to f$ in $W^{1,d}(\mathscr{M}; \mathscr{N})$ as $\varepsilon \to 0$. Indeed, by the previous steps, for sufficiently small ε, $\Pi \circ \overline{f}_\varepsilon$ is well-defined and Lipschitz. By a standard property of superposition operators in $W^{1,p}$, we have $\Pi \circ \overline{f}_\varepsilon \to \Pi \circ f = f$ in $W^{1,d}(\mathscr{M}; \mathscr{N})$ as $\varepsilon \to 0$. This completes the proof. □

Remark 6.3.35. For the record, we note that a variant of the proof of Lemma 6.3.34 leads to the the following result, that we will not use and whose detailed proof is presented in Section 6.8. □

Lemma 6.3.36. *Assume that* $0 < s \leq 1$ *and* $1 \leq p < +\infty$ *are such that* $sp \geq d$. *Then the space* $\operatorname{Lip}(\mathscr{M}; \mathscr{N})$ *is dense in* $W^{s,p}(\mathscr{M}; \mathscr{N})$.

In the above, when $0 < s < 1$, one naturally defines $W^{s,p}(\mathscr{M})$ as $\{f : \mathscr{M} \to \mathbb{R} : |f|_{W^{s,p}} < +\infty\}$, where

$$|f|^p_{W^{s,p}} = \int_{\mathscr{M}} \int_{\mathscr{M}} \frac{|f(x) - f(y)|^p}{\operatorname{dist}(x, y)^{d+sp}} \, \mathrm{d}\mathscr{H}^d(x) \, \mathrm{d}\mathscr{H}^d(y).$$

6.4 Estimate of $\mathscr{I}(f)$

Throughout this section: (a) $\mathscr{M}$ is a compact d-dimensional Lipschitz manifold oriented by a finite chart structure $\{(U_i, V_i, \varphi_i)\}_{i \in I}$; (b) $\mathscr{N}$ is a compact manifold; (c) ω is a smooth *closed* d-form on $\mathscr{N}$. We establish an analytical estimate on the integral invariants that we constructed in Section 6.3.7.

We have

$$W^{s,p}(\mathscr{M}) \hookrightarrow \operatorname{VMO}(\mathscr{M}), \quad \text{when } 0 < s < 1, 1 < p < +\infty, \text{ and } sp = d. \tag{6.4.1}$$

(To see this, it suffices to repeat the argument in H. Brezis and L. Nirenberg [BN95, Example 2, Case 2] and to rely on (6.3.4).)

From now on, we assume that $0 < s < 1$ *and* $sp = d$ (and thus the embedding in (6.4.1) holds). For $f \in W^{s,p}(\mathscr{M}; \mathscr{N})$, our purpose here is to control $|\mathscr{I}(f)|$ (defined in Corollary

6.3.29) by $|f|_{W^{s,p}}$. This significantly generalizes the corresponding result in J. Bourgain, H. Brezis, and P. Mironescu [BBM05]. (There, $\mathscr{M} = \mathbb{S}^d$, $\mathscr{N} = \mathbb{S}^d$, and ω is the standard volume form on $\mathbb{S}^d$.) The main result of this section is the following.

Theorem 6.4.1. *There exists a constant $C > 0$ depending on $\mathscr{M}$, $\mathscr{N}$, ω, s, and p such that*

$$|\mathscr{I}(f)| \leq C|f|^p_{W^{s,p}}, \quad \textit{for every } f \in W^{s,p}(\mathscr{M};\mathscr{N}). \tag{6.4.2}$$

Remark 6.4.2. For a refinement of the estimate (6.4.2), see Section 6.8. □

Proof of Theorem 6.4.1. In view of Corollary 6.3.32, it suffices to prove, instead of (6.4.2), the following seemingly weaker estimate

$$|\mathscr{I}(f)| \leq C(|f|^p_{W^{s,p}} + 1), \quad \text{for every } f \in W^{s,p}(\mathscr{M};\mathscr{N}). \tag{6.4.3}$$

In what follows, we fix a map $f \in W^{s,p}(\mathscr{M};\mathscr{N})$. Let, as in (6.2.9),

$$\begin{aligned} F(x,\varepsilon) = f_\varepsilon(x) &= K(x,\varepsilon)\int_{\mathscr{M}} \rho(x,\varepsilon,y)f(y)\,\mathrm{d}\mathscr{H}^d(y) \\ &= \int_{\mathscr{M}} \widetilde{\rho}(x,\varepsilon,y)f(y)\,\mathrm{d}\mathscr{H}^d(y), \quad \text{for every } x \in \mathscr{M}, \text{for every } \varepsilon > 0, \end{aligned} \tag{6.4.4}$$

where we have set

$$\widetilde{\rho}(x,\varepsilon,y) = K(x,\varepsilon)\rho(x,\varepsilon,y), \quad \text{for every } x, y \in \mathscr{M}, \text{for every } \varepsilon > 0. \tag{6.4.5}$$

(The relevance of considering F in the setting of Theorem 6.4.1 comes from Corollary 6.3.29 and the definition (6.2.32).) Let us note that F makes sense when $f\colon \mathscr{M} \to \mathscr{N}$ is merely a measurable map.

The next result, whose proof is postponed, collects some straightforward properties of $\widetilde{\rho}$. Since it does not rely on $\mathscr{M}$ being a Lipschitz manifold, it is stated in the more general setting of Section 6.2.

Lemma 6.4.3. *Assume that $\mathscr{M}$ is a compact doubling metric measure space. Then, we have*

$$\int_{\mathscr{M}} \widetilde{\rho}(x,\varepsilon,y)\,\mathrm{d}\mu(y) = 1, \quad \textit{for every } x \in \mathscr{M}, \textit{for every } \varepsilon > 0, \tag{6.4.6}$$

$$\begin{aligned} |\widetilde{\rho}(x,\varepsilon,y) - \widetilde{\rho}(x',\varepsilon',y)| &\leq Cg(x,x',\varepsilon,\varepsilon',y)[|\varepsilon-\varepsilon'| + \operatorname{dist}(x,x')], \\ &\qquad \textit{for every } x, x', y \in \mathscr{M}, \textit{for every } \varepsilon, \varepsilon' > 0, \end{aligned} \tag{6.4.7}$$

with $C = C_{\mathscr{M}} + 4(C_{\mathscr{M}})^2$ *and*

$$g(x,x',\varepsilon,\varepsilon',y) = \frac{\chi_{B_\varepsilon(x)\cup B_{\varepsilon'}(x')}(y)}{\varepsilon'\mu(B_{\varepsilon'}(x'))} + \frac{\mu(B_\varepsilon(x)\cup B_{\varepsilon'}(x'))\chi_{B_\varepsilon(x)}(y)}{\varepsilon'\mu(B_\varepsilon(x))\mu(B_{\varepsilon'}(x'))}. \tag{6.4.8}$$

Moreover, we have (with the same C*)*

$$\int_{\mathscr{M}} |\widetilde{\rho}(x,\varepsilon,y) - \widetilde{\rho}(x',\varepsilon',y)|\,\mathrm{d}\mu(y) \le \frac{4C}{\min\{\varepsilon,\varepsilon'\}}[|\varepsilon-\varepsilon'| + \operatorname{dist}(x,x')], \\ \textit{for every } x,\, x' \in \mathscr{M}, \textit{for every } \varepsilon,\, \varepsilon' > 0. \tag{6.4.9}$$

Granted Lemma 6.4.3, we proceed to the proof of Theorem 6.4.1.

Step 1. — There exists some constant $C_1 = C_1(\mathscr{M},\mathscr{N})$ such that

$$|F(x,\varepsilon) - F(x',\varepsilon')| \le \frac{C_1}{\min\{\varepsilon,\varepsilon'\}}[|\varepsilon-\varepsilon'| + \operatorname{dist}(x,x')], \\ \text{for every } x,\, x' \in \mathscr{M}, \text{ for every } \varepsilon,\, \varepsilon' > 0, \text{ for every } f\colon \mathscr{M} \to \mathscr{N}. \tag{6.4.10}$$

Indeed, we have, by Lemma 6.4.3,

$$\begin{aligned} |F(x,\varepsilon) - F(x',\varepsilon')| &\le \int_{\mathscr{M}} |\widetilde{\rho}(x,\varepsilon,y) - \widetilde{\rho}(x',\varepsilon',y)||f(y)|\,\mathrm{d}\mathscr{H}^d(y) \\ &\le \frac{4C}{\min\{\varepsilon,\varepsilon'\}}[|\varepsilon-\varepsilon'| + \operatorname{dist}(x,x')]\max_{z\in\mathscr{N}}|z|, \end{aligned}$$

whence (6.4.10).

We next define an *almost projection* on $\mathscr{N}$. For this purpose, we consider Π as in Proposition 1.4.2 and let $\widetilde{\Pi} \in C_c^\infty(\mathbb{R}^\nu;\mathbb{R}^\nu)$ be such that

$$\widetilde{\Pi}(z) = \Pi(z), \quad \text{for every } z \in \mathscr{N}_{\iota/2}. \tag{6.4.11}$$

Set

$$\widetilde{F} = \widetilde{\Pi} \circ F. \tag{6.4.12}$$

It is important to note the following. Let ε_1 be such that, for $\varepsilon < \varepsilon_1$, we have $f_\varepsilon(x) \in \mathscr{N}_{\iota/2}$, for every $x \in \mathscr{M}$ (see (6.2.31)). Then,

$$\widetilde{F}_\varepsilon = f^\varepsilon, \quad \text{for every } \varepsilon < \varepsilon_1 \tag{6.4.13}$$

(see (6.2.32)).

Combining (6.4.13) and Corollary 6.3.29, we find that

$$\mathcal{I}(f) = \mathcal{I}(f^\varepsilon) = \int_{\mathcal{M}} (f^\varepsilon)^\sharp \omega = \int_{\mathcal{M}} (\widetilde{F}_\varepsilon)^\sharp \omega, \quad \text{for every } \varepsilon < \varepsilon_1. \tag{6.4.14}$$

The following is a straightforward consequence of Step 1 and of the properties of $\widetilde{\Pi}$.

Step 2. — There exists some constant $C_2 = C_2(\mathcal{M}, \mathcal{N}, \widetilde{\Pi})$ such that

$$\begin{gathered}|\widetilde{F}(x,\varepsilon) - \widetilde{F}(x',\varepsilon')| \leq \frac{C_2}{\min\{\varepsilon,\varepsilon'\}}[|\varepsilon-\varepsilon'| + \operatorname{dist}(x,x')], \\ \text{for every } x, x' \in \mathcal{M}, \text{ for every } \varepsilon, \varepsilon' > 0, \text{ for every } f\colon \mathcal{M} \to \mathcal{N}.\end{gathered} \tag{6.4.15}$$

(In particular, $\widetilde{F}_\varepsilon$ is Lipschitz, for every $\varepsilon > 0$. Therefore, when $\varepsilon < \varepsilon_1$, the right-hand side of (6.4.14) is a standard integral of a bounded Borel function.)

We next define a convenient extension of ω. Since $\widetilde{\Pi}$ takes its values in $\mathcal{N}$ on $\mathcal{N}_{\iota/2}$, the form $\widetilde{\Pi}^\sharp\omega$ is well-defined on $\mathcal{N}_{\iota/2}$. (Let us recall that $\mathcal{N} \subset \mathbb{R}^\nu$.) Now, let $\psi\colon \mathbb{R}^\nu \to [0,1]$ be a smooth function, compactly supported in $\mathcal{N}_{\iota/2}$, and such that $\psi = 1$ on a neighborhood of $\mathcal{N}$. We may therefore set

$$\alpha = \psi\widetilde{\Pi}^\sharp\omega, \tag{6.4.16}$$

and this definition makes sense on the whole $\mathbb{R}^\nu$. We claim that

$$\alpha \text{ is a smooth extension of } \omega \text{ to } \mathbb{R}^\nu. \tag{6.4.17}$$

Indeed, on $\mathcal{N}$, we have $\widetilde{\Pi} = \mathrm{id}$. Therefore, for any $z \in \mathcal{N}$ and any $e_1, \ldots, e_d \in T_z\mathcal{N}$, it holds

$$\alpha(z)(e_1,\ldots,e_d) = \psi(\widetilde{\Pi}(z))\omega(\widetilde{\Pi}(z))(\mathrm{D}\widetilde{\Pi}(z)[e_1],\ldots,\mathrm{D}\widetilde{\Pi}(z)[e_d]) = \omega(z)(e_1,\ldots,e_d); \tag{6.4.18}$$

this proves the claim.

As a consequence of (6.4.17) and (6.4.13), we find that

$$(\widetilde{F}_\varepsilon)^\sharp\alpha = (f^\varepsilon)^\sharp\omega, \quad \text{for every } \varepsilon < \varepsilon_1. \tag{6.4.19}$$

We next combine (6.4.19), (6.4.14), (6.4.15) (which, in particular, implies that $\widetilde{F}$ is Lipschitz on $\mathcal{M} \times [\varepsilon, b]$, for every $0 < \varepsilon < b \leq +\infty$), and Proposition 6.3.26, and find that

$$\mathscr{I}(f) = \int_{\mathscr{M}} (\widetilde{F}_b)^\sharp \alpha - \int_{\mathscr{M}\times(\varepsilon,b)} \widetilde{F}^\sharp(\mathrm{d}\alpha), \tag{6.4.20}$$

for every $\varepsilon < \varepsilon_1$, for every $\varepsilon < b < +\infty$.

After these preliminaries, we are at the heart of the proof of Theorem 6.4.1 (Steps 3–5). It will be of interest to note, for each step, the assumptions on s and p. (The assumptions $0 < s < 1$ and $sp = d$ are a common roof to all these steps.)

In what follows, C_j denotes a finite constant depending only on $\mathscr{M}$, $\mathscr{N}$, $\widetilde{\Pi}$, s, p, and ω. We set

$$h(x) = \inf\{\varepsilon > 0 \colon \operatorname{dist}(F(x,\varepsilon), \mathscr{N}) \geq \iota/2\}. \tag{6.4.21}$$

Step 3. — If $f \colon \mathscr{M} \to \mathscr{N}$, we have

$$\int_{\mathscr{M}\times(0,+\infty)} |\widetilde{F}^\sharp(\mathrm{d}\alpha)| \lesssim \int_{\mathscr{M}} \frac{1}{[h(x)]^d} \,\mathrm{d}\mathscr{H}^d(x), \tag{6.4.22}$$

$$\lim_{b\to+\infty} \int_{\mathscr{M}} (\widetilde{F}_b)^\sharp \alpha = 0. \tag{6.4.23}$$

In particular (in view of (6.4.20)), when $sp = d$ we have

$$|\mathscr{I}(f)| \lesssim \int_{\mathscr{M}} \frac{1}{[h(x)]^{sp}} \,\mathrm{d}\mathscr{H}^d(x). \tag{6.4.24}$$

We next proceed to the proof of (6.4.23). By (6.4.15), we have

$$|\mathrm{D}\widetilde{F}_\varepsilon(x)| \lesssim \frac{1}{\varepsilon} \quad \text{for } \mathscr{H}^d\text{-a.e. } x \in \mathscr{M}, \text{ for every } \varepsilon > 0,$$

and thus

$$\left| \int_{\mathscr{M}} (\widetilde{F}_b)^\sharp \alpha \right| \lesssim \frac{1}{b^d}, \quad \text{for every } b > 0,$$

whence (6.4.23).

It remains to prove (6.4.22). For this purpose, we first note that, when $z \in \mathscr{N}$ and $e_1, \dots, e_{d+1} \in T_z\mathscr{N}$, we have (similarly to the proof of (6.4.18))

$$\begin{aligned}(\mathrm{d}\alpha)(z)(e_1, \dots, e_{d+1}) &= (\mathrm{d}(\widetilde{\Pi}^\sharp \omega))(z)(e_1, \dots, e_{d+1}) \\ &= (\mathrm{d}\omega)(\widetilde{\Pi}(z))(\mathrm{D}\widetilde{\Pi}(z)[e_1], \dots, \mathrm{D}\widetilde{\Pi}(z)[e_{d+1}]) = 0.\end{aligned} \tag{6.4.25}$$

Here, we use the fact that the differential commutes with the pullback, along with the fact that $\psi = 1$ on a neighborhood of $\mathscr{N}$.

We next consider the set

$$W = \{(x, \varepsilon) \in \mathscr{M} \times (0, +\infty) \colon \operatorname{dist}(F(x, \varepsilon), \mathscr{N}) < \iota/2\},$$

which is open (recall that F is continuous). Using (i) (6.4.25); (ii) the fact that $\widetilde{F}$ is locally Lipschitz; and (iii) the fact that (by definition of W) we have $\widetilde{F}(W) \subset \mathscr{N}$, we find that

$$\widetilde{F}^{\sharp}(\mathrm{d}\alpha) = 0 \text{ a.e. in } W. \tag{6.4.26}$$

Combining (6.4.26) with the definition (6.4.21) of $h(x)$, we find that

$$\int_{\mathscr{M} \times (0,+\infty)} |\widetilde{F}^{\sharp}(\mathrm{d}\alpha)| = \int_{\mathscr{M}} \int_{h(x)}^{+\infty} |\widetilde{F}^{\sharp}(\mathrm{d}\alpha)(x, \varepsilon)| \, \mathrm{d}\varepsilon \, \mathrm{d}\mathscr{H}^d(x). \tag{6.4.27}$$

On the other hand, using (6.4.15), we find that

$$|\widetilde{F}^{\sharp}(\mathrm{d}\alpha)(x, \varepsilon)| \lesssim \frac{1}{\varepsilon^{d+1}} \text{ for a.e. } (x, \varepsilon) \in \mathscr{M} \times (0, +\infty). \tag{6.4.28}$$

Inserting (6.4.28) into (6.4.27), we obtain (6.4.22).

Before proceeding further, let us note that the function h defined in (6.4.21) is measurable. Indeed, by (6.2.30), we know that $h(x)$, $x \in \mathscr{M}$, has a uniform lower bound $\widetilde{\varepsilon} > 0$. Therefore, for each $x \in \mathscr{M}$ we have $\operatorname{dist}(F(x, h(x)), \mathscr{N}) = \iota/2$, and $\operatorname{dist}(F(x, \varepsilon), \mathscr{N}) < \iota/2$ if $0 < \varepsilon < h(x)$. Using this fact, it is straightforward that h is l.s.c., and thus Borel measurable.

Step 4. — For $0 < s < 1$ and $1 \le p < +\infty$, we have, for every $f \in W^{s,p}(\mathscr{M}; \mathscr{N})$,

$$\frac{1}{[h(x)]^{sp}} \lesssim \int_0^{+\infty} \varepsilon^{p(1-s)-1} |\partial_\varepsilon F(x, \varepsilon)|^p \, \mathrm{d}\varepsilon \quad \text{for } \mathscr{H}^d\text{-a.e. } x \in \mathscr{M}. \tag{6.4.29}$$

In the proof of (6.4.29), it suffices to consider points $x \in \mathscr{M}$ such that (i) x is a Lebesgue point for f (and thus $F(x, \varepsilon) \to f(x)$ as $\varepsilon \to 0$); and (ii) $h(x) < +\infty$ (and thus $\operatorname{dist}(F(x, h(x)), \mathscr{N}) = \iota/2$). (For (i), we rely on the Lebesgue differentiation theorem for metric measure spaces satisfying the doubling condition; see e.g. [Fed69, Theorem 2.9.8].) We also note that (iii) $F(x, \cdot)$ is locally absolutely continuous. (This relies on the locally Lipschitz character of F, which does not require that $sp = d$; see Step 1.) For such x, we have (using (i) and (ii))

$$\begin{aligned}\lim_{a\to 0}|F(x,h(x)) - F(x,a)| &= |F(x,h(x)) - f(x)| \\ &\geq \operatorname{dist}(F(x,h(x)),\mathscr{N}) \geq \iota/2.\end{aligned} \tag{6.4.30}$$

From (6.4.30), we deduce (using (iii)) that

$$\int_0^{h(x)} |\partial_\varepsilon F(x,\varepsilon)|\,\mathrm{d}\varepsilon \geq \iota/2. \tag{6.4.31}$$

Combining (6.4.31) with Hölder's inequality, we obtain

$$\begin{aligned}(\iota/2)^p &\leq \left(\int_0^{h(x)} |\partial_\varepsilon F(x,\varepsilon)|\,\mathrm{d}\varepsilon\right)^p \\ &\leq \int_0^{h(x)} \varepsilon^{p(1-s)-1}|\partial_\varepsilon F(x,\varepsilon)|^p\,\mathrm{d}\varepsilon\left(\int_0^{h(x)} \varepsilon^{sp/(p-1)-1}\,\mathrm{d}\varepsilon\right)^{p-1} \\ &= \left(\frac{p-1}{sp}\right)^{p-1}[h(x)]^{sp}\int_0^{h(x)} \varepsilon^{p(1-s)-1}|\partial_\varepsilon F(x,\varepsilon)|^p\,\mathrm{d}\varepsilon\end{aligned}$$

(with the obvious modification when $p = 1$), whence (6.4.29).

Step 5. — When $0 < s < 1$ and $1 \leq p < +\infty$, we have, for every $f \in W^{s,p}(\mathscr{M};\mathbb{R}^n)$,

$$\int_{\mathscr{M}}\int_0^{+\infty} \varepsilon^{p(1-s)-1}|\partial_\varepsilon F(x,\varepsilon)|^p\,\mathrm{d}\varepsilon\,\mathrm{d}\mathscr{H}^d(x) \leq C_3|f|^p_{W^{s,p}} + C_4. \tag{6.4.32}$$

(This is well-known (with $C_4 = 0$) in the Euclidean case, see, e.g., the account of the theory of weighted Sobolev spaces in [MR15].)

The starting point in the proof of (6.4.32) is the following. Using (6.4.6)–(6.4.8), we find that, for $x \in \mathscr{M}$ and $-\varepsilon/2 < h < \varepsilon$, we have

$$\begin{aligned}\left|\frac{F(x,\varepsilon+h) - F(x,\varepsilon)}{h}\right| &= \left|\int_{\mathscr{M}} \frac{\widetilde{\rho}(x,\varepsilon+h,y) - \widetilde{\rho}(x,\varepsilon,y)}{h} f(y)\,\mathrm{d}\mathscr{H}^d(y)\right| \\ &= \left|\int_{\mathscr{M}} \frac{\widetilde{\rho}(x,\varepsilon+h,y) - \widetilde{\rho}(x,\varepsilon,y)}{h}(f(y) - f(x))\,\mathrm{d}\mathscr{H}^d(y)\right| \\ &\lesssim \frac{1}{\varepsilon}\fint_{B_{2\varepsilon}(x)} |f(y) - f(x)|\,\mathrm{d}\mathscr{H}^d(y).\end{aligned} \tag{6.4.33}$$

Combining (6.4.33) with (6.3.4), we find that, for some appropriate $r_0 > 0$, we have

$$|\partial_\varepsilon F(x,\varepsilon)| \lesssim \frac{1}{\varepsilon^{d+1}} \int_{B_{2\varepsilon}(x)} |f(x) - f(y)| \,\mathrm{d}\mathscr{H}^d(y), \\ \text{for every } x \in \mathscr{M}, \text{ for a.e. } 0 < \varepsilon \le r_0. \tag{6.4.34}$$

Using (6.4.34), (6.3.4), and Hölder's inequality, we find that

$$|\partial_\varepsilon F(x,\varepsilon)|^p \lesssim \frac{1}{\varepsilon^{d+p}} \int_{B_{2\varepsilon}(x)} |f(x) - f(y)|^p \,\mathrm{d}\mathscr{H}^d(y), \\ \text{for every } x \in \mathscr{M}, \text{ for a.e. } 0 < \varepsilon \le r_0. \tag{6.4.35}$$

On the other hand, (6.4.33) yields

$$|\partial_\varepsilon F(x,\varepsilon)| \lesssim \frac{1}{\varepsilon}, \quad \text{for every } x \in \mathscr{M}, \text{ for a.e. } \varepsilon \ge r_0. \tag{6.4.36}$$

Using (6.4.35) and (6.4.36), we obtain

$$\begin{aligned}
&\int_{\mathscr{M}} \int_0^{+\infty} \varepsilon^{p(1-s)-1} |\partial_\varepsilon F(x,\varepsilon)|^p \,\mathrm{d}\varepsilon \,\mathrm{d}\mathscr{H}^d(x) \\
&\lesssim \int_{\mathscr{M}} \int_0^{+\infty} \varepsilon^{-d-sp-1} \int_{B_{2\varepsilon}(x)} |f(x) - f(y)|^p \,\mathrm{d}\mathscr{H}^d(y) \,\mathrm{d}\varepsilon \,\mathrm{d}\mathscr{H}^d(x) + 1 \\
&\le \int_{\mathscr{M}} \int_{\mathscr{M}} \int_{\operatorname{dist}(x,y)/2}^{+\infty} \varepsilon^{-d-sp-1} |f(x) - f(y)|^p \,\mathrm{d}\varepsilon \,\mathrm{d}\mathscr{H}^d(y) \,\mathrm{d}\mathscr{H}^d(x) + 1 \\
&\lesssim \int_{\mathscr{M}} \int_{\mathscr{M}} \frac{|f(x) - f(y)|^p}{[\operatorname{dist}(x,y)]^{d+sp}} \,\mathrm{d}\mathscr{H}^d(y) \,\mathrm{d}\mathscr{H}^d(x) + 1 = |f|^p_{W^{s,p}} + 1.
\end{aligned}$$

Estimate (6.4.3) (and thus Theorem 6.4.1) follows from Steps 3–5. We note that the only place where we use the assumption $sp = d$ in the proof is to connect Steps 3 and 4 through (6.4.24). □

Proof of Lemma 6.4.3. By definition of K (see (6.2.8)), (6.4.6) is obvious.

We now proceed to the proof of (6.4.7). Set $B = B_\varepsilon(x)$ and $B' = B_{\varepsilon'}(x')$. By (6.2.8), we have

$$\begin{aligned} |\rho(x,\varepsilon,y) - \rho(x',\varepsilon',y)| &\le |\varepsilon - \varepsilon' - \operatorname{dist}(x,y) + \operatorname{dist}(x',y)| \chi_{B\cup B'}(y) \\ &\le [|\varepsilon - \varepsilon'| + \operatorname{dist}(x,x')] \chi_{B\cup B'}(y). \end{aligned} \tag{6.4.37}$$

On the other hand, (6.2.8), (6.4.37), and (6.2.12) yield

$$\begin{aligned} |K(x,\varepsilon)-K(x',\varepsilon')| &\le K(x,\varepsilon)K(x',\varepsilon')\int_{\mathscr{M}}|\rho(x,\varepsilon,y)-\rho(x',\varepsilon',y)|\,\mathrm{d}\mu(y) \\ &\le 4(C_{\mathscr{M}})^2\frac{\mu(B\cup B')}{\varepsilon\varepsilon'\mu(B)\mu(B')}[|\varepsilon-\varepsilon'|+\operatorname{dist}(x,x')]. \end{aligned} \tag{6.4.38}$$

Combining (6.4.37) and (6.4.38) with (6.2.10) and (6.2.12), we find that

$$\begin{aligned} &|\widetilde{\rho}(x,\varepsilon,y)-\widetilde{\rho}(x',\varepsilon',y)| \\ &\quad\le|\rho(x,\varepsilon,y)-\rho(x',\varepsilon',y)|K(x',\varepsilon')+\rho(x,\varepsilon,y)|K(x,\varepsilon)-K(x',\varepsilon')| \\ &\quad\le\left(2C_{\mathscr{M}}\frac{\chi_{B\cup B'}(y)}{\varepsilon'\mu(B')}+4(C_{\mathscr{M}})^2\frac{\mu(B\cup B')\chi_B(y)}{\varepsilon'\mu(B)\mu(B')}\right)[|\varepsilon-\varepsilon'|+\operatorname{dist}(x,x')], \end{aligned}$$

whence (6.4.7) with g as in (6.4.8).

Finally, we prove (6.4.9). Without loss of generality, we may assume that $\mu(B)\le\mu(B')$. Integrating (6.4.7) in y and using (6.4.8), we find that

$$\begin{aligned} \int_{\mathscr{M}}|\widetilde{\rho}(x,\varepsilon,y)-\widetilde{\rho}(x',\varepsilon',y)|\,\mathrm{d}\mu(y) &\le C\frac{2\mu(B\cup B')}{\varepsilon'\mu(B')}[|\varepsilon-\varepsilon'|+\operatorname{dist}(x,x')] \\ &\le C\frac{4\mu(B')}{\varepsilon'\mu(B')}[|\varepsilon-\varepsilon'|+\operatorname{dist}(x,x')], \end{aligned}$$

whence (6.4.9). □

Remark 6.4.4. Let $0<s<1$ and $1<p<+\infty$ be such that $sp=d$. Let ω be a smooth closed d-form on $\mathscr{N}$. Combining (6.4.20), (6.4.22), (6.4.23), (6.4.29) and (6.4.32), we have the following explicit formula:

$$\mathscr{I}(f)=-\int_{\mathscr{M}\times(0,+\infty)}\widetilde{F}^\sharp(\mathrm{d}\alpha),\quad\text{for every } f\in W^{s,p}(\mathscr{M};\mathscr{N}).\qquad\square \tag{6.4.39}$$

6.5 Additional properties of $f^\sharp\omega$

Throughout this section: (a) $\mathscr{M}$ is a compact d-dimensional Lipschitz manifold oriented by a finite chart structure $\{(U_i,V_i,\varphi_i)\}_{i\in I}$; (b) $\mathscr{N}$ is a closed manifold; (c) ω is a smooth closed d-form on $\mathscr{N}$.

6.5.1 An explicit formula for $\mathscr{I}(f)$ when $f \in \mathrm{VMO}$

As a continuation of our excursion into the land of Sobolev maps, we prove that (6.4.39) still holds for VMO maps.

Proposition 6.5.1. *Let α be a smooth compactly supported extension of ω to $\mathbb{R}^\nu$. Let $f \in \mathrm{VMO}(\mathscr{M};\mathscr{N})$. Let F, respectively $\widetilde{F}$, be as in (6.4.4), respectively (6.4.12). Then,*

$$\mathscr{I}(f) = -\int_{\mathscr{M}\times(0,+\infty)} \widetilde{F}^\sharp(d\alpha). \tag{6.5.1}$$

Proof. An inspection of the proof of Theorem 6.4.1 shows that the specific extension α of ω we take plays no role in the proof, and thus (6.4.39) holds for any such α. Moreover, (6.4.39) holds for any $f \in \mathrm{Lip}(\mathscr{M};\mathscr{N})$.

Let now $f \in \mathrm{VMO}(\mathscr{M};\mathscr{N})$. Let ε_1 be as in (6.2.31). Using (i) Corollary 6.3.29; and (ii) the fact that f^ε is Lipschitz when $0 < \varepsilon \le \varepsilon_1$ (see Step 1 in the proof of Theorem 6.4.1); (iii) the proof of Theorem 6.4.1, we find that

$$\mathscr{I}(f) = \mathscr{I}(f^\varepsilon) = \int_{\mathscr{M}} (f^\varepsilon)^\sharp\omega = -\int_{\mathscr{M}\times(\varepsilon,+\infty)} \widetilde{F}^\sharp(d\alpha), \quad \text{for every } 0 < \varepsilon \le \varepsilon_1. \tag{6.5.2}$$

In order to obtain (6.5.1) from (6.5.2), it suffices to prove that $F^\sharp(d\alpha)$ is integrable on $\mathscr{M} \times (0,+\infty)$. For this purpose, we note that, clearly, the number $h(x)$ introduced in (6.4.21) satisfies

$$h(x) \ge \varepsilon_1, \quad \text{for every } x \in \mathscr{M}. \tag{6.5.3}$$

Combining (6.5.3) with (6.4.26) and (6.4.28), we obtain the domination

$$|\widetilde{F}^\sharp(d\alpha)(x,\varepsilon)| \lesssim \frac{1}{\varepsilon^{k+1}}\chi_{(\varepsilon_1,+\infty)}(\varepsilon) \quad \text{for a.e. } (x,\varepsilon) \in \mathscr{M}\times(0,+\infty),$$

which implies the integrability of $\widetilde{F}^\sharp(d\alpha)$ on $\mathscr{M}\times(0,+\infty)$. □

Remark 6.5.2. There is a lot of freedom in the choice of the extension F yielding $\widetilde{F}$. For example, one can prove that (6.5.1) still holds for F defined as in Lemma 6.5.7 below. □

6.5.2 Action on the de Rham cohomology classes

An immediate consequence of Proposition 6.5.1 is the following.

Corollary 6.5.3. *If ω is exact, then*

$$\mathcal{I}(f) = 0, \quad \textit{for every } f \in \mathrm{VMO}(\mathcal{M};\mathcal{N}).$$

Proof. Let η be a $(d-1)$-form such that $\mathrm{d}\eta = \omega$. With the notation after Step 2 in the proof of Theorem 6.4.1, set $\alpha = \mathrm{d}(\psi\widetilde{\Pi}^\sharp\eta)$. Since $\psi = 1$ in an open neighborhood Y of $\mathcal{N}$, we have, in Y,

$$\alpha = \psi\,\mathrm{d}(\widetilde{\Pi}^\sharp\eta) = \psi\,\widetilde{\Pi}^\sharp(\mathrm{d}\eta) = \psi\,\widetilde{\Pi}^\sharp\omega,$$

and thus (6.4.17) holds.

Using (6.5.1) and the definition of α, we find that

$$\begin{aligned}\mathcal{I}(f) &= -\int_{\mathcal{M}\times(0,+\infty)} \widetilde{F}^\sharp(\mathrm{d}\alpha)\\ &= -\int_{\mathcal{M}\times(0,+\infty)} \widetilde{F}^\sharp(\mathrm{d}^2(\psi\widetilde{\Pi}^\sharp\eta)) = 0, \quad \text{for every } f \in \mathrm{VMO}(\mathcal{M};\mathcal{N}).\end{aligned}$$ □

Combining Corollary 6.3.29 and Corollary 6.5.3, we obtain the following

Corollary 6.5.4. *The quantity $\mathcal{I}(f)$, with $f \in \mathrm{VMO}(\mathcal{M};\mathcal{N})$, depends only on the homotopy class of f and the (de Rham) cohomology class of ω.*

6.5.3 A digression: the distribution $f^\sharp\omega$

This section is in the spirit of H. Brezis and Nguyen H.-M. [BN11]. When $f : \mathcal{M} \to \mathcal{N}$ is Lipschitz, $f^\sharp\omega$ can be identified with a bounded Borel function ($\mathcal{H}^d$-a.e. defined on $\mathcal{M}$), and then $\int_{\mathcal{M}} f^\sharp\omega$ is merely the integral of this function with respect to $\mathcal{H}^d$. Therefore, if ξ is a real Borel integrable test function on $\mathcal{M}$, then one may consider the integral $\int_{\mathcal{M}} \xi\, f^\sharp\omega$. (Similar considerations apply to the case where $f \in W^{1,d}(\mathcal{M};\mathcal{N})$ and ξ is bounded; see Section 6.3.8.) We discuss here the possibility of giving a robust meaning to the latter integral, possibly under more restrictive assumptions on ξ. This is a generalization of the case where $\mathcal{M} = \mathcal{N} = \mathbb{S}^d$ and ω is the standard volume form on $\mathbb{S}^d$, investigated in [BN11]. (However, strictly speaking the results below are not generalizations of the results in [BN11].) Our purpose here is to illustrate how the ideas used in the proof of Theorem 6.4.1 can be adapted to this context, and also to provide heuristics for Section 6.6. The results we present below are otherwise off topic, and therefore the proofs are rather sketchy.

For simplicity, in addition to the assumptions (a)–(c) at the beginning of Section 6.5, we make here the extra assumption (d) $\mathcal{M}$ is connected. Also, in order to slightly simplify the statement of

Lemma 6.5.7 below, we make the assumption (e) the constant K_2 in (6.3.1) equals 1. (The latter assumption can be achieved by a scale change.)

Remark 6.5.5. A preliminary observation is that, even in the smooth case, $\int_{\mathscr{M}} \xi\, f^\sharp \omega$ is *not* a homotopical invariant. To illustrate this assertion, assume, e.g., that $\mathscr{M} = \mathscr{N}$ contains a flat ball, that we identify with $\mathbb{B}^d$, the unit ball in $\mathbb{R}^d$. Consider a d-form ω that coincides, on $\mathbb{B}^d$, with the standard volume form. Let $\xi \in C^\infty_{\mathrm{c}}(\mathbb{B}^d; [0,1]) \setminus \{0\}$. If f, $g \in C^\infty(\mathscr{M}; \mathbb{B}^d)$, then clearly f and g are homotopic. Choose now $f = \psi\, \mathrm{id}$, where $\psi \in C^\infty_{\mathrm{c}}(\mathbb{B}^d)$ and $\psi = 1$ on $\operatorname{supp} \xi$, and $g = 0$. By the above, f and g are homotopic. However, we have

$$\int_{\mathscr{M}} \xi\, f^\sharp \omega = \int_{\mathscr{M}} \xi > 0,$$

while $\int_{\mathscr{M}} \xi\, g^\sharp \omega = 0$. □

We next present two results in the spirit of Theorem 6.4.1. We start with the easier case where $d \geq 2$.

Theorem 6.5.6. *Assume $d \geq 2$. Let $0 < s < 1$ and $1 < p < +\infty$ be such that $sp = d$. Let $\xi \in \operatorname{Lip}(\mathscr{M}; \mathbb{R})$. Then, the mapping*

$$\operatorname{Lip}(\mathscr{M}; \mathscr{N}) \ni f \mapsto \int_{\mathscr{M}} \xi\, f^\sharp \omega$$

can be extended by density to $(W^{s,p} \cap W^{1-1/d,d})(\mathscr{M}; \mathscr{N})$.

The extension, still denoted $f \mapsto \int_{\mathscr{M}} \xi\, f^\sharp \omega$, satisfies

$$\begin{aligned} \left| \int_{\mathscr{M}} \xi\, f^\sharp \omega \right| &\leq C_1 |f|^p_{W^{s,p}} \|\xi\|_{L^\infty} + C_2 |f|^d_{W^{1-1/d,d}} |\xi|_{\mathrm{Lip}}, \\ &\text{for every } f \in (W^{s,p} \cap W^{1-1/d,d})(\mathscr{M}; \mathscr{N}), \text{ for every } \xi \in \operatorname{Lip}(\mathscr{M}; \mathbb{R}), \end{aligned} \tag{6.5.4}$$

for some constants $C_1 > 0$ and $C_2 > 0$ depending on s, p, ω, $\mathscr{M}$, and $\mathscr{N}$.

We note the extra assumption $f \in W^{1-1/d,d}(\mathscr{M}; \mathscr{N})$, which was not needed in Theorem 6.4.1.

Sketch of proof. Let us start by guessing the analogue of (6.4.39) in this context. We use notation similar to the one in the proof of Theorem 6.4.1. Let F be an extension of f to be defined later and set $\widetilde{F} = \widetilde{\Pi} \circ F$. Set $\widetilde{\xi}(x, \varepsilon) = \xi(x)$, for every $x \in \mathscr{M}$, for every $\varepsilon > 0$.

With (6.4.23) in mind, we formally have the following chain of equalities:

$$\begin{aligned}\int_{\mathscr{M}} \xi f^\sharp\omega &= -\int_{\mathscr{M}\times(0,+\infty)} \mathrm{d}[\widetilde{\xi}\,\widetilde{F}^\sharp\alpha] \\ &= -\int_{\mathscr{M}\times(0,+\infty)} \mathrm{d}\widetilde{\xi} \wedge \widetilde{F}^\sharp\alpha - \int_{\mathscr{M}\times(0,+\infty)} \widetilde{\xi}\,\widetilde{F}^\sharp(\mathrm{d}\alpha).\end{aligned} \tag{6.5.5}$$

The strategy of the rigorous proof is now clear; see Steps 1–5 below.

Step 1. — Definition of an appropriate extension operator $f \mapsto F$. This is the content of the following auxiliary result, inspired by the proof of Lemma 6.3.34. We consider the notation in (6.3.55)–(6.3.56).

Lemma 6.5.7. *Let $\varepsilon_1 = \varepsilon_1(\mathscr{M}) > 0$ be such that the open sets*

$$U'_i = \{x \in U_i \colon B^d_{\varepsilon_1}((\varphi_i)^{-1}(x)) \subset V_i\} = \varphi_i(\{v \in V_i \colon \operatorname{dist}(v, (V_i)^c) > \varepsilon_1\})$$

cover $\mathscr{M}$.

For $f \in L^1(\mathscr{M};\mathbb{R}^\nu)$ and $v \in V_i$, set $f_i(v) = f \circ \varphi_i(v)$. Consider a Lipschitz partition of unity $\{\xi_i\}_{i\in I}$ subordinated to the cover $\{U'_i\}_{i\in I}$ of $\mathscr{M}$. Let $\rho \in C^\infty_c(\mathbb{B}^d)$ be a mollifier. For $0 < \varepsilon \le \varepsilon_1$ and $x \in \mathscr{M}$, set

$$F(\cdot, \varepsilon) = \sum_i \xi_i\,\rho_\varepsilon * [f_i \circ (\varphi_i)^{-1}]. \tag{6.5.6}$$

(With the natural convention that $\xi_i(x)\rho_\varepsilon * f_i((\varphi_i)^{-1}(x)) = 0$ if $x \notin U'_i$.)

For $x \in \mathscr{M}$ and the other non-negative values of ε, we set

$$F(x,\varepsilon) = \begin{cases} f(x), & \text{if } \varepsilon = 0, \\ \displaystyle\fint_{\mathscr{M}} f, & \text{if } \varepsilon \ge 2\varepsilon_1, \\ \displaystyle\Big(2 - \frac{\varepsilon}{\varepsilon_1}\Big)F(x,\varepsilon_1) + \Big(\frac{\varepsilon}{\varepsilon_1} - 1\Big)\fint_{\mathscr{M}} f, & \text{if } \varepsilon_1 < \varepsilon < 2\varepsilon_1. \end{cases}$$

The linear operator $L^1(\mathscr{M};\mathbb{R}^\nu) \ni f \mapsto F$ has the following properties (with constants independent of f).

(1) *If f is Lipschitz, then so is F.*

(2) *For $\varepsilon_1 \le \varepsilon \le 2\varepsilon_1$, we have*

$$\begin{aligned}\partial_\varepsilon F(x,\varepsilon) &= \frac{1}{\varepsilon_1}\Big(\fint_{\mathscr{M}} f - F(x,\varepsilon_1)\Big)\\ &= -\frac{1}{\varepsilon_1}\sum_i \xi_i(x)\rho_{\varepsilon_1} * \Big(f - \fint_{\mathscr{M}} f\Big)\circ \varphi_i((\varphi_i)^{-1}(x))\end{aligned}$$

and therefore

$$|\partial_\varepsilon F(x,\varepsilon)| \le C_3 \int_{\mathscr{M}}\int_{\mathscr{M}} |f(y)-f(z)|\,\mathrm{d}\mathscr{H}^d(y)\,\mathrm{d}\mathscr{H}^d(z), \quad \text{for every } x\in\mathscr{M}, \text{for every } \varepsilon_1 \le \varepsilon \le 2\varepsilon_1. \tag{6.5.7}$$

(3) *For $0<\varepsilon\le\varepsilon_1$ and $\mathscr{H}^d$-a.e. $x\in\mathscr{M}$, we have*

$$|\mathrm{D}F(x,\varepsilon)| \le \frac{C_4}{\varepsilon}\fint_{B_\varepsilon(x)} |f(y)-f(x)|\,\mathrm{d}\mathscr{H}^d(y). \tag{6.5.8}$$

(4) *If $f\colon \mathscr{M}\to\mathscr{N}$, then, for every $x\in\mathscr{M}$ and $0<\varepsilon\le\varepsilon_1$,*

$$\operatorname{dist}(F(x,\varepsilon),\mathscr{N}) \le C_5 M_\varepsilon(f).$$

Sketch of proof. The proof of (1) follows readily from the fact that $\mathrm{Lip}\cap L^\infty$ is an algebra. The proof of (2) is a straightforward computation. Property (4) is proved in 6.3.58.

To prove (3), we calculate the differential of $F(x,\varepsilon)$ via the Leibniz rule. For the term involving $\mathrm{D}\xi_i$, we rely on the fact that $\sum_i \mathrm{D}\xi_i = 0$ a.e. to obtain that, for a.e. $x\in\mathscr{M}$, it holds

$$\begin{aligned}\Big|\sum_i [\mathrm{D}\xi_i(x)]\,[(\rho_\varepsilon * \overline{f}_i)((\varphi_i)^{-1}(x))]\Big| &= \Big|\sum_i [\mathrm{D}\xi_i(x)][(\rho_\varepsilon * \overline{f}_i)((\varphi_i)^{-1}(x)) - f(x)]\Big|\\ &\le \sum_i \fint_{B^d_\varepsilon((\varphi_i)^{-1}(x))} |\overline{f}_i(v)-f(x)|\,\mathrm{d}v\\ &\le \fint_{B_\varepsilon(x)} |f(y)-f(x)|\,\mathrm{d}\mathscr{H}^d(y),\end{aligned} \tag{6.5.9}$$

where we have used the extra assumption (e) in the last inequality.

For the term involving $f_i * \mathrm{D}\rho_\varepsilon$ (where D stands for the gradient in both x and ε of $(x,\varepsilon)\mapsto\rho_\varepsilon(x)$), we rely on the fact that the integral of $\mathrm{D}\rho_\varepsilon$ is zero to deduce that

$$(\overline{\mathrm{D}}\rho_\varepsilon * f_i)((\varphi_i)^{-1}(x)) = \int_{B^d_\varepsilon((\varphi_i)^{-1}(x))} \mathrm{D}\rho_\varepsilon((\varphi_i)^{-1}(x)-v)(f_i(v)-f(x))\,\mathrm{d}v. \tag{6.5.10}$$

We obtain (6.5.8) from (6.5.9) and (6.5.10) combined with (i) the fact that $\{\xi_i\}_{i\in I}$ is a partition of unity; (ii) the fact that φ_i is bi-Lipschitz; and (iii) the estimate $|\mathrm{D}\rho_\varepsilon| \lesssim \varepsilon^{-d-1}$. □

Step 2. — Justification of (6.5.5) when $f \in \mathrm{Lip}(\mathscr{M};\mathscr{N})$. The starting point is the following result.

Proposition 6.5.8. *Let $F\colon \mathscr{M}\times[a,b]\to\mathscr{N}$ and $Z\colon \mathscr{M}\times[a,b]\to\mathbb{R}$ be Lipschitz maps. Then*

$$\int_{\mathscr{M}\times(a,b)} \mathrm{d}Z\wedge F^\sharp\omega + \int_{\mathscr{M}\times(a,b)} Z\,F^\sharp(\mathrm{d}\omega) = \int_{\mathscr{M}} Z_b\,(F_b)^\sharp\omega - \int_{\mathscr{M}} Z_a\,(F_a)^\sharp\omega.$$

Similarly, if $F\colon \mathscr{M}\times[a,b]\to\mathbb{R}^n$ and $Z\colon \mathscr{M}\times[a,b]\to\mathbb{R}$ are Lipschitz maps, and if α is a smooth d-form on $\mathbb{R}^\nu$ with bounded coefficients, then

$$\int_{\mathscr{M}\times(a,b)} \mathrm{d}Z\wedge F^\sharp\alpha + \int_{\mathscr{M}\times(a,b)} Z\,F^\sharp(\mathrm{d}\alpha) = \int_{\mathscr{M}} Z_b\,(F_b)^\sharp\alpha - \int_{\mathscr{M}} Z_a\,(F_a)^\sharp\alpha.$$

This is a cousin of Proposition 6.3.26, and its proof is a straightforward variant of the one of Proposition 6.3.26.

By Lemma 6.5.7 (1), when f is Lipschitz, so is F (and thus $\widetilde{F}$). We are therefore in position to apply Proposition 6.5.8 to $\widetilde{\xi}$, $\widetilde{F}$, and α. Using (6.4.17) and the fact that, by definition of F, $\widetilde{F}(x,\varepsilon)$ is constant for $x\in\mathscr{M}$ and $\varepsilon\geq 2\varepsilon_1$, we find that

$$\begin{aligned}\int_{\mathscr{M}} \xi\, f^\sharp\omega &= -\int_{\mathscr{M}\times(0,2\varepsilon_1)} \mathrm{d}\widetilde{\xi}\wedge\widetilde{F}^\sharp\alpha - \int_{\mathscr{M}\times(0,2\varepsilon_1)} \widetilde{\xi}\,\widetilde{F}^\sharp(\mathrm{d}\alpha)\\ &= -\int_{\mathscr{M}\times(0,+\infty)} \mathrm{d}\widetilde{\xi}\wedge\widetilde{F}^\sharp\alpha - \int_{\mathscr{M}\times(0,+\infty)} \widetilde{\xi}\,\widetilde{F}^\sharp(\mathrm{d}\alpha).\end{aligned}$$

This completes Step 2.

Step 3. — Justification of (6.5.4) when $f\in\mathrm{Lip}(\mathscr{M};\mathscr{N})$. By (6.5.7), (6.5.8), and the definitions of F and $\widetilde{F}$, we have, for $\mathscr{H}^d$-a.e. $x\in\mathscr{M}$,

$$|\mathrm{D}\widetilde{F}(x,\varepsilon)| \lesssim \frac{1}{\varepsilon}\fint_{B_\varepsilon(x)} |f(y)-f(x)|\,\mathrm{d}\mathscr{H}^d(y) \text{ if } 0<\varepsilon<\varepsilon_1, \tag{6.5.11}$$

$$\begin{aligned}|\mathrm{D}\widetilde{F}(x,\varepsilon)| \lesssim{}& \int_{\mathscr{M}} |f(y)-f(x)|\,\mathrm{d}\mathscr{H}^d(y)\\ &+\int_{\mathscr{M}}\int_{\mathscr{M}} |f(y)-f(z)|\,\mathrm{d}\mathscr{H}^d(y)\,\mathrm{d}\mathscr{H}^d(z) \text{ if } \varepsilon_1\leq\varepsilon<2\varepsilon_1,\end{aligned} \tag{6.5.12}$$

$$\mathrm{D}\widetilde{F}(x,\varepsilon) = 0 \text{ if } \varepsilon\geq 2\varepsilon_1. \tag{6.5.13}$$

Combining (6.5.11)–(6.5.13) with the proof of (6.4.32), we obtain the following estimate.

Lemma 6.5.9. *Let* $0 < r < 1$ *and* $1 \le q < +\infty$. *Then*

$$\int_{\mathscr{M}\times(0,+\infty)} \varepsilon^{q(1-r)-1}|\mathrm{D}\widetilde{F}(x,\varepsilon)|^q \lesssim \int_{\mathscr{M}}\int_{\mathscr{M}} \frac{|f(x)-f(y)|^q}{[\operatorname{dist}(x,y)]^{d+rq}}\,\mathrm{d}\mathscr{H}^d(y)\,\mathrm{d}\mathscr{H}^d(x).$$

Applying Lemma 6.5.9 with $r = 1 - 1/d$ and $q = d$, we obtain the estimate

$$\left|\int_{\mathscr{M}\times(0,+\infty)} \mathrm{d}\widetilde{\xi} \wedge \widetilde{F}^\sharp\alpha\right| \lesssim |f|^d_{W^{1-1/d,d}}|\xi|_{\mathrm{Lip}}. \tag{6.5.14}$$

On the other hand, using Lemma 6.5.7 (4) and repeating the proofs of (6.4.22) and (6.4.29) (proofs that are robust with respect to the definition of F), we find that

$$\left|\int_{\mathscr{M}\times(0,+\infty)} \widetilde{\xi}\,\widetilde{F}^\sharp(\mathrm{d}\alpha)\right| \lesssim \int_{\mathscr{M}\times(0,+\infty)} \varepsilon^{p(1-s)-1}|\partial_\varepsilon\widetilde{F}(x,\varepsilon)|^p\|\xi\|_{L^\infty}. \tag{6.5.15}$$

From (6.5.15) and Lemma 6.5.9 with $r = s$ and $q = p$, we obtain

$$\left|\int_{\mathscr{M}\times(0,+\infty)} \widetilde{\xi}\,\widetilde{F}^\sharp(\mathrm{d}\alpha)\right| \lesssim |f|^p_{W^{s,p}}\|\xi\|_{L^\infty}. \tag{6.5.16}$$

We complete Step 3 via (6.5.14) and (6.5.16).

Step 4. — Continuity of the right-hand side of (6.5.5) in $(W^{s,p}\cap W^{1-1/d,d})(\mathscr{M};\mathscr{N})$. We essentially rely on the converse to the dominated convergence theorem. Consider f_n, $f\colon \mathscr{M}\to\mathscr{N}$ such that $f_n \to f$ in $W^{s,p}\cap W^{1-1/d,d}$ as $n\to+\infty$.

There exists some maps $G, H\colon \mathscr{M}\times\mathscr{M}\to[0,+\infty]$ such that

$$\int_{\mathscr{M}}\int_{\mathscr{M}} \frac{|G(x,y)|^d}{[\operatorname{dist}(x,y)]^{2d-1}}\,\mathrm{d}\mathscr{H}^d(y)\,\mathrm{d}\mathscr{H}^d(x) < +\infty, \tag{6.5.17}$$
$$\int_{\mathscr{M}}\int_{\mathscr{M}} \frac{|H(x,y)|^p}{[\operatorname{dist}(x,y)]^{2d}}\,\mathrm{d}\mathscr{H}^d(y)\,\mathrm{d}\mathscr{H}^d(x) < +\infty, \tag{6.5.18}$$

and, up to a subsequence,

$$|f_n(x) - f_n(y)| \le G(x,y), \tag{6.5.19}$$
$$|f_n(x) - f_n(y)| \le H(x,y). \tag{6.5.20}$$

Let F_n, $\widetilde{F}_n$ be the corresponding maps associated with f_n. It is straightforward that, for $\mathscr{H}^d$-a.e. $x \in \mathscr{M}$ and every $\varepsilon > 0$, we have

$$\mathrm{D}\widetilde{F}_n(x,\varepsilon) \to \mathrm{D}\widetilde{F}(x,\varepsilon). \tag{6.5.21}$$

Combining dominated convergence with (6.5.17), (6.5.19), (6.5.21), (6.5.11)–(6.5.13), and the proof of Lemma 6.5.9 with $r = 1 - 1/d$ and $q = d$, we find that

$$\int_{\mathscr{M}\times(0,+\infty)} \mathrm{d}\widetilde{\xi} \wedge \widetilde{F}_n^\sharp\alpha \to \int_{\mathscr{M}\times(0,+\infty)} \mathrm{d}\widetilde{\xi} \wedge \widetilde{F}^\sharp\alpha, \quad \text{for every } \xi \in \mathrm{Lip}(\mathscr{M};\mathbb{R}). \tag{6.5.22}$$

On the other hand, with h_n associated with f_n as in (6.4.21), we have, by (6.4.26), the proof of (6.4.29), and (6.5.11)–(6.5.13),

$$|\widetilde{F}_n^\sharp(\mathrm{d}\alpha)(x,\varepsilon)| \lesssim \frac{1}{\varepsilon^{d+1}}\chi_{(h_n(x),+\infty)}(\varepsilon), \quad \text{for a.e. } x,\ \varepsilon, \tag{6.5.23}$$

$$\frac{1}{[h_n(x)]^{sp}} \lesssim \int_0^{+\infty} \varepsilon^{p(1-s)-1}|\partial_\varepsilon \widetilde{F}_n(x,\varepsilon)|^p\,\mathrm{d}\varepsilon, \quad \text{for } \mathscr{H}^d\text{-a.e. } x \in \mathscr{M}. \tag{6.5.24}$$

Combining dominated convergence with (6.5.23), (6.5.24), (6.5.18), (6.5.20), and the proof of Lemma 6.5.9 with $r = s$ and $q = p$, we find that

$$\int_{\mathscr{M}\times(0,+\infty)} \widetilde{\xi}\,\widetilde{F}_n^\sharp(\mathrm{d}\alpha) \to \int_{\mathscr{M}\times(0,+\infty)} \widetilde{\xi}\,\widetilde{F}^\sharp(\mathrm{d}\alpha), \quad \text{for every } \xi \in L^\infty(\mathscr{M};\mathbb{R}). \tag{6.5.25}$$

We complete Step 4 via (6.5.22) and (6.5.25).

Step 5. — Density of $\mathrm{Lip}(\mathscr{M};\mathscr{N})$ in $(W^{s,p} \cap W^{1-1/d,d})(\mathscr{M};\mathscr{N})$. Thanks to Lemma 6.5.7 (4) and the embedding $W^{s,p} \hookrightarrow \mathrm{VMO}$, for small ε, we have $\widetilde{\Pi} \circ F(\cdot,\varepsilon)\colon \mathscr{M} \to \mathscr{N}$. We also have $\widetilde{\Pi} \circ f = f$. We complete Step 5 by combining the next two results. (The first one is straightforward, and the second one is an easy consequence of the converse to the dominated convergence theorem.)

Lemma 6.5.10. *Let $0 < r < 1$ and $1 \le q < +\infty$. Let $f \in W^{r,q}(\mathscr{M};\mathbb{R}^\nu)$. Let F be as in (6.5.6). Then $F(\cdot,\varepsilon) \to f$ in $W^{r,q}$ as $\varepsilon \to 0$.*

Lemma 6.5.11. *Let $0 < r < 1$ and $1 \le q < +\infty$. Let $\Phi \in \mathrm{Lip}(\mathbb{R}^\nu;\mathbb{R}^\ell)$. Then the mapping $f \mapsto \Phi \circ f$ is continuous from $W^{r,q}(\mathscr{M};\mathbb{R}^\nu)$ to $W^{r,q}(\mathscr{M};\mathbb{R}^\ell)$.* □

Formally, when $d = 1$, the assumption on f in Theorem 6.5.6 becomes $f \in (W^{s,p} \cap L^1)(\mathscr{M};\mathscr{N})$. However, for such f the proof of Theorem 6.5.6 does not work anymore, since, already in the case where $\mathscr{M}$ is flat, the above extension F of an L^1 function need not have a gradient in L^1. (This is a well-known phenomenon, see, e.g., J. Peetre [Pee79].)

The educated guess in the next statement comes from the estimate (6.5.11). (For more insight, see [MR15, Theorem 1.15].) Set

$$|f|_X = \int_{\mathscr{M}} \int_{\mathscr{M}} \frac{|f(x) - f(y)|}{\operatorname{dist}(x, y)} \,\mathrm{d}\mathscr{H}^d(x)\,\mathrm{d}\mathscr{H}^d(y),$$
$$X = \{f : \mathscr{M} \to \mathbb{R}^\nu : |f|_X < +\infty\}.$$

Theorem 6.5.12. *Assume $d = 1$. Let $0 < s < 1$ and $1 < p < +\infty$ be such that $sp = 1$. Let $\xi \in \operatorname{Lip}(\mathscr{M}; \mathbb{R})$. Then, the mapping*

$$\operatorname{Lip}(\mathscr{M}; \mathscr{N}) \ni f \mapsto \int_{\mathscr{M}} \xi\, f^\sharp \omega$$

can be extended by density to $(W^{s,p} \cap X)(\mathscr{M}; \mathscr{N})$.

The extension, still denoted $f \mapsto \int_{\mathscr{M}} \xi\, f^\sharp \omega$, satisfies

$$\left| \int_{\mathscr{M}} \xi\, f^\sharp \omega \right| \le C_1 |f|^p_{W^{s,p}} \|\xi\|_{L^\infty} + C_2 |f|_X |\xi|_{\mathrm{Lip}},$$
$$\text{for every } f \in (W^{s,p} \cap X)(\mathscr{M}; \mathscr{N}), \text{ for every } \xi \in \operatorname{Lip}(\mathscr{M}; \mathbb{R}),$$

for some constants $C_1 > 0$ and $C_2 > 0$ depending on s, p, ω, $\mathscr{M}$, and $\mathscr{N}$.

Theorem 6.5.12 follows by repeating the proof of Theorem 6.5.6.

6.5.4 A further digression: help from topology and lifting

We next discuss, mostly at a formal level, an alternative, and potentially more powerful, approach to the existence of the distribution $f^\sharp \omega$. We also present one specific instance where this approach is successful, see Theorem 6.5.13 below. In great generality, a careful analysis of some cases where this approach can be rigorously implemented will be presented in [DX].

As in the previous section, our objective is to give a robust meaning to $\int_{\mathscr{M}} \xi\, f^\sharp \omega$. In order to simplify the presentation of the main idea, we assume that $\mathscr{M}$ is a ball $B \subset \mathbb{R}^d$ and ξ is compactly supported in B. (The general case can be reduced to this one, via a partition of unity and working in chart domains.) Consider an embedded manifold $\mathscr{E}$ and a smooth map $\Theta : \mathscr{E} \to \mathscr{N}$ with the two following crucial properties: (a) (*killing* property) the closed form $\Theta^\sharp \omega$ is exact: there exists some $(d-1)$-form γ such that $\Theta^\sharp \omega = \mathrm{d}\gamma$; (b) (lifting property) every *sufficiently smooth* map $f : B \to \mathscr{N}$ has a *sufficiently smooth* lifting $\widetilde{f} : B \to \mathscr{E}$ (i.e., $\widetilde{f}$ satisfies $\Theta \circ \widetilde{f} = f$).

A typical example occurs when $d = 1$, $\mathscr{E}$ is the universal cover of $\mathscr{N}$, and Θ is the corresponding covering map. Indeed, since $\mathscr{E}$ is simply connected, the smooth closed 1-form $\Theta^\sharp \omega$ is automatically exact. On the other hand, if $f \in W^{s,p}(B; \mathscr{N})$ (with B an interval), where $0 < s < 1$ and $sp \geq 1$, then f has a lifting $\widetilde{f} \in W^{s,p}(B;\mathscr{E})$ (see [BBM00] and [BC07]).

The following formal calculation shows the help one can expect from the existence of $\mathscr{E}$ and Θ:

$$\begin{aligned}\int_B \xi\, f^\sharp\omega &= \int_B \xi\,(\Theta\circ\widetilde{f})^\sharp\omega = \int_B \xi\,\widetilde{f}^\sharp(\Theta^\sharp\omega)\\ &= \int_B \xi\,\widetilde{f}^\sharp(\mathrm{d}\gamma) = (-1)^d\int_B \widetilde{f}^\sharp\gamma\wedge\mathrm{d}\xi.\end{aligned} \tag{6.5.26}$$

We are now in a situation similar to the one in the proof of Theorem 6.5.6: we can *define* the left-hand side of (6.5.26) as the right-hand side of (6.5.26), provided the latter integral makes sense. We note that, in principle, we are now in a better position than initially, since γ is a $(d-1)$-form and thus the right-hand side of (6.5.26) is defined when, e.g., $\widetilde{f} \in W^{1,d-1}(B;\mathscr{E})$ and

$$\text{there exists a compact set } K \subset \mathscr{E} \text{ such that } \widetilde{f}(B) \subset K \tag{6.5.27}$$

— this is to be compared with the natural condition for the existence of the left-hand side of (6.5.26), which is $f \in W^{1,k}(B;\mathscr{N})$.

Actually, when $d \geq 3$, one can even go beyond $W^{1,d-1}(B;\mathscr{E})$, by adapting the main idea of the proof of Theorem 6.5.6, as follows. Let $\widetilde{f} \in W^{1-1/d,d}(B;\mathscr{E}))$ satisfy (6.5.27) — by the Gagliardo–Nirenberg inequalities, this condition is weaker than $\widetilde{f} \in W^{1,d-1} \cap (B;\mathscr{E})$. Take an extension $\widetilde{F} \in W^{1,d} \cap L^\infty$ of $\widetilde{f}$, and let $\widetilde{\xi}(x,\varepsilon) = \xi(x)$, for every $x \in B$, for every $\varepsilon > 0$. Assuming that $\mathscr{E}$ is embedded in $\mathbb{R}^{\tilde{\nu}}$, consider a smooth compactly supported $(d-1)$-form $\widetilde{\gamma}$ on $\mathbb{R}^{\tilde{\nu}}$ that coincides with γ on K. Then, at least formally,

$$\begin{aligned}\int_B \xi\, f^\sharp\omega &= (-1)^d\int_B \widetilde{f}^\sharp\gamma\wedge\mathrm{d}\xi = (-1)^{d+1}\int_{\partial(B\times(0,+\infty))}(\widetilde{F}^\sharp\widetilde{\gamma})\wedge\mathrm{d}\widetilde{\xi}\\ &= (-1)^{d+1}\int_{B\times(0,+\infty)}[\widetilde{F}^\sharp(\mathrm{d}\widetilde{\gamma})]\wedge\mathrm{d}\widetilde{\xi},\end{aligned} \tag{6.5.28}$$

and, as above, we may *define* the left-hand side of (6.5.26) or (6.5.28) as the right-hand side of (6.5.28), potentially obtaining in this way the existence of the distribution $\xi \mapsto \int_B \xi\, f^\sharp\omega$ for maps f of lower regularity than expected. For more insight, see [DX].

When $d = 1$, the general philosophy presented above yields a 0-form γ, i.e., a function, and thus $\widetilde{f}^\sharp\gamma = \gamma(\widetilde{f})$ is a function. This suggests that natural function spaces leading to a robust distribution $\xi \mapsto \int_B \xi\, f^\sharp\omega$ *involve no derivatives of* f.

We next illustrate the effectiveness of this approach when $d = 1$, $\mathscr{N} = \mathbb{S}^1$, $\mathscr{E} = \mathbb{R}$, and $\Theta(t) = e^{it}$, for every $t \in \mathbb{R}$. In this case, any 1-form ω on $\mathscr{N}$ is automatically closed, and its pullback $\Theta^\sharp\omega$ is automatically exact. Since, in this setting, $\mathscr{M}$ is a Lipschitz closed curve, we assume, for simplicity, that $\mathscr{M} = \mathbb{S}^1$. (The general case may be easily reduced to this one.) In this case, we have the following result, suggested by the above discussion.

Theorem 6.5.13. *Let ω be a smooth 1-form on $\mathbb{S}^1$.*

(1) *Let $\xi \in W^{1,1}(\mathbb{S}^1;\mathbb{R})$. Then the mapping*

$$C^1(\mathbb{S}^1;\mathbb{S}^1) \ni f \mapsto \int_{\mathbb{S}^1} \xi\, f^\sharp\omega \tag{6.5.29}$$

has a unique extension by continuity plus density to $C(\mathbb{S}^1;\mathbb{S}^1)$ (with the uniform convergence metric).

(2) *Let $\xi\colon \mathbb{S}^1 \to \mathbb{R}$ be such that ξ' belongs to the Hardy space $\mathscr{H}^1(\mathbb{S}^1)$. Then the mapping (6.5.29) has a unique extension by continuity plus density to* VMO$(\mathbb{S}^1;\mathbb{S}^1)$ *(with the metric induced by the* BMO $\cap\, L^1$ *convergence).*

Remark 6.5.14. 1. Let us note that, in this very special situation, there is no need for a partition of unity and we do not make any support assumption on ξ.

2. When ω is the canonical volume form on $\mathbb{S}^1$, Theorem 6.5.13 is due to H. Brezis and Nguyen H.-M. [BN11, Definition 2], and (6.5.31) below coincides with formula (7.2) presented in H. Brezis and Nguyen H.-M. [BN11, Remark 14]. □

Proof. We denote by $z = e^{i\theta}$ a generic point on $\mathbb{S}^1$. Let $\omega = \alpha(z)\,\omega_{\mathbb{S}^1}$ be a smooth form on $\mathbb{S}^1$, with $\alpha\colon \mathbb{S}^1 \to \mathbb{R}$ smooth and $\omega_{\mathbb{S}^1}(e^{i\theta}) = d\theta$ the standard volume form of $\mathbb{S}^1$. Let $\beta(\theta) = \alpha(e^{i\theta})$ and let B be a (fixed) primitive of β. Clearly, we have

$$B(\theta + 2\ell\pi) = B(\theta) + \ell \int_{\mathbb{S}^1} \omega, \quad \text{for every } \theta \in \mathbb{R}, \text{ for every } \ell \in \mathbb{Z}. \tag{6.5.30}$$

Let $\psi(\theta) = \xi(e^{i\theta})$. We first assume that $f \in C^1(\mathbb{S}^1;\mathbb{S}^1)$. Let $\varphi \in C^1(\mathbb{R};\mathbb{R})$ be such that $f(e^{\imath\theta}) = e^{i\varphi(\theta)}$, for every $\theta \in \mathbb{R}$. Then, for each $\theta_0 \in \mathbb{R}$, we have

$$\begin{aligned}
\int_{\mathbb{S}^1} \xi\, f^\sharp\omega &= \int_{\theta_0}^{2\pi+\theta_0} \psi(\theta)[\alpha(\varphi(\theta))]\,\varphi'(\theta)\,d\theta = \int_{\theta_0}^{2\pi+\theta_0} \psi(\theta)[B(\varphi(\theta))]'\,d\theta \\
&= \Big[\psi(\theta)\,B(\varphi(\theta))\Big]_{\theta_0}^{2\pi+\theta_0} - \int_{\theta_0}^{2\pi+\theta_0} \psi'(\theta)\,[B(\varphi(\theta))]\,d\theta \\
&= \deg(f)\,\psi(\theta_0)\int_{\mathbb{S}^1}\omega - \int_{\theta_0}^{2\pi+\theta_0} \psi'(\theta)\,[B(\varphi(\theta))]\,d\theta,
\end{aligned} \tag{6.5.31}$$

where we have used (6.5.30) and the fact that $\varphi(2\pi+\theta_0)-\varphi(\theta_0)=2\pi\deg(f)$. It is clear, from (6.5.31), that the last line in (6.5.31) does not depend on φ. This can also be derived from the fact that, for two possible choices φ_1 and φ_2 of φ, $B(\varphi_1)-B(\varphi_2)$ is constant (by (6.5.30)), combined with the fact that $\int_{\theta_0}^{2\pi+\theta_0}\xi'=0$.

Proof of item (1). Given $f\in C(\mathbb{S}^1;\mathbb{S}^1)$, let $\varphi\in C(\mathbb{R};\mathbb{R})$ be such that $f(e^{i\theta})=e^{i\varphi(\theta)}$, for every $\theta\in\mathbb{R}$. If $(f_n)_{n\in\mathbb{N}}\subset C^1(\mathbb{S}^1;\mathbb{S}^1)$ is such that $f_n\to f$ uniformly, then there exist $\varphi_n\in C^1(\mathbb{R};\mathbb{R})$ such that $f_n(e^{i\theta})=e^{i\varphi_n(\theta)}$, for every $\theta\in\mathbb{R}$, for each n, and $\varphi_n\to\varphi$ uniformly. Clearly, (6.5.31) implies that

$$\int_{\mathbb{S}^1}\xi\,(f_n)^\sharp\omega\to\deg(f)\,\psi(\theta_0)\int_{\mathbb{S}^1}\omega-\int_{\theta_0}^{2\pi+\theta_0}\psi'(\theta)\,[B(\varphi(\theta))]\,d\theta.$$

This proves item (1). Moreover, since the right-hand side of (6.5.31) is well-defined for $f\in C(\mathbb{S}^1;\mathbb{S}^1)$ and does not depend on the choice of a continuous lifting φ or of a point θ_0, we can take it as the *definition* of $\int_{\mathbb{S}^1}\xi\,f^\sharp\omega$ for continuous f.

Proof of item (2). This is slightly more involved. To start with, it will be convenient to rewrite, for $f\in C^1(\mathbb{S}^1;\mathbb{S}^1)$, the identity (6.5.31) in a form involving only maps well-defined on $\mathbb{S}^1$ (which is not the case for φ and $B\circ\varphi$).

Step 1. — An alternative form of (6.5.31). Assume that f is C^1. We write

$$f(z)=z^{\deg(f)}e^{i\overline{\varphi}(z)},\quad\text{for every } z\in\mathbb{S}^1,\tag{6.5.32}$$

where $\overline{\varphi}\in C^1(\mathbb{S}^1;\mathbb{R})$. The connection between $\overline{\varphi}$ and φ above is that, up to a constant integer multiple of 2π, we have

$$\varphi(\theta)=\deg(f)\,\theta+\overline{\varphi}(e^{i\theta}),\quad\text{for every }\theta\in\mathbb{R}.$$

We also write $\alpha=\alpha_0+\overline{\alpha}$, where $\alpha_0=\fint_{\mathbb{S}^1}\omega$ and $\overline{\alpha}=\alpha-\alpha_0$. Let C denote a primitive of $\theta\mapsto\overline{\alpha}(e^{i\theta})$. By contrast with B, the function C is 2π-periodic, and thus the map

$$e^{i\theta}\mapsto D(e^{i\theta})=C(\theta)$$

is well-defined and smooth.

Applying (6.5.31) with

$$B(\theta)=C(\theta)+\alpha_0\theta=D(e^{i\theta})+\alpha_0\theta,\quad\text{for every }\theta\in\mathbb{R},$$

and

$$\varphi(\theta) = \deg(f)\theta + \overline{\varphi}(\mathrm{e}^{\mathrm{i}\theta}), \quad \text{for every } \theta \in \mathbb{R},$$

we obtain

$$\begin{aligned}
\int_{\mathbb{S}^1} \xi\, f^\sharp \omega &= \deg(f)\, \psi(\theta_0) \int_{\mathbb{S}^1} \omega - \int_{\theta_0}^{2\pi+\theta_0} \psi'(\theta)\, [B(\varphi(\theta))]\, \mathrm{d}\theta \\
&= 2\pi \deg(f) \psi(\theta_0)\, \alpha_0 \\
&\quad - \int_{\theta_0}^{2\pi+\theta_0} \psi'(\theta)\, [D(f(\mathrm{e}^{\mathrm{i}\theta})) + \alpha_0 \times (\deg(f)\, \theta + \overline{\varphi}(\mathrm{e}^{\mathrm{i}\theta}))]\, \mathrm{d}\theta \qquad (6.5.33) \\
&= \alpha_0 \deg(f) \int_{\mathbb{S}^1} \xi - \int_{\mathbb{S}^1} \xi' D(f) - \alpha_0 \int_{\mathbb{S}^1} \xi' \overline{\varphi} \\
&= \frac{1}{2\pi} \deg(f) \int_{\mathbb{S}^1} \omega \int_{\mathbb{S}^1} \xi - \int_{\mathbb{S}^1} \xi' D(f) - \frac{1}{2\pi} \int_{\mathbb{S}^1} \omega \int_{\mathbb{S}^1} \xi' \overline{\varphi}.
\end{aligned}$$

Step 2. — Density. The space $C^1(\mathbb{S}^1; \mathbb{S}^1)$ is dense in $\mathrm{VMO}(\mathbb{S}^1; \mathbb{S}^1)$ (with the $\mathrm{BMO} \cap L^1$ convergence). Indeed, $\mathrm{Lip}(\mathbb{S}^1; \mathbb{S}^1)$ is dense in $\mathrm{VMO}(\mathbb{S}^1; \mathbb{S}^1)$ (see, e.g., the construction in the proof of [BM21, Corollary 15.5]). By a standard smoothing argument, this implies that $C^1(\mathbb{S}^1; \mathbb{S}^1)$ is dense in $\mathrm{VMO}(\mathbb{S}^1; \mathbb{S}^1)$. (See also [BN11, Lemma 4].)

Step 3. — Existence of a lifting. If $f \in \mathrm{VMO}(\mathbb{S}^1; \mathbb{S}^1)$, then f has a well-defined winding number. This follows from the considerations in Section 6.2.2, using the fact that the winding number accounts for the homotopy class of continuous maps from $\mathbb{S}^1$ to $\mathbb{S}^1$. Moreover, there exists some lifting $\overline{\varphi} \in \mathrm{VMO}(\mathbb{S}^1; \mathbb{R})$, unique up to a constant integer multiple of 2π, such that (6.5.32) holds (see [BN95, Theorem 3, Remark 10 (iii)]). In particular, if $f \in C^k$, then $\overline{\varphi} \in C^k$, and thus $\overline{\varphi}$ is a classical lifting of $z \mapsto f(z)/z^{\deg(f)}$. (This follows by uniqueness.) In addition, if $f_n, f \in \mathrm{VMO}(\mathbb{S}^1; \mathbb{S}^1)$ and $f_n \to f$ in $\mathrm{BMO} \cap \mathscr{L}^1$, then, for large n we have $\deg(f_n) = \deg(f)$ (by Corollary 6.2.12) and we may choose the corresponding liftings $\overline{\varphi}_n$ such that $\overline{\varphi}_n \to \overline{\varphi}$ in $\mathrm{BMO} \cap L^1$. (For the latter fact, see Lemma 6.5.15 below.)

Step 4. — Conclusion. Let $f \in \mathrm{VMO}(\mathbb{S}^1; \mathbb{S}^1)$. Consider a sequence $(f_n)_{n \in \mathbb{N}}$ in $C^1(\mathbb{S}^1; \mathbb{S}^1)$ such that $f_n \to f$ in $\mathrm{BMO} \cap L^1$ and the same holds for corresponding liftings $\overline{\varphi}_n$ and $\overline{\varphi}$. Using (i) Corollary 6.2.12; (ii) the fact that $D(f_n) \to D(f)$ in BMO [BN95, Lemma A.8]; and (iii) the fact that BMO and $\mathscr{H}^1$ are in duality, we find that

$$\begin{aligned}
&\frac{1}{2\pi} \deg(f_n) \int_{\mathbb{S}^1} \omega \int_{\mathbb{S}^1} \xi - \int_{\mathbb{S}^1} \xi' D(f_n) - \frac{1}{2\pi} \int_{\mathbb{S}^1} \omega \int_{\mathbb{S}^1} \xi' \overline{\varphi}_n \\
&\quad \to \frac{1}{2\pi} \deg(f) \int_{\mathbb{S}^1} \omega \int_{\mathbb{S}^1} \xi - \langle \xi', D(f) \rangle - \frac{1}{2\pi} \left(\int_{\mathbb{S}^1} \omega \right) \times \langle \xi', \overline{\varphi} \rangle,
\end{aligned}$$

where $\langle \cdot , \cdot \rangle$ stands for the duality pairing between $\mathscr{H}^1$ and BMO.

Therefore, the last line in (6.5.33), (j) is well-defined for $f \in \mathrm{VMO}(\mathbb{S}^1;\mathbb{S}^1)$ (if we interpret the second and the third integral as duality pairings); (jj) is continuous with respect to the $\mathrm{BMO} \cap L^1$ convergence; and (jjj) can be taken as *definition* of $\int_{\mathbb{S}^1} \xi\, f^\sharp \omega$ for $f \in \mathrm{VMO}$.

□

We next complete Step 3 in the proof of Theorem 6.5.13.

Lemma 6.5.15. *Let f_n, $f \in \mathrm{VMO}(\mathbb{S}^1;\mathbb{S}^1)$ be such that $f_n \to f$ in $\mathrm{BMO} \cap L^1$. Then, for sufficiently large n, there exist $\overline{\varphi}_n$, $\overline{\varphi} \in \mathrm{VMO}(\mathbb{S}^1;\mathbb{R})$ such that*

$$f_n(z) = z^{\deg(f)} \mathrm{e}^{\mathrm{i}\overline{\varphi}_n(z)},\ f(z) = z^{\deg(f)} \mathrm{e}^{\mathrm{i}\overline{\varphi}(z)}, \quad \textit{for every } z \in \mathbb{S}^1, \textit{for each } n,$$

and

$$\overline{\varphi}_n \to \overline{\varphi} \textit{ in } \mathrm{BMO} \cap L^1.$$

Proof. Let $g_n = f_n/f : \mathbb{S}^1 \to \mathbb{S}^1$. Since $f_n \to f$ in $\mathrm{BMO} \cap L^1$, we have $g_n \to 1$ in $\mathrm{BMO} \cap L^1$. (Apply [BN95, Lemma A.8] to the map $(z, w) \mapsto z\overline{w}$.) By Corollary 6.2.12, for large n, we have $\deg(g_n) = 0$, and thus we may write $g_n = \mathrm{e}^{\mathrm{i}\widetilde{\varphi}_n}$, with $\widetilde{\varphi}_n \in \mathrm{VMO}(\mathbb{S}^1;\mathbb{R})$ [BN95, Theorem 3]. Moreover, for large n, we may choose $\widetilde{\varphi}_n$ such that

$$|\widetilde{\varphi}_n|_{\mathrm{BMO}} \le 4|g_n|_{\mathrm{BMO}} \tag{6.5.34}$$

([BN95, Theorem 4]).

Set $c_n = ⨍_{\mathbb{S}^1} \widetilde{\varphi}_n$. Combining (6.2.5) with (6.5.34), we find that

$$\|g_n - \mathrm{e}^{\mathrm{i}c_n}\|_{L^1} \le \|\mathrm{e}^{\mathrm{i}\widetilde{\varphi}_n} - \mathrm{e}^{\mathrm{i}c_n}\|_{L^1} \le \|\widetilde{\varphi}_n - c_n\|_{L^1} \lesssim |g_n|_{\mathrm{BMO}}. \tag{6.5.35}$$

Therefore, we have $\mathrm{e}^{\mathrm{i}c_n} \to 1$ as $n \to +\infty$, and, after adding to each $\overline{\varphi}_n$ (and c_n) a suitable integer multiple of 2π, we may assume that $c_n \to 0$. Going back to (6.5.35), we find that $\widetilde{\varphi}_n \to 0$ in $\mathrm{BMO} \cap L^1$. Finally, the conclusions of the lemma hold with $\overline{\varphi}_n = \overline{\varphi} + \widetilde{\varphi}_n$, where $\overline{\varphi} \in \mathrm{VMO}(\mathbb{S}^1;\mathbb{R})$ is such that $f(z) = z^{\deg(f)} \mathrm{e}^{\mathrm{i}\overline{\varphi}(z)}$, for every $z \in \mathbb{S}^1$. □

6.6 A higher dimensional case

6.6.1 Heuristics

In Sections 6.3 and 6.4, we have considered a situation where $\dim \mathscr{M}$ and $d = \deg \omega$ coincide. A typical more general situation consists of considering maps $f : \mathscr{M} \times W \to \mathscr{N}$, where $W \subset \mathbb{R}^\ell$ is an (open) set of parameters. Let $0 < s < 1$ and $1 < p < +\infty$ be such that $sp = d$. Assuming that $f(\cdot, w) \in W^{s,p}$ for a.e. w, one may consider the map $w \mapsto \mathscr{I}(f(\cdot, w))$, establish its properties, and estimate its *size*. In view of the applications we have in mind, we investigate here a similar, but slightly different, situation.

In what follows, we consider: (a) a smooth *closed* d-form ω on $\mathscr{N}$; (b) $0 < s < 1$ and $1 < p < +\infty$ such that $sp = d$; (c) an integer $N > d$.

In order to simplify the presentation, we consider only maps *that live in a compact set*. We could consider for example, as in other chapters, the space $W^{s,p}(\Omega; \mathscr{N})$, with $\Omega \subset \mathbb{R}^N$ a smooth bounded open set. In this chapter, for the sake of technical simplicity, we instead choose to work in the space

$$W_1^{s,p}(\mathbb{R}^N; \mathscr{N}) = \{u \in \dot{W}^{s,p}(\mathbb{R}^N; \mathscr{N}) \colon u \text{ is constant outside } K = K(u) \subset \mathbb{B}^N\},$$

where K is a compact set. (However, all the results below have counterparts for the space $W^{s,p}(\Omega; \mathscr{N})$.)

The purpose of this section is to give a robust meaning to the action of the d-form $u^\sharp\omega$ on appropriate test forms, with $u \in W_1^{s,p}(\mathbb{R}^N; \mathscr{N})$, and to exhibit the homotopical information encoded by this action, at least for *nice* u's.

An initial remark is that $u^\sharp\omega$, which is formally a d-form, may act, up to the action of the Hodge $*$-operator, either on d-forms, or on $(N-d)$-forms. For convenience matters, it is customary to choose the latter perspective.

We now present some heuristics, provided by the next formal calculation, inspired by (6.5.5). If $\xi \in C_c^\infty(\mathbb{R}^N; \Lambda^{N-d})$, then we formally have

$$\begin{aligned}\int_{\mathbb{R}^N} u^\sharp\omega \wedge \xi = -\int_{\mathbb{R}^N\times(0,+\infty)} \mathrm{d}[\widetilde{U}^\sharp\alpha \wedge \widetilde{\xi}] = -\int_{\mathbb{R}^N\times(0,+\infty)} \widetilde{U}^\sharp(\mathrm{d}\alpha) \wedge \widetilde{\xi} \\ + (-1)^{d+1}\int_{\mathbb{R}^N\times(0,+\infty)} \widetilde{U}^\sharp\alpha \wedge \mathrm{d}\widetilde{\xi},\end{aligned} \tag{6.6.1}$$

where $\widetilde{U}$ is a suitable extension of u, to be defined more precisely later on.

As explained in Section 6.5.3, in order to treat the latter integral in (6.6.1), the assumption $u \in W_1^{s,p}(\mathbb{R}^N; \mathscr{N})$ is not sufficient. For example, when $d \geq 2$, we have to make the extra assumption $u \in W_1^{1-1/d,d}(\mathbb{R}^N; \mathscr{N})$ (see Theorem 6.5.6). In order to work in the minimal space $u \in W_1^{s,p}(\mathbb{R}^N; \mathscr{N})$, it is natural to require that the latter integral in (6.6.1) vanishes. This is the case if $\mathrm{d}\widetilde{\xi} = 0$, and can be achieved if $\mathrm{d}\xi = 0$ (take $\widetilde{\xi}(x, \varepsilon) = \xi(x)$, for every $x \in \mathbb{R}^N$, for every $\varepsilon > 0$). In $\mathbb{R}^N$, the assumption $\mathrm{d}\xi = 0$ is equivalent to ξ being exact. With this in mind, we do not investigate below the action of $u^\sharp\omega$ on general $(N-d)$-forms, but only on exterior (differentials of) $(N-d-1)$-forms.

In view of the above discussion, it is natural to consider the operator (at least formally) given by

$$\langle Tu, \zeta\rangle = \int_{\mathbb{R}^N} u^\sharp\omega \wedge \mathrm{d}\zeta, \quad \text{for every } \zeta \in \mathrm{Lip}(\mathbb{R}^N; \Lambda^{N-d-1}). \tag{6.6.2}$$

To connect the definition (6.6.2) with the informal exposition in the introduction, we mention more specifically that our definition of T amounts to

$$Tu = (-1)^{d+1} * \mathrm{d}(u^\sharp\omega) \quad \text{in the sense of distributions.}$$

To justify the above, writing $\alpha = u^\sharp\omega$ and using standard identities from exterior calculus (see, e.g., [Fed69, 1.7.8]), we find that

$$(\mathrm{d}\alpha) \wedge \zeta = (* * \mathrm{d}\alpha) \wedge (* * \zeta) = \langle * * \mathrm{d}\alpha, *\zeta\rangle = \langle *\mathrm{d}\alpha, \zeta\rangle.$$

On the other hand, we have

$$0 = \int_{\mathbb{R}^N} \mathrm{d}(\alpha \wedge \zeta) = \int_{\mathbb{R}^N} (\mathrm{d}\alpha) \wedge \zeta + (-1)^d \int_{\mathbb{R}^N} \alpha \wedge \mathrm{d}\zeta,$$

whence the claimed identity.

A crucial property in what follows is the density of $W^{1,d}_1(\mathbb{R}^N; \mathcal{N})$ into $W^{s,p}_1(\mathbb{R}^N; \mathcal{N})$. Although not stated in these terms, this property was implicitly obtained in H. Brezis and P. Mironescu [BM15]. Indeed, the proof of Theorem 3 in [BM15] explicitly exhibits, for a given $u \in W^{s,p}_1(\mathbb{R}^N; \mathcal{N})$, a sequence $(u_n)_{n\in\mathbb{N}}$ in $W^{1,d}_1(\mathbb{R}^N; \mathcal{N})$ such that $u_n - u \to 0$ in $W^{s,p}(\mathbb{R}^N)$. Moreover, this sequence satisfies the additional properties (j) $u_n \in W^{1,q}_1(\mathbb{R}^N; \mathcal{N})$, for every $1 \leq q < d + 1$; and (jj) $u_n = u$ in $\mathbb{R}^N \setminus \mathbb{B}^N$. For further use, we note that, by the Gagliardo–Nirenberg inequalities, if $d \geq 2$, then $W^{1,d}_1(\mathbb{R}^N; \mathcal{N}) \subset W^{s,p}_1(\mathbb{R}^N; \mathcal{N})$. This fails when $d = 1$, but we have $W^{1,q}_1(\mathbb{R}^N; \mathcal{N}) \subset W^{s,p}_1(\mathbb{R}^N; \mathcal{N})$, for every $q > 1$.

Clearly, when $u \in W^{1,d}_1(\mathbb{R}^N; \mathcal{N})$, (j) $u^\sharp\omega$ is naturally defined a.e. as the pullback of ω through $\mathrm{d}u$; (jj) $u^\sharp\omega \in L^1(\mathbb{R}^N; \Lambda^d)$; and (jjj) $\langle Tu, \zeta\rangle$ is well-defined and satisfies the obvious bound

$$|\langle Tu, \zeta\rangle| \leq C\|\mathrm{D}u\|^d_d |\zeta|_{\mathrm{Lip}},$$

where the finite constant C depends only on ω. For such u, Tu was considered by F. Bethuel, J.-M. Coron, F. Demengel, and F. Hélein [BCDH91], with the purpose of characterizing the closure of smooth maps in the space of $W^{1,d}$ mappings. In the same functional setting, some of the properties of T were investigated by M. Giaquinta and collaborators — see for example M. Giaquinta, G. Modica, and J. Souček [GMS98b, Chapter 5.4], M. Giaquinta and D. Mucci [GM05b], M. Giaquinta and G. Modica [GM01] — and in a different direction by G. Alberti, S. Baldo, and G. Orlandi [ABO03]. These ideas have their roots in the work notably by F. Bethuel [Bet90], F. Almgren, W. Browder, and E. Lieb [ABL88], and H. Brezis, J.-M. Coron, and E. Lieb [BCL86].

Our main purpose in this section is first to give a robust meaning to Tu when $u \in W^{s,p}_1(\mathbb{R}^N;\mathscr{N})$ and to obtain the corresponding estimate, and then to exploit this object to obtain a characterization of the closure of smooth maps in $W^{s,p}_1(\mathbb{R}^N;\mathscr{N})$ for a large class of target manifolds $\mathscr{N}$. When $\mathscr{N}=\mathbb{S}^d$ and ω is the standard volume form, the first part of this program was completed by J. Bourgain, H. Brezis, and P. Mironescu [BBM05] when $N=d+1$, and for a general $N\geq d+1$ by P. Bousquet and P. Mironescu [BM14]. The second part of this program was addressed by D. Mucci [Muc24] in the special case where $\mathscr{N}=\mathbb{S}^d$.

6.6.2 Existence of a robust map T

Recall that we consider: (a) a smooth *closed* d-form ω on $\mathscr{N}$; (b) $0<s<1$ and $1<p<+\infty$ such that $sp=d$; (c) an integer $N>d$.

The first main result in this section is

Theorem 6.6.1. *Let $d\geq 2$.*

(1) *The map T, defined in (6.6.2) for $u\in W^{1,d}_1(\mathbb{R}^N;\mathscr{N})$, has an (unique) extension by continuity to $W^{s,p}_1(\mathbb{R}^N;\mathscr{N})$.*

(2) *The extension, still denoted T, satisfies*

$$|\langle Tu,\zeta\rangle|\leq C|u|^p_{W^{s,p}}|\zeta|_{\mathrm{Lip}},\quad \textit{for every } u\in W^{s,p}_1(\mathbb{R}^N;\mathscr{N}),\ \zeta\in \mathrm{Lip}(\mathbb{R}^N;\Lambda^{N-d-1}), \tag{6.6.3}$$

for some constant $C>0$ depending on s, p, N, and ω.

(3) *Let $\widetilde{\Pi}$ be as in (6.4.11) and $\alpha\in C^\infty_c(\mathbb{R}^\nu;\Lambda^d)$ be an extension of ω. We have the following formula:*

$$\langle Tu,\zeta\rangle = -\int_{\mathbb{R}^N\times(0,+\infty)}\widetilde{U}^\sharp(\mathrm{d}\alpha)\wedge \mathrm{d}\widetilde{\zeta},\quad \textit{for every } u\in W^{s,p}_1(\mathbb{R}^N;\mathscr{N}),\ \textit{for every } \zeta\in\mathrm{Lip}(\mathbb{R}^N;\Lambda^{N-d-1}). \tag{6.6.4}$$

Here, $\widetilde{U}=\widetilde{\Pi}\circ U$, with U defined by (6.6.5) below, and $\widetilde{\zeta}(x,t)=\zeta(x)$, for every $x\in\mathbb{R}^N$, for every $t\geq 0$.

The proof of Theorem 6.6.1 relies on the following cousin of Proposition 6.5.8.

Proposition 6.6.2. *For $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$, let $i_t(x) = (x,t)$. Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded open set. For $U \in \mathrm{Lip}(\Omega \times [a,b]; \mathbb{R}^\nu)$ and $\xi \in \mathrm{Lip}_\mathrm{c}(\Omega \times [a,b]; \Lambda^{N-d-1})$, we have*

$$\int_{\Omega\times(a,b)} U^\sharp(\mathrm{d}\alpha) \wedge \mathrm{d}\xi = \int_\Omega (U_b)^\sharp\alpha \wedge \mathrm{d}(i_b^\sharp \xi) - \int_\Omega (U_a)^\sharp \alpha \wedge \mathrm{d}(i_b^\sharp\xi).$$

In particular, if $\zeta \in \mathrm{Lip}_\mathrm{c}(\Omega; \Lambda^{N-d-1})$, then

$$\int_{\Omega\times(a,b)} U^\sharp(\mathrm{d}\alpha) \wedge \mathrm{d}\widetilde{\zeta} = \int_\Omega (U_b)^\sharp\alpha \wedge \mathrm{d}\zeta - \int_\Omega (U_a)^\sharp\alpha \wedge \mathrm{d}\zeta.$$

The proof of Proposition 6.6.2 is a straightforward variant of the one of Proposition 6.3.26.

Proof of Theorem 6.6.1. Let $\rho \in C_\mathrm{c}^\infty(\mathbb{B}^N)$ be a mollifier (in $\mathbb{R}^N$). For $\varepsilon > 0$, set

$$U(x,\varepsilon) = \rho_\varepsilon * u(x). \tag{6.6.5}$$

For every $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, the map U is smooth and

$$|\mathrm{D}U(x,\varepsilon)| \lesssim \frac{1}{\varepsilon} \fint_{B_\varepsilon^N(x)} |f(y) - f(x)|\,\mathrm{d}y, \quad \text{for every } x \in \mathbb{R}^N, \text{ for every } \varepsilon > 0. \tag{6.6.6}$$

We define the associated map $\widetilde{U}$ by $\widetilde{U} = \widetilde{\Pi} \circ U$.

The existence of an extension of T to $W_1^{s,p}(\mathbb{R}^N; \mathscr{N})$ relies on Steps 1 and 2 below.

Step 1. — For $u \in W_1^{1,d}(\mathbb{R}^N; \mathscr{N})$, we have

$$\int_{\mathbb{R}^N\times(0,+\infty)} \left|\widetilde{U}^\sharp(\mathrm{d}\alpha)\right| < +\infty. \tag{6.6.7}$$

Indeed, since $u \in L^\infty \cap W_1^{1,d}$, by the Gagliardo–Nirenberg inequality, we have

$$u \in W_1^{1-1/(d+1),d+1}(\mathbb{R}^N; \mathscr{N}). \tag{6.6.8}$$

By standard trace theory, (6.6.8) implies that

$$U \in \dot{W}^{1,d+1}(\mathbb{R}^N \times (0,+\infty)). \tag{6.6.9}$$

We obtain (6.6.7) from (6.6.9) and the fact that $|\widetilde{U}^\sharp(\mathrm{d}\alpha)(x,\varepsilon)| \lesssim |\mathrm{D}U(x,\varepsilon)|^{d+1}$.

Step 2. — For $u \in W_1^{1,d}(\mathbb{R}^N; \mathcal{N})$, we have

$$\int_{\mathbb{R}^N} u^\sharp \omega \wedge \mathrm{d}\zeta = -\int_{\mathbb{R}^N \times (0,+\infty)} \widetilde{U}^\sharp(\mathrm{d}\alpha) \wedge \mathrm{d}\widetilde{\zeta}. \tag{6.6.10}$$

Indeed, let $0 < a < b < +\infty$. Let $\phi = \phi(x) \in C_c^\infty(B_{b+2}^N(0))$, $x \in \mathbb{R}^N$, be such that $\phi = 1$ in $\overline{B_{b+1}^N}(0)$. Since $F(\cdot, t)$ is constant in $(\mathbb{R}^N \setminus \overline{B_{b+1}^N}(0)) \times [a, b]$, we have, by Proposition 6.6.2 (applied with $\Omega = B_{b+2}^N(0)$),

$$\begin{aligned}
\int_{\mathbb{R}^N \times (a,b)} \widetilde{U}^\sharp(\mathrm{d}\alpha) \wedge \mathrm{d}\widetilde{\zeta} &= \int_{B_{b+1}^N(0) \times (a,b)} \widetilde{U}^\sharp(\mathrm{d}\alpha) \wedge \mathrm{d}\widetilde{\zeta} \\
&= \int_{B_{b+1}^N(0) \times (a,b)} \widetilde{U}^\sharp(\mathrm{d}\alpha) \wedge \mathrm{d}(\widetilde{\phi\zeta}) \\
&= \int_{B_{b+2}^N(0) \times (a,b)} \widetilde{U}^\sharp(\mathrm{d}\alpha) \wedge \mathrm{d}(\widetilde{\phi\zeta}) \\
&= \int_{B_{b+2}^N(0)} (\widetilde{U}_b)^\sharp \alpha \wedge \mathrm{d}(\phi\zeta) - \int_{B_{b+2}^N(0)} (\widetilde{U}_a)^\sharp \alpha \wedge \mathrm{d}(\phi\zeta) \\
&= \int_{B_{b+1}^N(0)} (\widetilde{U}_b)^\sharp \alpha \wedge \mathrm{d}\zeta - \int_{B_{b+1}^N(0)} (\widetilde{U}_a)^\sharp \alpha \wedge \mathrm{d}\zeta \\
&= \int_{\mathbb{R}^N} (\widetilde{U}_b)^\sharp \alpha \wedge \mathrm{d}\zeta - \int_{\mathbb{R}^N} (\widetilde{U}_a)^\sharp \alpha \wedge \mathrm{d}\zeta.
\end{aligned} \tag{6.6.11}$$

We notice that

$$\left| \int_{\mathbb{R}^N} (\widetilde{U}_b)^\sharp \alpha \wedge \mathrm{d}\zeta \right| \lesssim |\zeta|_{\mathrm{Lip}} \int_{\mathbb{R}^N} |\mathrm{D}U(x, b)|^d \,\mathrm{d}x. \tag{6.6.12}$$

By (6.6.6) and the fact that $u = c_u$ outside $\mathbb{B}^N$ for some constant c_u, for $b > 1$, we have

$$|\mathrm{D}U(x, b)| \lesssim \begin{cases} 0, & \text{if } |x| \geq 1 + b, \\ 1/b^{N+1}, & \text{if } 1 < |x| < 1 + b, \\ 1/b, & \text{if } |x| \leq 1. \end{cases} \tag{6.6.13}$$

We justify, e.g., the second estimate. If $1 < |x| < 1 + b$, we have, by (6.6.6),

$$\begin{aligned}
|\mathrm{D}U(x, b)| &\lesssim \frac{1}{b^{N+1}} \int_{B_b^N(x)} |u(x) - u(y)| \,\mathrm{d}y = \frac{1}{b^{N+1}} \int_{B_b^N(x)} |c_u - u(y)| \,\mathrm{d}y \\
&\leq \frac{1}{b^{N+1}} \int_{\mathbb{B}^N} |c_u - u(y)| \,\mathrm{d}y \lesssim \frac{1}{b^{N+1}}.
\end{aligned}$$

Combining (6.6.12) and (6.6.13), one gets

$$\left|\int_{\mathbb{R}^N} (\widetilde{U}_b)^\sharp \alpha \wedge \mathrm{d}\zeta\right| \lesssim \left(\frac{b^N}{b^{d+Nd}} + \frac{1}{b^d}\right)|\zeta|_{\mathrm{Lip}} \to 0 \quad \text{as } b \to +\infty. \tag{6.6.14}$$

We next note that

$$U_\varepsilon \to u \quad \text{in } W^{1,d}(\mathbb{R}^N) \text{ as } \varepsilon \to 0. \tag{6.6.15}$$

Combining (6.6.15) with the $W^{1,d}$-continuity of the superposition with Lipschitz functions (see, e.g., [BM21, Theorem 15.6]), we find that

$$\widetilde{\Pi} \circ U_\varepsilon \to \widetilde{\Pi} \circ u \quad \text{in } W^{1,d}(\mathbb{R}^N) \text{ as } \varepsilon \to 0,$$

i.e.,

$$\widetilde{U}(\cdot, \varepsilon) \to u \quad \text{in } W^{1,d}(\mathbb{R}^N) \text{ as } \varepsilon \to 0. \tag{6.6.16}$$

From (6.6.16) and the L^p-continuity of the superposition with Carathéodory functions (see, e.g., [Rin18, Theorem 2.13]), we have

$$(\widetilde{U}_\varepsilon)^\sharp \alpha \to u^\sharp \alpha \quad \text{in } L^1 \text{ as } \varepsilon \to 0. \tag{6.6.17}$$

Finally, (6.6.10) follows from Step 1, (6.6.11), (6.6.14), (6.6.17), and the fact that, for every extension α of ω, we have

$$u^\sharp \alpha = u^\sharp \omega \text{ a.e.}$$

(This last property follows from the chain rule for the superposition of a smooth map and a Sobolev map.) Step 2 is completed.

In view of (6.6.10), it is natural to define, for $u \in W^{s,p}_1(\mathbb{R}^N; \mathcal{N})$, $\langle Tu, \zeta\rangle$ as the quantity on the right hand side of (6.6.10). This requires first to prove that this quantity makes sense and is finite. The proof of these facts is reminiscent of the one of Theorem 6.4.1.

For this purpose, we first settle a measurability issue by introducing $\widetilde{h}(x)$, a convenient substitute of $h(x)$ defined as in (6.4.21). To motivate the definition of $\widetilde{h}(x)$ below, we note that, if $x \in \mathbb{R}^N$ is a Lebesgue point of f, then $h(x) > 0$. (Since we are no longer in the setting where $W^{s,p}$ is embedded into VMO, we cannot use (6.2.30) anymore to conclude that h has a uniform lower bound.) Assuming further that $h(x) < +\infty$, we therefore have

$$\frac{\iota}{2} = \operatorname{dist}(U(x, h(x)), \mathcal{N}) \leq |U(x, h(x)) - u(x)| \leq \int_0^{h(x)} |\partial_\varepsilon U(x, \varepsilon)| \,\mathrm{d}\varepsilon. \tag{6.6.18}$$

With (6.6.18) in mind, we set

$$V(x, \varepsilon) = |\partial_\varepsilon U(x, \varepsilon)|, \quad \text{for every } x \in \mathbb{R}^N, \text{for every } \varepsilon > 0, \quad c = \iota/2, \tag{6.6.19}$$

$$\widetilde{h}(x) = \begin{cases} 0, & \text{if } \int_0^1 V(x, \varepsilon)\,\mathrm{d}\varepsilon = +\infty, \\ +\infty, & \text{if } \int_0^t V(x, \varepsilon)\,\mathrm{d}\varepsilon < c \text{ for every } t > 0, \\ \inf\Big\{t > 0 \colon \int_0^t V(x, \varepsilon)\,\mathrm{d}\varepsilon \geq c\Big\}, & \text{otherwise.} \end{cases} \tag{6.6.20}$$

By the above, we have

$$\widetilde{h}(x) \leq h(x) \quad \text{for a.e. } x \in \mathbb{R}^N. \tag{6.6.21}$$

The measurability of $\widetilde{h}(x)$ is an easy consequence of the following result.

Lemma 6.6.3. *Let X be a metric space. Let $g \colon X \times (0, +\infty) \to [0, +\infty)$ be continuous. For $0 < c < +\infty$, define*

$$\widetilde{g}(x) = \begin{cases} 0, & \textit{if } \int_0^1 g(x, \varepsilon)\,\mathrm{d}\varepsilon = +\infty, \\ +\infty, & \textit{if } \int_0^t g(x, \varepsilon)\,\mathrm{d}\varepsilon < c, \textit{for every } t > 0, \\ \inf\Big\{t > 0 \colon \int_0^t g(x, \varepsilon)\,\mathrm{d}\varepsilon \geq c\Big\}, & \textit{otherwise.} \end{cases} \tag{6.6.22}$$

Then $\widetilde{g}(x)$ is a Borel function.

Granted Lemma 6.6.3, the function $\widetilde{h}$ is Borel.

Step 3. — For $u \in W^{s,p}_1(\mathbb{R}^N; \mathcal{N})$, the form $\widetilde{U}^\sharp(\mathrm{d}\alpha)$ is integrable over $\mathbb{R}^N \times (0, +\infty)$, and

$$\int_{\mathbb{R}^N \times (0, +\infty)} \big|\widetilde{U}^\sharp(\mathrm{d}\alpha)\big| \lesssim |u|^p_{W^{s,p}}. \tag{6.6.23}$$

Repeating the proof of (6.4.28) (relying on (6.6.6) instead of (6.4.15)) and using (6.6.21), we have, for a.e. $x \in \mathbb{R}^N$,

$$\begin{aligned}\int_0^{+\infty} |\widetilde{U}^\sharp(\mathrm{d}\alpha)(x,\varepsilon)|\,\mathrm{d}\varepsilon &\le \int_{h(x)}^{+\infty} |\widetilde{U}^\sharp(\mathrm{d}\alpha)(x,\varepsilon)|\,\mathrm{d}\varepsilon \\ &\le \int_{\widetilde{h}(x)}^{+\infty} |\widetilde{U}^\sharp(\mathrm{d}\alpha)(x,\varepsilon)|\,\mathrm{d}\varepsilon \\ &\lesssim \frac{1}{[\widetilde{h}(x)]^{sp}}.\end{aligned} \tag{6.6.24}$$

On the other hand, by Hölder's inequality, we have

$$\left(\int_0^t |\mathrm{D}U(x,\varepsilon)|\,\mathrm{d}\varepsilon\right)^p \le t^{sp}\left(\frac{p-1}{sp}\right)^{p-1}\int_0^t \varepsilon^{p(1-s)-1}|\mathrm{D}U(x,\varepsilon)|^p\,\mathrm{d}\varepsilon. \tag{6.6.25}$$

By the standard theory of weighted Sobolev spaces (see, e.g., [MR15, Theorem 1.2]), we have

$$\int_{\mathbb{R}^N\times(0,+\infty)} \varepsilon^{p(1-s)-1}|\mathrm{D}U(x,\varepsilon)|^p\,\mathrm{d}x\,\mathrm{d}\varepsilon \lesssim |u|^p_{W^{s,p}}. \tag{6.6.26}$$

Combining (6.6.25), (6.6.26), and the definition of $\widetilde{h}$, we find that $\widetilde{h}(x) > 0$ for a.e. $x \in \mathbb{R}^N$. On the other hand, if $0 < \widetilde{h}(x) < +\infty$, then clearly

$$\int_0^{\widetilde{h}(x)} |\partial_\varepsilon U(x,\varepsilon)|\,\mathrm{d}\varepsilon = \frac{\iota}{2}. \tag{6.6.27}$$

Combining (6.6.27) and (6.6.25), we find that

$$\frac{1}{[\widetilde{h}(x)]^{sp}} \lesssim \int_0^{+\infty} \varepsilon^{p(1-s)-1}|\partial_\varepsilon U(x,\varepsilon)|^p\,\mathrm{d}\varepsilon \quad \text{for a.e. } x \in \mathbb{R}^N. \tag{6.6.28}$$

The estimate (6.6.23) follows from (6.6.24), (6.6.26), and (6.6.28), hence completing Step 3.

Finally, the existence and uniqueness of the extension of T rely on the density property of $W^{1,d}_1(\mathbb{R}^N;\mathcal{N})$ in $W^{s,p}_1(\mathbb{R}^N;\mathcal{N})$ (see the discussion in Section 6.6.1) and the next step.

Step 4. — The map $W^{s,p}_1(\mathbb{R}^N;\mathcal{N}) \ni u \mapsto \widetilde{U}^\sharp(\mathrm{d}\alpha) \in L^1(\mathbb{R}^N\times(0,+\infty);\Lambda^{d+1})$ is continuous in the following sense: if $u_n, u \in W^{s,p}_1(\mathbb{R}^N\times(0,+\infty);\mathcal{N})$ are such that $|u_n - u|_{W^{s,p}} \to 0$ and $c_{u_n} \to c_u$, then

$$\widetilde{U}^\sharp_n(\mathrm{d}\alpha) \to \widetilde{U}^\sharp(\mathrm{d}\alpha) \text{ in } L^1(\mathbb{R}^N\times(0,+\infty)). \tag{6.6.29}$$

In order to prove (6.6.29), we first note that

$$(u_n - c_{u_n}) - (u - c_u) \to 0 \quad \text{in } L^p. \tag{6.6.30}$$

This follows using (i) $|u_n - u|_{W^{s,p}} \to 0$; (ii) the fact that $(u_n - c_{u_n}) - (u - c_u)$ is supported in $\mathbb{B}^N$; and (iii) the Poincaré type inequality $\|v\|_p \lesssim |g|_{W^{s,p}}$, valid for g supported in $\mathbb{B}^N$.

Using (6.6.30) we find that, up to a subsequence, $u_n \to u$ a.e. and then, by dominated convergence,

$$\mathrm{D}^t \widetilde{U}_n \to \mathrm{D}^t \widetilde{U} \quad \text{uniformly on compacts of } \mathbb{R}^N \times (0, +\infty), \text{ for every } t \in \mathbb{N}_*. \tag{6.6.31}$$

Using (6.6.31) with $t = 1$ yields

$$\widetilde{U}_n^\sharp(\mathrm{d}\alpha) \to \widetilde{U}^\sharp(\mathrm{d}\alpha) \quad \text{pointwise}. \tag{6.6.32}$$

On the other hand, let $\widetilde{h}_n$ be associated with u_n as in (6.6.20). By the proof of (6.4.28), we have

$$|\widetilde{U}_n^\sharp(\mathrm{d}\alpha)(x, \varepsilon)| \lesssim \frac{1}{\varepsilon^{d+1}} \chi_{\{\varepsilon > \widetilde{h}_n(x)\}}, \quad \text{for every } x \in \mathbb{R}^N, \text{ for every } \varepsilon > 0.$$

We next claim that there exists an $L^1(\mathbb{R}^N)$ function $H = H(x)$ such that, up to a subsequence,

$$\frac{1}{[\widetilde{h}_n(x)]^{sp}} \leq H(x). \tag{6.6.33}$$

Indeed, by (6.6.26), we have

$$\varepsilon^{1-s-1/p} \mathrm{D}U_n(x, \varepsilon) \to \varepsilon^{1-s-1/p} \mathrm{D}U(x, \varepsilon) \quad \text{in } L^p(\mathbb{R}^N \times (0, +\infty)).$$

By the converse to the dominated convergence theorem, up to a subsequence, there exists some $J = J(x, \varepsilon) \in L^p(\mathbb{R}^N \times (0, +\infty))$ such that

$$\varepsilon^{1-s-1/p} |\mathrm{D}U_n(x, \varepsilon)| \leq J(x, \varepsilon), \quad \text{for every } n, \text{ for every } x \in \mathbb{R}^N, \text{ for every } \varepsilon > 0.$$

For this subsequence, (6.6.28) yields

$$\frac{1}{[\widetilde{h}_n(x)]^{sp}} \lesssim \int_0^{+\infty} [J(x, \varepsilon)]^p \, \mathrm{d}\varepsilon = H(x),$$

so that (6.6.33) holds, as claimed.

Combining (6.6.32)–(6.6.33), we obtain (6.6.29), possibly up to a subsequence. However, the uniqueness of the limit in (6.6.29) implies that (6.6.29) holds for the full sequence.

The conclusions of the theorem follow from Steps 1–4. □

We next consider the case $d = 1$. As explained in Section 6.6.1, some care is needed to define initially Tu, because of the non-embedding $W_1^{1,1}(\mathbb{R}^N;\mathscr{N}) \not\subset W_1^{s,p}(\mathbb{R}^N;\mathscr{N})$. With this in mind, we have the following version of Theorem 6.6.1.

Theorem 6.6.4. *Let $d = 1$. Let $1 < q < 2$.*

(1) *The map T, defined in (6.6.2) for $u \in W_1^{1,q}(\mathbb{R}^N;\mathscr{N})$, has an (unique) extension by continuity to $W_1^{s,p}(\mathbb{R}^N;\mathscr{N})$.*

(2) *The extension, still denoted T, satisfies*

$$|\langle Tu,\zeta\rangle| \le C|u|^p_{W^{s,p}}|\zeta|_{\mathrm{Lip}},$$
for every $u \in W_1^{s,p}(\mathbb{R}^N;\mathscr{N})$, for every $\zeta \in \mathrm{Lip}(\mathbb{R}^N;\Lambda^{N-2})$,

for some constant $C > 0$ depending on s, p, N, and ω.

(3) *Let $\widetilde{\Pi}$ be as in Section 6.4 and $\alpha \in C_c^\infty(\mathbb{R}^\nu;\Lambda^1)$ be an extension of ω. We have the following formula:*

$$\langle Tu,\zeta\rangle = -\int_{\mathbb{R}^N\times(0,+\infty)} \widetilde{U}^\sharp(\mathrm{d}\alpha)\wedge \mathrm{d}\widetilde{\zeta},$$
for every $u \in W_1^{s,p}(\mathbb{R}^N;\mathscr{N})$, for every $\zeta \in \mathrm{Lip}(\mathbb{R}^N;\Lambda^{N-2})$.

Here, $\widetilde{U} = \widetilde{\Pi}\circ U$, with U defined by (6.6.5), and $\widetilde{\zeta}(x,t) = \zeta(x)$, for every $x \in \mathbb{R}^N$, for every $t \ge 0$.

Proof. The proof is essentially the same as the one of Theorem 6.6.1. The only difference occurs in Steps 1 and 2, where we rely on the Gagliardo–Nirenberg embedding $W_1^{1,q}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N) \subset W^{1/2,2}(\mathbb{R}^N)$, valid when $q > 1$ (but wrong when $q = 1$). □

We next prove Lemma 6.6.3.

Proof of Lemma 6.6.3. We only need to prove the lemma in the case where g is positive. Indeed, if the lemma holds for positive maps and g is only assumed to be nonnegative, we let $g_n(x,\varepsilon) = g(x,\varepsilon) + 1/n$, $n \ge 1$. Then, $\widetilde{g}_n \nearrow \widetilde{g}$, and by the lemma for positive functions, the $\widetilde{g}_n$'s are Borel functions. Therefore, $\widetilde{g}$ is a Borel function as well.

Hence, we assume that g is positive, and we will prove the lemma by constructing a sequence $(\widetilde{g}_n)_{n\in\mathbb{N}}$ of continuous functions such that $\widetilde{g}_n \to \widetilde{g}$ pointwise.

Step 1. — Define

$$g_n(x,\varepsilon) = \begin{cases} 1, & \text{if } \varepsilon < 1/n, \\ n, & \text{if } \varepsilon > n, \\ g(x,\varepsilon), & \text{otherwise,} \end{cases}$$

and let $\widetilde{g}_n$ be associated with g_n as in (6.6.22). Since $g(\cdot,\varepsilon)$ is continuous, we clearly have

$$\int_0^1 g_n(x,\varepsilon)\,\mathrm{d}\varepsilon < +\infty \quad \text{and} \quad \lim_{t\to+\infty} \int_0^t g_n(x,\varepsilon)\,\mathrm{d}\varepsilon = +\infty.$$

By (6.6.22) and the fact that $g > 0$, this implies that $0 < \widetilde{g}_n(x) < +\infty$ is the only number such that

$$\int_0^{\widetilde{g}_n(x)} g(x,\varepsilon)\,\mathrm{d}\varepsilon = c. \tag{6.6.34}$$

Combining (6.6.34) with the continuity of g, one easily obtains that $\widetilde{g}_n$ is continuous.

Step 2. — We prove that $\lim_{n\to+\infty} \widetilde{g}_n(x) = \widetilde{g}(x)$, which implies that $\widetilde{g}$ is a Borel function. To prove this, we have to consider the three cases occurring in the definition (6.6.22).

Assume first that $\int_0^1 g(x,t)\,\mathrm{d}\varepsilon = +\infty$ (and thus $\widetilde{g}(x) = 0$). For large n, we have $\widetilde{g}_n(x) > 1/n$ and

$$\int_{1/n}^{\widetilde{g}_n(x)} g_n(x,\varepsilon)\,\mathrm{d}\varepsilon = c - \frac{1}{n}. \tag{6.6.35}$$

On the other hand, for any given $t > 0$ and large n (depending on t), we have

$$\int_{1/n}^{t} g_n(x,\varepsilon)\,\mathrm{d}\varepsilon = \int_{1/n}^{t} g(x,\varepsilon)\,\mathrm{d}\varepsilon > c. \tag{6.6.36}$$

For such n, we have $\widetilde{g}_n(x) < t$. (This follows from (6.6.35) and (6.6.36).) Therefore, in this case we have $\widetilde{g}_n(x) \to 0$.

The case where $\widetilde{g}(x) = +\infty$ is similar, since for any fixed $0 < M < +\infty$ and large n (depending on M), we have

$$\int_{1/n}^{M} g_n(x,\varepsilon)\,\mathrm{d}\varepsilon = \int_{1/n}^{M} g(x,\varepsilon)\,\mathrm{d}\varepsilon < c - \frac{1}{n},$$

and thus, for such n, we have $\widetilde{g}_n(x) > M$.

Finally, assume that $0 < \widetilde{g}(x) < +\infty$, and thus $\int_0^{\widetilde{g}(x)} g(x,\varepsilon)\,\mathrm{d}\varepsilon = c$. If $t < \widetilde{g}(x)$, then, for large n, we have

$$\int_{1/n}^{t} g_n(x,\varepsilon)\,\mathrm{d}\varepsilon = \int_{1/n}^{t} g(x,\varepsilon)\,\mathrm{d}\varepsilon < \int_{0}^{t} g(x,\varepsilon)\,\mathrm{d}\varepsilon < c - \frac{1}{n},$$

and thus, for such n, we have $\widetilde{g}_n(x) > t$. Similarly, if $M > \widetilde{g}(x)$ then, for large n, we have $\widetilde{g}_n(x) < M$. □

By analogy with Corollary 6.5.4, we have the following Corollary.

Corollary 6.6.5. *Two cohomologous forms yield the same T.*

Proof. Let T_ω be the operator T associated with ω. Let $\omega_1 = \omega + \mathrm{d}\eta$, with $\eta \in C^\infty(\mathscr{N};\Lambda^{d-1})$, be an element of de Rham cohomology class $[\omega]$. If α is a extension of ω, we claim that $\alpha_1 = \alpha + \mathrm{d}(\psi\,\widetilde{\Pi}^\sharp\eta)$ (with ψ as in (6.4.16)) is an extension of ω_1. Indeed, this amounts to proving that $\mathrm{d}(\psi\,\widetilde{\Pi}^\sharp\eta)$ is an extension of $\mathrm{d}\eta$. In turn, this property is obtained as follows. We have

$$\mathrm{d}(\psi\,\widetilde{\Pi}^\sharp\eta) = \mathrm{d}\psi \wedge \widetilde{\Pi}^\sharp\eta + \psi\,\mathrm{d}\widetilde{\Pi}^\sharp\eta. \tag{6.6.37}$$

By the proof of (6.4.17) and the facts that $\psi = 1$ and $\mathrm{d}\psi = 0$ near $\mathscr{N}$, we find that the right-hand side of (6.6.37) is indeed an extension of $\mathrm{d}\eta$.

By Theorem 6.6.1 and the fact that clearly $\mathrm{d}\alpha_1 = \mathrm{d}\alpha$, we have

$$\langle T_{\omega_1}u,\zeta\rangle = -\int_{\mathbb{R}^N\times(0,+\infty)} \widetilde{U}^\sharp(\mathrm{d}\alpha_1)\wedge \mathrm{d}\widetilde{\zeta} = -\int_{\mathbb{R}^N\times(0,+\infty)} \widetilde{U}^\sharp(\mathrm{d}\alpha)\wedge \mathrm{d}\widetilde{\zeta} = \langle T_\omega u,\zeta\rangle. \quad \square$$

6.6.3 T hears singularities

In this section, we consider: (a) a smooth *closed* d-form ω on $\mathscr{N}$; (b) an integer $N > d$. In this setting, we provide, for nice u's, an explicit formula for $\langle Tu,\zeta\rangle$ in terms of the homotopy classes carried by the singular set of u.

We start by defining adapted nice u's. Consider the class

$$\begin{aligned}\mathscr{R}_1 = \{u\colon \mathbb{R}^N \to \mathscr{N} &: u \text{ is constant in } \mathbb{R}^N\setminus\mathbb{B}^N,\ u\in C^\infty(\overline{\mathbb{B}^N}\setminus\mathscr{S}_u),\\ &\mathscr{S}_u \text{ is a } (N-d-1)\text{-closed, oriented submanifold}\\ &\text{of } \mathbb{B}^N,\ |\mathrm{D}u(x)| \le C_u/\operatorname{dist}(x,\mathscr{S}_u), \text{ for every } x\in\mathbb{R}^N\}.\end{aligned}$$

(It is important to note that both the manifold $\mathscr{S}_u$ and the finite constant C_u depend on the nice map u.) Adapting the arguments in Chapter 3, one may prove that $\mathscr{R}_1$ is dense in $W^{s,p}_1(\mathbb{R}^N;\mathscr{N})$ when $0 < s \le 1$ and $d \le sp < d+1$. (See Theorem 3.1.2 for a similar statement in the function space $W^{s,p}((-1,1)^N;\mathscr{N})$.) However, since density is not relevant for the main result of this section, we overlook this property and focus on the calculation of Tu for $u \in \mathscr{R}_1$.

Let $u \in \mathscr{R}_1$ and let $\mathscr{S}_1, \dots, \mathscr{S}_\ell$ be the connected components of $\mathscr{S} = \mathscr{S}_u$. Consider, for $z \in \mathscr{S}_i$, the affine normal space $N_z\mathscr{S}_i$ to $\mathscr{S}_i$ (passing through z), with the natural orientation induced by the one of $\mathscr{S}_i$, i.e., we ask that a direct basis of $T_z\mathscr{S}_i$, completed with a direct basis of $N_z\mathscr{S}_i$, forms a direct basis of $\mathbb{R}^N$. Let $S_\varepsilon(z)$ be the sphere of radius ε of $N_z\mathscr{S}_i$ centered at z, with the orientation induced by the one of $N_z\mathscr{S}_i$. It is straightforward (using the fact that, on a sphere S, $f \mapsto \int_S f^\sharp\omega$ is a homotopical invariant; see Corollary 6.3.29) that the quantity $\int_{S_\varepsilon(z)} u^\sharp\omega$ does not depend on small ε (smallness depending only on $\mathscr{S}$) or on z. With this in mind, we may set

$$c_i = \int_{S_\varepsilon(z)} u^\sharp\omega, \quad \text{for every } z \in \mathscr{S}_i, \text{ for every } 0 < \varepsilon < \overline{\varepsilon} = \overline{\varepsilon}(\mathscr{S}).$$

Our result is the following.

Theorem 6.6.6. *Let $u \in \mathscr{R}_1$ and define c_i as above. Then*

$$\langle Tu, \zeta\rangle = (-1)^{d(N+1)+1} \sum_{i=1}^{\ell} c_i \int_{\mathscr{S}_i} \zeta, \quad \textit{for every } \zeta \in \operatorname{Lip}(\mathbb{R}^N;\Lambda^{N-d-1}). \tag{6.6.38}$$

The above result was obtained for a slightly different, less nice, dense class of maps, by M. Giaquinta, G. Modica, and J. Souček [GMS98b, Section 4.2, Theorem 1]; see also R. L. Jerrard and H. M. Soner [JS02, Theorem 1.2], G. Alberti, S. Baldo, and G. Orlandi [ABO03, Theorem 3.8], and P. Bousquet [Bou07, Proposition 1]. Their proofs require more advanced geometric measure theory arguments than the proof we present below, which merely relies on an iterated application of the Stokes formula.

To prove the theorem, we first consider a special case.

Lemma 6.6.7. *Let (x,y), with $x \in \mathbb{R}^{N-d-1}$ and $y \in \mathbb{R}^{d+1}$, denote a point in $\mathbb{R}^N$. Let $u = u(x,y)$, with $x \in \Omega$ and $y \in B^{d+1}_r(0) \setminus \{0\}$, be a smooth map such that $u(x,y) \in \mathscr{N}$ and*

$$|Du(x,y)| \le C_u |y|^{-1}, \quad \textit{for every } x \in \Omega, \textit{ for every } y \in B^{d+1}_r(0) \setminus \{0\}. \tag{6.6.39}$$

Let $\zeta \in \mathrm{Lip}_c(\Omega \times B_r^{d+1}(0); \Lambda^{N-d-1})$. *Then*

$$\begin{aligned}\int_{\Omega\times B_r^{d+1}(0)} u^\sharp\omega \wedge \mathrm{d}\zeta &= (-1)^{d(N+1)+1} \int_{S_\varepsilon} u(x_0,\cdot)^\sharp\omega \int_\Omega \zeta_x(\cdot,0) \\ &= (-1)^{d(N+1)+1} \int_{S_\varepsilon} u(x_0,\cdot)^\sharp\omega \int_{\Omega\times\{0\}} \zeta, \\ &\textit{for every } x_0 \in \Omega, \textit{for every } 0<\varepsilon<r.\end{aligned} \tag{6.6.40}$$

Here, (i) S_ε *is the sphere of radius* ε *of* $\mathbb{R}^{d+1}$ *centered at* 0; *and (ii)* $\zeta_x(x,y)$ *is the coefficient of* $\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{N-d-1}$ *in* ζ.

Proof. We first note that it suffices to prove (6.6.40) when $\zeta \in C_c^\infty$. (The general case is then obtained by smoothing, using dominated convergence in the first and the third integral.)

Using (i) the estimate (6.6.39); (ii) the fact that the degree of ω is less than $d+1$; (iii) Stokes' formula; and (iv) the fact that ζ has compact support in $\Omega \times B_r^{d+1}(0)$; we find that

$$\begin{aligned}\int_{\Omega\times B_r^{d+1}(0)} u^\sharp\omega \wedge \mathrm{d}\zeta &= (-1)^d \lim_{\varepsilon\to 0} \int_{\Omega\times\{\varepsilon<y<r\}} \mathrm{d}(u^\sharp\omega\wedge\zeta) \\ &= (-1)^{d+1} \lim_{\varepsilon\to 0} \int_{\Omega\times S_\varepsilon} u^\sharp\omega\wedge\zeta.\end{aligned} \tag{6.6.41}$$

Next, we write, with $\zeta_{\alpha,\beta} = \zeta_{\alpha,\beta}(x,y)$,

$$\zeta = \sum_{\substack{\alpha\subset[\![1,N-d-1]\!],\, \beta\subset[\![1,d+1]\!] \\ \#\alpha+\#\beta=N-d-1}} \zeta_{\alpha,\beta}\, \mathrm{d}x^\alpha \wedge \mathrm{d}y^\beta \tag{6.6.42}$$

(with the convention that the indices in α and β are taken in the natural order). Let us note that $\zeta_x = \zeta_{\alpha,\beta}$, with $\alpha = \{1,\ldots,N-d-1\}$ and $\beta = \emptyset$.

The decomposition (6.6.42) is suggesting that we have to show that the contribution in (6.6.41) of the coefficient $\zeta_{\alpha,\beta}$ converges to 0 as $\varepsilon \to 0$ when $\beta \neq \emptyset$. Without loss of generality, we may assume that $\alpha = \{1,2,\ldots,a\}$ (possibly with $a=0$) and $\beta = \{1,2,\ldots,b\}$, with $b \geq 1$. Set $\beta' = \{1,2,\ldots,b-1\}$ and let

$$\eta = y^b \cdot u^\sharp\omega \wedge (\zeta_{\alpha,\beta}\, \mathrm{d}x^\alpha \wedge \mathrm{d}y^{\beta'}).$$

Then η is an $(N-2)$-form in $\mathbb{R}^N$, smooth in $\mathbb{R}^{N-d-1} \times (\mathbb{R}^{d+1} \setminus \{0\})$, and such that $\eta = 0$

near $\partial(\Omega \times S_\varepsilon)$. Using Stokes' formula again, we find that

$$\begin{aligned} 0 = \int_{\Omega\times S_\varepsilon} \mathrm{d}\eta = (-1)^d \int_{\Omega\times S_\varepsilon} y^b \cdot u^\sharp\omega \wedge \mathrm{d}\zeta_{\alpha,\beta} \wedge \mathrm{d}x^\alpha \wedge \mathrm{d}y^{\beta'} \\ + (-1)^{N-2} \int_{\Omega\times S_\varepsilon} u^\sharp\omega \wedge (\zeta_{\alpha,\beta}\,\mathrm{d}x^\alpha \wedge \mathrm{d}y^\beta). \end{aligned} \tag{6.6.43}$$

Using the fact that $|\mathrm{D}u(x,y)| \lesssim |y|^{-1}$ and $\zeta \in C_c^\infty(\Omega \times \{|y| < r\}; \Lambda^{N-d-1})$, we have

$$\left| \int_{\Omega\times S_\varepsilon} y^b \cdot u^\sharp\omega \wedge \mathrm{d}\zeta_{\alpha,\beta} \wedge \mathrm{d}x^\alpha \wedge \mathrm{d}y^{\beta'} \right| \lesssim \int_{\Omega\times S_\varepsilon} \varepsilon \cdot \frac{1}{\varepsilon^d} = O(\varepsilon). \tag{6.6.44}$$

From (6.6.43) and (6.6.44), we find that, with $\alpha = \{1, \dots, N-d-1\}$, we have

$$\int_{\Omega\times S_\varepsilon} u^\sharp\omega \wedge \zeta = \int_{\Omega\times S_\varepsilon} u^\sharp\omega \wedge (\zeta_x\,\mathrm{d}x^\alpha) + O(\varepsilon). \tag{6.6.45}$$

Since $|\zeta_x(x,y) - \zeta_x(x,0)| \le \varepsilon \|\mathrm{D}\zeta_x\|_{L^\infty}$ for any $x \in \Omega$ and $y \in S_\varepsilon$, (6.6.45) implies that

$$\begin{aligned} \int_{\Omega\times S_\varepsilon} u^\sharp\omega \wedge \zeta &= \int_{\Omega\times S_\varepsilon} u^\sharp\omega \wedge (\zeta_x(x,0)\,\mathrm{d}x^\alpha) + O(\varepsilon) \\ &= \int_{\Omega\times S_\varepsilon} (u(x,\cdot)^\sharp\omega) \wedge (\zeta_x(x,0)\,\mathrm{d}x^\alpha) + O(\varepsilon) \\ &= (-1)^{d(N-d-1)} \int_\Omega \zeta_x(x,0) \left(\int_{S_\varepsilon} u(x,\cdot)^\sharp\omega \right) \mathrm{d}x + O(\varepsilon), \end{aligned} \tag{6.6.46}$$

where the last equality follows from the Fubini theorem. Combining (6.6.41) and (6.6.46), we obtain (6.6.40), since, by standard (smooth) homotopy arguments, the integral $\int_{S_\varepsilon} u(x,\cdot)^\sharp\omega$ does not depend on $x \in \Omega$ and $\varepsilon < r$. □

Proof of Theorem 6.6.6. As in the proof of Lemma 6.6.7, we may assume that ζ is smooth. Without loss of generality, we may also assume that $\operatorname{supp}\zeta \subset \mathbb{B}^N$.

Consider a finite cover of $\overline{\mathbb{B}^N}$ with open sets U_j such that, for each j, either $U_j \cap \mathscr{S} = \varnothing$ or there exists an orientation preserving diffeomorphism $\Phi_j \colon \{|x| < 1\} \times \{|y| < 1\} \to U_j$ such that $\Phi_j^{-1}(\mathscr{S} \cap U_j) = \{(x,0) \colon |x| < 1\}$. (Here, as in Lemma 6.6.7, we have $x \in \mathbb{R}^{N-d-1}$ and $y \in \mathbb{R}^{d+1}$.) Using a partition of unity subordinated to the cover (U_j) and the linearity of (6.6.38) with respect to ζ, we may assume that ζ is compactly supported in some U_j.

If $\mathscr{S} \cap U_j = \varnothing$, then

$$\int_{\mathbb{R}^N} u^\sharp\omega \wedge \mathrm{d}\zeta = (-1)^d \int_{U_j} \mathrm{d}(u^\sharp\omega \wedge \zeta) = 0.$$

If $\mathcal{S} \cap U_j \neq \emptyset$, let i be such that $\mathcal{S} \cap U_j \subset \mathcal{S}_i$. Using (i) the estimate $|Df(x)| \leq C_u/\operatorname{dist}(x, \mathcal{S}_u)$; (ii) the fact that the degree of ω is less than $d+1$; and (iii) standard properties of the exterior differential calculus, we find that

$$\begin{aligned}\int_{\mathbb{R}^N} u^\sharp\omega \wedge d\zeta &= \int_{U_j} u^\sharp\omega \wedge d\zeta \\ &= \int_{\{|x|<1\}\times\{|y|<1\}} [(u \circ \Phi_j)^\sharp\omega] \wedge d(\Phi_j^\sharp\zeta).\end{aligned}$$

We deduce from Lemma 6.6.7 that, for $|x_0| < 1$ and $\varepsilon < 1$,

$$\begin{aligned}&\int_{\{|x|<1\}\times\{|y|<1\}} [(u \circ \Phi_j)^\sharp\omega] \wedge d(\Phi_j^\sharp\zeta) \\ &\qquad = (-1)^{d(N+1)+1} \int_{S_\varepsilon} (u \circ \Phi_j(x_0, \cdot))^\sharp\omega \int_{\{|x|<1\}\times\{0\}} \Phi_j^\sharp\zeta.\end{aligned} \tag{6.6.47}$$

By change of variables, the latter integral in (6.6.47) equals

$$\int_{\mathcal{S}\cap U_j} \zeta = \int_{\mathcal{S}_i} \zeta. \tag{6.6.48}$$

Therefore, (6.6.38) follows from (6.6.47) (with $x_0 = 0$) and (6.6.48) provided

$$\int_{S_\varepsilon} (u \circ \Phi_j(0, \cdot))^\sharp\omega = c_i. \tag{6.6.49}$$

For this purpose, consider, for $z \in \mathcal{S}$, an orientation preserving isometry $T = T_z$ of $\mathbb{R}^{d+1}$ onto $N_z\mathcal{S}$ such that $T(0) = z$. We now obtain (6.6.49) from

$$\int_{S_\varepsilon} (u \circ \Phi_j(0, \cdot))^\sharp\omega = \int_{\Phi(\{0\}\times S_\varepsilon)} u^\sharp\omega$$

and standard (smooth) homotopy arguments, using the fact that, for small ε, the embeddings

$$S_\varepsilon \ni y \mapsto \Phi_j(0, y), \text{ respectively } S_\varepsilon \ni y \mapsto T_{\Phi_j(0,0)}(y)$$

of $\Phi_j(\{0\} \times S_\varepsilon)$, respectively $S_\varepsilon(\Phi_j(0,0))$ (viewed as a positively oriented sphere on $N_{\Phi_j(0,0)}\mathcal{S}$) are isotopical in $\mathbb{R}^N \setminus \mathcal{S}$. □

6.6.4 Slicing

In this section, we consider: (a) a smooth *closed* d-form ω on $\mathcal{N}$; (b) $0 < s < 1$ and $1 < p < +\infty$ such that $sp = d$; (c) an integer $N \geq d+1$ (most often, $N > d+1$).

We start with a formal calculation that will provide insight for the main results in this section. Let $N > d+1$ and write $N = \ell + l$, with $\ell \geq d+1$ and $l \geq 1$. Let $u \in W^{1,d}_1(\mathbb{R}^\ell; \mathcal{N})$ and consider a Lipschitz form of the type $\zeta = \eta \, \mathrm{d}x^\alpha = \eta(x) \, \mathrm{d}x^\alpha$, with $x \in \mathbb{R}^\ell$ and $\alpha \subset [\![1, \ell]\!]$, $\#\alpha = \ell - d - 1$. Then,

$$\langle Tu, \zeta \rangle = \int_{\mathbb{R}^\ell} u^\sharp \omega \wedge \mathrm{d}\zeta = \int_{\mathbb{R}^\ell} u^\sharp \omega \wedge \mathrm{d}\eta \wedge \mathrm{d}x^\alpha.$$

Consider next $u = u(x, y) \in W^{1,d}_1(\mathbb{R}^N; \mathcal{N})$, with $x \in \mathbb{R}^\ell$ and $y \in \mathbb{R}^l$, and a Lipschitz form of the type

$$\zeta = \eta \, \mathrm{d}x^\alpha \wedge \mathrm{d}y = \eta(x, y) \, \mathrm{d}x^\alpha \wedge \mathrm{d}y, \tag{6.6.50}$$

with $\#\alpha = \ell - k - 1$ and $\mathrm{d}y = \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^l$. Using the identity

$$u^\sharp \omega \wedge \mathrm{d}[\eta \, \mathrm{d}x^\alpha \wedge dy] = [(u(\cdot, y))^\sharp \omega] \wedge \mathrm{d}(\eta(\cdot, y) \, \mathrm{d}x^\alpha) \wedge \mathrm{d}y,$$

and the Fubini theorem, we find that, for ζ as in (6.6.50), we have

$$\langle Tu, \zeta \rangle = \int_{\mathbb{R}^l} \langle Tu(\cdot, y), \eta(\cdot, y) \, \mathrm{d}x^\alpha \rangle \, \mathrm{d}y. \tag{6.6.51}$$

Our first purpose in this section is to extend the validity of (6.6.51) to $u \in W^{s,p}_1(\mathbb{R}^N; \mathcal{N})$, allowing also permutations of the coordinates x^i and y^j.

Consider a partition $[\![1, N]\!] = A \sqcup B$, with $A = \{i_1 < i_2 < \cdots < i_\ell\}$, $B = \{j_1 < j_2 < \cdots < j_l\}$. (The above calculations correspond to the choice $A = [\![1, \ell]\!]$, $B = [\![\ell+1, N]\!]$.) Given a point $z = (z^1, \ldots, z^N) \in \mathbb{R}^N$, let $x = (z^{i_1}, \ldots, z^{i_\ell}) \sim (x^1, \ldots, x^\ell) \in \mathbb{R}^\ell$, $y = (z^{j_1}, \ldots, z^{j_l}) \sim (y^1, \ldots, y^l) \in \mathbb{R}^l$, and identify z with (x, y). We associate with each partition (A, B) a *signature* $\sigma = \sigma(B) \in \{-1, 1\}$ through the formula

$$\mathrm{d}x^1 \wedge \cdots \mathrm{d}x^\ell \wedge \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^l = \sigma(B) \, \mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^N. \tag{6.6.52}$$

Let $\alpha \subset [\![1, \ell]\!]$ be such that $\#\alpha = \ell - d - 1$ and consider an *elementary* Lipschitz form of the type

$$\zeta = \eta \, \mathrm{d}x^\alpha \wedge \mathrm{d}y = \eta(x, y) \, \mathrm{d}x^\alpha \wedge \mathrm{d}y. \tag{6.6.53}$$

It is important to note that every Lipschitz form is the sum of at most $\binom{N}{N-d-1}$ Lipschitz forms as in (6.6.53), and thus Proposition 6.6.8 below provides a *slicing* or *disintegration* formula for $\langle Tf, \zeta\rangle$ for *any* ζ.

We next note that, if $u \in W_1^{s,p}(\mathbb{R}^N; \mathscr{N})$, then, for a.e. $y = (y^1, \dots, y^l) \in \mathbb{R}^l$, the partial function $u(\cdot, y)$ belongs to $W_1^{s,p}(\mathbb{R}^\ell; \mathscr{N})$, and thus the distribution $Tu(\cdot, y)$ makes sense and acts on forms $\xi \in \mathrm{Lip}(\mathbb{R}^\ell; \Lambda^{\ell-d-1})$.

Proposition 6.6.8. *Let $N > d+1$. Let $u \in W_1^{s,p}(\mathbb{R}^N; \mathscr{N})$ and let ζ be as in* (6.6.53)*. Then, the map*

$$\mathbb{R}^l \ni y \mapsto G_u(y) = \langle Tu(\cdot, y), \eta(\cdot, y)\, \mathrm{d}x^\alpha\rangle$$

is defined a.e. and is (*Lebesgue*) *integrable.*

Moreover, we have, with $\sigma = \sigma(B)$ as in (6.6.52),

$$\begin{aligned}\langle Tu, \eta\, \mathrm{d}x^\alpha \wedge \mathrm{d}y\rangle &= \sigma \int_{\mathbb{R}^l} \langle Tu(\cdot, y), \eta(\cdot, y)\, \mathrm{d}x^\alpha\rangle\, \mathrm{d}y \\ &= \sigma \int_{\mathbb{R}^l} \langle Tu(\cdot, y), \eta(\cdot, y)\, \mathrm{d}x^\alpha\rangle\, \mathrm{d}\mathscr{H}^\nu(y).\end{aligned} \tag{6.6.54}$$

In the special case where $\mathscr{N} = \mathbb{S}^1$ and ω is the standard volume form, formula (6.6.54) was proved by P. Mironescu, E. Russ, and Y. Sire [MRS20, Section 3.4, (3.64)].

Proof. We present the proof when $d > 1$. The case $d = 1$ is similar; we start from $u \in W_1^{1,q}$, with $1 < q < 2$, instead of $u \in W_1^{1,1}$. We divide the proof into two steps.

Step 1. — Formula (6.6.54) holds for $u \in W_1^{1,d}(\mathbb{R}^N; \mathscr{N})$. Indeed, arguing as in the proof of (6.6.51) and using (i) Theorem 6.6.1; (ii) the identity

$$u^\sharp\omega \wedge \mathrm{d}\eta \wedge \mathrm{d}x^\alpha \wedge \mathrm{d}y = u(\cdot, y)^\sharp\omega \wedge \mathrm{d}[\eta(\cdot, y)\, \mathrm{d}x^\alpha] \wedge \mathrm{d}y;$$

(iii) the definition of σ in (6.6.52); and (iv) the Fubini theorem, we find that

$$\begin{aligned}\langle Tu, \eta\, \mathrm{d}x^\alpha \wedge \mathrm{d}y\rangle &= \int_{\mathbb{R}^N} u^\sharp\omega \wedge \mathrm{d}\eta \wedge \mathrm{d}x^\alpha \wedge \mathrm{d}y \\ &= \int_{\mathbb{R}^N} u(\cdot, y)^\sharp\omega \wedge \mathrm{d}[\eta(\cdot, y)\, \mathrm{d}x^\alpha] \wedge \mathrm{d}y \\ &= \sigma \int_{\mathbb{R}^l} \left(\int_{\mathbb{R}^\ell} u(\cdot, y)^\sharp\omega \wedge \mathrm{d}[\eta(\cdot, y)\, \mathrm{d}x^\alpha] \right) \mathrm{d}\mathscr{H}^l(y) \\ &= \sigma \int_{\mathbb{R}^l} \langle Tu(\cdot, y), \eta(\cdot, y)\, \mathrm{d}x^\alpha\rangle\, \mathrm{d}\mathscr{H}^l(y).\end{aligned}$$

Incidentally, the Fubini theorem implies that G_u is Lebesgue integrable.

Step 2. — If $(u_n)_{n\in\mathbb{N}}$ in $W_1^{1,d}(\mathbb{R}^N;\mathcal{N})$, $u \in W_1^{s,p}(\mathbb{R}^N;\mathcal{N})$, and $u_n \to u$ in $W_1^{s,p}$, then $G_{u_n} \to G_u$ in $L^1(\mathbb{R}^l)$. Indeed, it suffices to obtain the conclusion up to a subsequence (then use Theorem 6.6.1 on the left-hand side of (6.6.54)). The argument is similar to the one used in Step 4 in the proof of Theorem 6.6.1. There exists a null set $A \subset \mathbb{R}^l$ and a function $F \in L^p(\mathbb{R}^l)$ such that, possibly up to a subsequence, we have

$$u_n(\cdot, y) \to u(\cdot, y) \text{ in } W_1^{s,p}(\mathbb{R}^\ell), \quad \text{for each } y \in \mathbb{R}^l \setminus A, \tag{6.6.55}$$

$$|u_n(\cdot, y)|_{W^{s,p}} \le F(y), \quad \text{for each } y, \text{ for each } n. \tag{6.6.56}$$

Combining (6.6.55) and Theorem 6.6.1, we have $G_{u_n}(y) \to G_u(y)$, for every $y \in \mathbb{R}^l \setminus A$. On the other hand, (6.6.56) and (6.6.3) imply that $|G_{u_n}(y)| \lesssim [F(y)]^p$, for every y, for every n, whence the conclusion of Step 2. □

Consider next a general $\zeta \in \mathrm{Lip}(\mathbb{R}^N;\Lambda^{N-d-1})$. Then, we may write $\zeta = \sum_\gamma \zeta_\gamma \, dz^\gamma = \sum_\gamma \zeta_\gamma(z) \, dz^\gamma$. Here the sum is taken over $\gamma \subset [\![1, N]\!]$ such that $\#\gamma = N - d - 1$. We may rewrite

$$\zeta = \sum_\alpha \sum_\beta \eta_{\alpha,\beta} \, dz^\alpha \wedge dz^\beta = \sum_\alpha \sum_\beta \eta_{\alpha,\beta}(z) \, dz^\alpha \wedge dz^\beta, \tag{6.6.57}$$

where (i) the sums are over $\alpha \subset [\![1, N]\!]$ such that $\#\alpha = l - d - 1$, respectively $\beta \subset [\![1, N]\!]$ such that $\#\beta = l$; and (ii)

$$\eta_{\alpha,\beta} = \begin{cases} (C_l)^{-1}\sigma(\alpha,\beta)\zeta_{\alpha\sqcup\beta}, & \text{if } \alpha \cap \beta = \varnothing, \\ 0, & \text{if } \alpha \cap \beta \ne \varnothing, \end{cases}$$

where $C_l = \binom{N-d-1}{l}$ and $\sigma(\alpha,\beta) \in \{-1, 1\}$ is the sign such that $dz^\alpha \wedge dz^\beta = \sigma(\alpha,\beta) \, dz^{\alpha\sqcup\beta}$.

The following Corollary is a direct consequence of Proposition 6.6.8 and the identity (6.6.57).

Corollary 6.6.9. *With the notation above, we have*

$$\begin{aligned} \langle Tu, \zeta\rangle &= \sum_\beta \sigma(\beta) \int_{\mathbb{R}^l} \Big\langle Tu(\cdot, z^\beta), \sum_\alpha \eta_{\alpha,\beta}(\cdot, z^\beta) \, dz^\alpha \Big\rangle \, dz^\beta \\ &= \frac{1}{\binom{N-d-1}{l}} \sum_\alpha \sum_\beta \sigma(\alpha,\beta)\sigma(\beta) \int_{\mathbb{R}^l} \langle Tu(\cdot, z^\beta), \zeta_{\alpha\sqcup\beta}(\cdot, z^\beta) \, dz^\alpha\rangle \, dz^\beta. \end{aligned}$$

With the help of the slicing property, one can prove the following dimensional reduction property. *In the next result, we consider the setting of Proposition 6.6.8.*

Proposition 6.6.10. *Let $\rho = \rho(y)$ be a standard mollifier and set $\rho_{\varepsilon,y_0}(y) = \rho_\varepsilon(y - y_0)$, for each $y, y_0 \in \mathbb{R}^l$. Then, for a.e. $y_0 \in \mathbb{R}^l$, we have, with $\sigma = \sigma(B)$ as in (6.6.52),*

$$\langle Tu(\cdot, y_0), \xi\rangle = \sigma \lim_{\varepsilon\to 0}\langle Tu, \xi \wedge (\rho_{\varepsilon,y_0}\,\mathrm{d}y)\rangle, \quad \textit{for every } \xi \in \mathrm{Lip}(\mathbb{R}^\ell; \Lambda^{\ell-d-1}). \tag{6.6.58}$$

Proof. Using Proposition 6.6.8 and the fact that, once B is fixed, the signature $\sigma(B)$ does not depend on the choice of $\alpha \subset A$, we find that

$$\langle Tu, \xi \wedge (\rho_{\varepsilon,y_0}\,\mathrm{d}y)\rangle = \sigma \int_{\mathbb{R}^l} \langle Tu(\cdot, y), \xi\rangle \rho_{\varepsilon,y_0}(y)\,\mathrm{d}y. \tag{6.6.59}$$

Let $\widetilde{\rho} = \widetilde{\rho}(x)$ be a standard mollifier in $\mathbb{R}^\ell$. Using the notation in Section 6.6.2, set (with α as in (6.4.16)):

$$u_y(x) = u(x, y),\, U_y(x, \widetilde{\varepsilon}) = u_y\widetilde{\rho}_{\widetilde{\varepsilon}},\, \widetilde{U}_y = \widetilde{\Pi} \circ U_y,\, H_y = \widetilde{U}_y(\mathrm{d}\alpha),$$
$$x \in \mathbb{R}^\ell, y \in \mathbb{R}^l, \widetilde{\varepsilon} > 0.$$

Then there exists a null set $A \subset \mathbb{R}^l$ such that $u(\cdot, y) \in W^{s,p}_1(\mathbb{R}^\ell)$, for every $y \in \mathbb{R}^l \setminus A$. Formula (6.6.23) in Step 3 in the proof of Theorem 6.6.1 implies that, for every $y \in \mathbb{R}^l \setminus A$, we have

$$\int_{\mathbb{R}^\ell\times(0,+\infty)} |H_y|\,\mathrm{d}x\,\mathrm{d}\widetilde{\varepsilon} \lesssim |u(\cdot, y)|^p_{W^{s,p}}. \tag{6.6.60}$$

Combining (6.6.60) and the Besov type inequality

$$\int_{\mathbb{R}^l} |u(\cdot, y)|^p_{W^{s,p}}\,\mathrm{d}y \lesssim |u|^p_{W^{s,p}} \tag{6.6.61}$$

(see, e.g., [BM21, Corollary 15.1] or [Leo23, Theorem 6.35]), we obtain

$$\mathbb{R}^l \ni y \mapsto H_y \in \mathscr{L}^1(\mathbb{R}^l; L^1(\mathbb{R}^\ell \times (0, +\infty))).$$

We next note that, by (6.6.59) and (6.6.4), we have

$$
\begin{aligned}
|\langle Tu, \xi \wedge (\rho_{\varepsilon,y_0}\, \mathrm{d}y)\rangle - \sigma\langle Tu(\cdot, y_0), \xi\rangle| \\
&\lesssim \fint_{B^l_\varepsilon(y_0)} |\langle Tu(\cdot, y), \xi\rangle - \langle Tu(\cdot, y_0), \xi\rangle|\, \mathrm{d}y \\
&\lesssim |\xi|_{\mathrm{Lip}} \fint_{B^l_\varepsilon(y_0)} \int_{\mathbb{R}^\ell\times(0,+\infty)} |H_y - H_{y_0}|\, \mathrm{d}x\, \mathrm{d}\varepsilon'\, \mathrm{d}y \\
&= |\xi|_{\mathrm{Lip}} \fint_{B^l_\varepsilon(y_0)} \|H_y - H_{y_0}\|_{L^1}\, \mathrm{d}y.
\end{aligned}
\tag{6.6.62}
$$

We finally invoke the vector-valued Lebesgue differentiation theorem (see, e.g., [HKST15, Section 3.4]), which implies that, for a.e. $y_0 \in \mathbb{R}^l$, we have

$$
\lim_{\varepsilon\to 0} \fint_{B^l_\varepsilon(y_0)} \|H_y - H_{y_0}\|_{L^1}\, \mathrm{d}y = 0. \tag{6.6.63}
$$

Combining (6.6.63) with (6.6.62), one obtains (6.6.58). □

Remark 6.6.11. In the above proof, the exceptional null set A depends on the choice of the closed d-form ω. We claim that, actually, we may pick *the same* null set for *every* ω. Indeed, by Corollary 6.6.5, the exceptional set A depends only on the de Rham cohomology class $[\omega]$. On the other hand, it is clear that the mapping $\omega \mapsto T_\omega$ (with ω closed d-form) is linear. The claim follows by combining these considerations with the fact that the d-th de Rham cohomology group of $\mathscr{N}$ is of finite dimension (since $\mathscr{N}$ is compact). □

We next present a version of slicing in the case where $N = d + 1$. As discussed in Section 6.5.3, this requires considering test forms ζ whose restriction to d-dimensional slices are constant. Proposition 6.6.13 below is such a possible result (others could be considered) and is fitted to our main result, Theorem 6.6.22.

To start with, we note the following lemma.

Lemma 6.6.12. *Let $u \in W^{s,p}(\mathbb{R}^{d+1})$ and set $Q^{d+1}_r = (-r, r)^{d+1}$. Then, for every $P \in \mathbb{R}^{d+1}$, we have*

$$
u_{|P+\partial Q^{d+1}_r} \in W^{s,p}(P + \partial Q^{d+1}_r)\ \text{for a.e. } r > 0. \tag{6.6.64}
$$

Moreover, for a.e. $r > 0$, we have $u_{|P+\partial Q^{d+1}_r} \in W^{s,p}(P + \partial Q^{d+1}_r)$ for a.e. $P \in \mathbb{R}^{d+1}$.

Proof. Let $0 < a < b < +\infty$. Using the fact that the set $\{x \in \mathbb{R}^{d+1} : a \leq |x - P| \leq b\}$ is bi-Lipschitz homeomorphic to $\partial Q^{d+1}_1 \times [0, 1]$ and a standard cousin of (6.6.61), we find

that

$$\int_a^b |u_{|P+\partial Q_r^{d+1}}|^p_{W^{s,p}(P+\partial Q_r^{d+1})}\,\mathrm{d}r \lesssim |u|^p_{W^{s,p}}, \tag{6.6.65}$$

whence (6.6.64).

The second part follows from the Tonelli theorem. □

Proposition 6.6.13. *Let $u \in W_1^{s,p}(\mathbb{R}^{d+1};\mathcal{N})$ and $\psi\colon (0,+\infty) \to \mathbb{R}$ be Lipschitz, with* $\operatorname{supp}\psi' \subset (0,+\infty)$*. Then, for every $P \in \mathbb{R}^{d+1}$, we have*

$$\langle Tu, \psi(|\cdot - P|_\infty)\rangle = (-1)^d \int_0^{+\infty} \psi'(r)\mathcal{I}(u_{|P+\partial Q_r^{d+1}})\,\mathrm{d}r. \tag{6.6.66}$$

Here, (a) $\mathcal{I}(u_{|P+\partial Q_r^{d+1}}) = \mathcal{I}_{P+\partial Q_r^{d+1},\omega}(u_{|P+\partial Q_r^{d+1}})$ is defined in Corollary 6.3.29 (with $\mathcal{M} = P + \partial Q_r^{d+1}$); and (b) the orientation on $P + \partial Q_r^{d+1}$ is as in Example 6.3.16.

Proof. With no loss of generality, we may assume that $P = 0$. Assume that $\operatorname{supp}\psi' \subset (a,b)$ for some $a, b > 0$.

Step 1. — Proof of (6.6.66) when $u \in W_1^{1,d}(\mathbb{R}^{d+1};\mathcal{N})$. In this case, $u^\sharp\omega$ can be written as $\sum_\ell \beta_\ell \,\widehat{\mathrm{d}x^\ell}$, with $\beta_\ell = \beta_\ell(x) \in L^1_c(\mathbb{R}^{d+1})$ and $\widehat{\mathrm{d}x^\ell}$ as in (6.3.35). Let

$$\Omega_{\ell,\pm} = \{x = (x^1,\ldots,x^{d+1}) \in \mathbb{R}^{d+1}\colon \max_{j\neq\ell}|x^j| < \pm x^\ell\},\quad 1 \le \ell \le d+1.$$

Then,

$$\begin{aligned}\langle Tu, \psi(|\cdot|_\infty)\rangle &= \sum_\ell \int_{\mathbb{R}^{d+1}} \beta_\ell \,\widehat{\mathrm{d}x^\ell} \wedge \mathrm{d}[\psi(|\cdot|_\infty)]\\ &= \sum_\ell \Big(\int_{\Omega_{\ell,+}} \beta_\ell \psi'(x^\ell)\,\widehat{\mathrm{d}x^\ell}\wedge \mathrm{d}x^\ell - \int_{\Omega_{\ell,-}} \beta_\ell\psi'(-x^\ell)\,\widehat{\mathrm{d}x^\ell}\wedge\mathrm{d}x^\ell\Big).\end{aligned}$$

By the Fubini theorem,

$$\begin{aligned}\int_{\Omega_{\ell,+}} \beta_\ell\psi'(x^\ell)\,\widehat{\mathrm{d}x^\ell}\wedge\mathrm{d}x^\ell &= (-1)^{d-\ell+1}\int_{\Omega_{\ell,+}} \beta_\ell\psi'(x^\ell)\\ &= (-1)^{d-\ell+1}\int_a^b \psi'(x^\ell)\Big(\int_{F_{\ell,x^\ell,+}} \beta_\ell(\cdot,x^\ell)\Big)\,\mathrm{d}x^\ell,\end{aligned}$$

where

$$F_{\ell,x^\ell,\pm} = \{\widehat{x^\ell} = (x^1,\ldots,x^{\ell-1},x^{\ell+1},\ldots,x^{d+1}) \in \mathbb{R}^d\colon (x^1,\ldots,x^\ell,\ldots,x^{d+1}) \in \Omega_{\ell,\pm}\}.$$

A similar identity holds on $\Omega_{\ell,-}$. Taking the sum over ℓ and using (i) Definition 6.3.24; (ii) equation (6.3.36) in Example 6.3.18 (choosing $C = P + Q_r^{d+1}$ and $\alpha_{\ell,\pm}(x) = \beta_\ell(x^1, \ldots, x^{\ell-1}, \pm x^\ell, x^{\ell+1}, \ldots, x^{k+1})$); and (iii) Proposition 6.3.33, we find that

$$
\begin{aligned}
\langle Tu, \psi(|\cdot|_\infty)\rangle &= \sum_\ell (-1)^{d-\ell+1} \int_a^b \psi'(x^\ell)\Big(\int_{F_{\ell,x^\ell,+}} \beta_\ell(\cdot, x^\ell)\Big)\mathrm{d}x^\ell \\
&\quad - \sum_\ell (-1)^{d-\ell+1} \int_{-b}^{-a} \psi'(-x^\ell)\Big(\int_{F_{\ell,x^\ell,-}} \beta_\ell(\cdot, -x^\ell)\Big)\mathrm{d}x^\ell \\
&= (-1)^d \int_a^b \psi'(r)\Big(\int_{\partial Q_r^{d+1}} (u_{|\partial Q_r^{d+1}})^\sharp \omega\Big)\,\mathrm{d}r \\
&= (-1)^d \int_a^b \psi'(r)\mathscr{I}(u_{|\partial Q_r^{d+1}})\,\mathrm{d}r.
\end{aligned}
$$

Step 2. — Proof of (6.6.66) for a general $u \in W_1^{s,p}(\mathbb{R}^{d+1}; \mathscr{N})$. Consider some $d < q < d+1$ and a sequence $(u_n)_{n\in\mathbb{N}}$ in $W_1^{1,q}(\mathbb{R}^{d+1}; \mathscr{N})$ such that

$$
u_n \to u \text{ in } W_1^{s,p}(\mathbb{R}^{d+1}; \mathscr{N}).
$$

By Lemma 6.6.12 and a standard argument, possibly up to a subsequence, we have, for a.e. $r > 0$,

$$
u_{n|\partial Q_r^{d+1}} \to u_{|\partial Q_r^{d+1}} \text{ in } W^{s,p}(\partial Q_r^{d+1}; \mathscr{N}). \tag{6.6.67}
$$

Set

$$
F_n(r) = \int_{\partial Q_r^{d+1}} (u_{n|\partial Q_r^{d+1}})^\sharp \omega \quad \text{and} \quad F(r) = \mathscr{I}(u_{|\partial Q_r^{d+1}})
$$

(which are well-defined for a.e. $r > 0$). Using (i) (6.6.67); (ii) the embedding $W^{s,p} \hookrightarrow \mathrm{VMO}$; (iii) Proposition 6.3.33; and (iv) Corollary 6.3.29, we find that

$$
F_n(r) = \mathscr{I}(u_{n|\partial Q_r^{d+1}}) \to F(r) \quad \text{for a.e. } r > 0. \tag{6.6.68}
$$

In view of Theorem 6.6.1 and Step 1, in order to obtain (6.6.66), it suffices to prove that $F_n \to F$ in $L^1((a, b))$. For this purpose, consider $\Phi_r \colon \partial Q_1^{d+1} \to \partial Q_r^{d+1}$, $\Phi_r(x) = rx$. By Corollary 6.3.31, we have

$$
F_n(r) = \mathscr{I}(u_{n|\partial Q_r^{d+1}} \circ \Phi_r).
$$

Combining this with Theorem 6.4.1 with $\mathscr{M} = \partial Q_1$, we obtain

$$|F_n(r)| \leq C\big|u_{n|\partial Q_r^{d+1}} \circ \Phi_r\big|^p_{W^{s,p}(\partial Q_1^{d+1})} = C\big|u_{n|\partial Q_r^{d+1}}\big|^p_{W^{s,p}(\partial Q_r^{d+1})}. \tag{6.6.69}$$

Combining (6.6.65), (6.6.68), (6.6.69), and the converse to the dominated convergence theorem, we obtain the desired conclusion $F_n \to F$ in $L^1((a,b))$. □

Using a special choice of ψ, we obtain the following variant of Proposition 6.6.13 adapted to boundaries of cubes.

Proposition 6.6.14. *Let $u \in W^{s,p}_1(\mathbb{R}^{d+1};\mathscr{N})$. For $\varepsilon > 0$ and $0 < \eta \leq \varepsilon/2$, let $\psi_\eta = \psi_{\varepsilon,\eta}$ be defined by*

$$\psi_\eta(r) = \psi_{\eta,\varepsilon}(r) = \begin{cases} 1, & \text{if } r \leq \varepsilon - \eta, \\ (\varepsilon - r)/\eta, & \text{if } \varepsilon - \eta \leq r \leq \varepsilon, \\ 0, & \text{if } r \geq \varepsilon. \end{cases}$$

Let $\eta_n \to 0$. Then, for a.e. $\varepsilon > 0$, we have

$$\mathscr{I}(u_{|P+\partial Q_\varepsilon^{d+1}}) = \lim_{n\to+\infty} (-1)^{d+1}\langle Tu, \psi_{\eta_n}(|\cdot - P|_\infty)\rangle, \quad \textit{for a.e. } P \in \mathbb{R}^{d+1}. \tag{6.6.70}$$

Proof. Let $G(P,\varepsilon) = \mathscr{I}(u_{|P+\partial Q_\varepsilon^{d+1}})$ and

$$G_n(P,\varepsilon) = \frac{1}{\eta_n}\int_{\varepsilon-\eta_n}^{\varepsilon} \mathscr{I}(u_{|P+\partial Q_r^{d+1}})\,\mathrm{d}r = (-1)^{d+1}\langle Tu, \psi_{\eta_n}(|\cdot - P|_\infty)\rangle,$$

where the equality follow from Proposition 6.6.13. Therefore, (6.6.70) amounts to

$$\lim_{n\to+\infty} G_n(P,\varepsilon) = G(P,\varepsilon), \tag{6.6.71}$$

for a.e. $\varepsilon > 0$ and, once ε is fixed, for a.e. $P \in \mathbb{R}^{d+1}$.

Fix $P \in \mathbb{R}^{d+1}$. By the proof of Proposition 6.6.13, $\varepsilon \mapsto \mathscr{I}(u_{|P+\partial Q_\varepsilon^{d+1}})$ is in $L^1_{\mathrm{loc}}((0,+\infty))$. The Lebesgue differentiation theorem then implies that, for a.e. $\varepsilon > 0$, we have

$$\lim_{n\to+\infty} G_n(P,\varepsilon) = G(P,\varepsilon). \tag{6.6.72}$$

Next, we set $\widetilde{G}(P,\varepsilon) = \liminf_{n\to+\infty} G_n(P,\varepsilon)$ and let

$$A = \{(P,\varepsilon) \in \mathbb{R}^{d+1} \times (0,+\infty) : G(P,\varepsilon) \neq \widetilde{G}(P,\varepsilon)\}.$$

We note that, for a.e. $\varepsilon > 0$, we have $u \in \mathrm{VMO}(P + \partial Q_\varepsilon^{d+1}; \mathscr{N})$, and in this case (by Proposition 6.5.1)

$$G(P,\varepsilon) = \mathscr{I}(u_{|P+\partial Q_\varepsilon^{d+1}}) = \int_{(P+\partial Q_\varepsilon^{d+1})\times(0,+\infty)} (\widetilde{\Pi} \circ U_{P,\varepsilon})^\sharp(\mathrm{d}\alpha),$$

where, as in (6.4.4), we let

$$U_{P,\varepsilon}(x,t) = \int_{P+\partial Q_\varepsilon^{d+1}} \widetilde{\rho}(x,t,y)u(y)\,\mathrm{d}\mathscr{H}^d(y).$$

Using (6.2.8) (with $\mathscr{M} = P + \partial Q_\varepsilon^{d+1}$), we find that $U_{P,\varepsilon}(x,t)$ is measurable with respect to (P,ε,x,t), and that G is measurable with respect to (x,ε). Similarly, G_n and $\widetilde{G}(P,\varepsilon)$ are measurable with respect to (P,ε), and $A = (G - \widetilde{G})^{-1}\{0\}$ is a Borel set. For fixed $P \in \mathbb{R}^{d+1}$, we have

$$\int_0^{+\infty} \chi_A(P,\varepsilon)\,\mathrm{d}\varepsilon = 0$$

(by (6.6.72)). We find that

$$\int_0^{+\infty}\int_{\mathbb{R}^{d+1}} \chi_A(P,\varepsilon)\,\mathrm{d}P\,\mathrm{d}\varepsilon = \int_{\mathbb{R}^{d+1}}\int_0^{+\infty} \chi_A(P,\varepsilon)\,\mathrm{d}\varepsilon\,\mathrm{d}P = 0,$$

and thus, for a.e. $\varepsilon > 0$, we have $\liminf_{n\to+\infty} G_n(P,\varepsilon) = G(P,\varepsilon)$ for a.e. $P \in \mathbb{R}^{d+1}$. Similarly, for a.e. $\varepsilon > 0$, we have $\limsup_{n\to+\infty} G_n(P,\varepsilon) = G(P,\varepsilon)$ for a.e. $P \in \mathbb{R}^{d+1}$. This completes the proof of (6.6.71). □

6.6.5 Approximation with maps induced by skeletons

In this section, we consider: (a) $0 < s < 1$ and $1 < p < +\infty$ such that $1 \le d \le sp < d+1$; (b) an integer $N > d$. In this setting, we will present several results related to the approximation of maps in $W_1^{s,p}(\mathbb{R}^N; \mathscr{N})$. Most of these results (or at least variants of them) were essentially established (but possibly not stated) by H. Brezis and P. Mironescu [BM15]. Here, we adapt the statements therein to our setting and provide only the missing arguments.

For $\varepsilon > 0$ and a point P in the cube $Q_\varepsilon^N = [-\varepsilon,\varepsilon)^N$, let $\mathscr{K}_{\varepsilon,P}^N$ be the (underlying set of the) cubication of $\mathbb{R}^N$ with diameter 2ε and P as one of its centers. We also recall that we denote by $\mathscr{K}_{\varepsilon,P}^l$ the (underlying set of the) l-skeleton of $\mathscr{K}_{\varepsilon,P}^N$.

We next discuss the properties of the restrictions of $W^{s,p}$ maps $u\colon \mathbb{R}^N \to \mathbb{R}^\nu$ to *generic* skeletons $\mathscr{K}_{\varepsilon,P}^l$. We recall that, for this purpose, we make the convenient choice of viewing u not as an equivalence class, but as an everywhere defined Borel function. For such u, the following holds [BM01, Appendix E]: For *every* $\varepsilon > 0$, for *almost every* $P \in Q^N_\varepsilon$, and for *every* cube $C^{d+1} \in K^{d+1}_{\varepsilon,P}$, we have

$$u_{|\partial C^{d+1}} \in W^{s,p}(\partial C^{d+1}) \subset \mathrm{VMO}(\partial C^{d+1}). \tag{6.6.73}$$

As a consequence of (6.6.73), if, in addition, $u \colon \mathbb{R}^N \to \mathcal{N}$, then $u_{|\partial C^{d+1}}$ has a well-defined homotopy class in $\mathrm{VMO}(\partial C^{d+1}; \mathcal{N})$ (see (6.2.34)).

We are now in position to state the main result of this section.

Theorem 6.6.15. *Let $f \in W^{s,p}_1(\mathbb{R}^N; \mathcal{N})$. If there exist $c_0 > 0$ and a sequence $\varepsilon_n \to 0$ such that the set*

$$A_n = \{P \in Q_{\varepsilon_n} \colon u_{|\partial C^{d+1}} \textit{ is } W^{s,p}(\partial C^{d+1}; \mathcal{N}) \textit{ and nullhomotopic for each } C^{d+1} \in K^{d+1}_{\varepsilon_n,P}\}$$

satisfies $|A_n|/|Q_{\varepsilon_n}| > c_0$ for every n, then

$$u \in \overline{C^\infty_1(\mathbb{R}^N; \mathcal{N})}^{W^{s,p}_1}.$$

Before proceeding to the proof of Theorem 6.6.15, we introduce some definitions used in the proof.

If $g \colon \mathscr{K}^d_{\varepsilon,P} \to \mathbb{R}^\nu$, we recall that its homogeneous extension to the cubes in $K^{d+1}_{\varepsilon,P}$, that we denote here by $H^{d+1}(g) \colon \mathscr{K}^{d+1}_{\varepsilon,P} \to \mathbb{R}^\nu$, is defined as follows. Let $x \in C^{d+1} \in K^{d+1}_{\varepsilon,P}$. If x is not the center $\mathcal{O}$ of C^{d+1}, we let

$$y = \mathcal{O} + \frac{\varepsilon(x - \mathcal{O})}{|x - \mathcal{O}|_\infty} \in \mathscr{K}^d_{\varepsilon,P}$$

and set

$$H^{d+1}(g)(x) = g(y).$$

The definition does not depend on the choice of the cube $C^{d+1} \in K^{d+1}_{\varepsilon,P}$ such that $x \in C^{d+1}$, and $H^{d+1}(g)$ is locally 0−homogeneous, in the sense that $H^{d+1}(g)$ is constant along the *ray* $(\mathcal{O}, y]$. We also note that $H^{d+1}(g)$ is well-defined except at the centers of the cubes in $\mathscr{K}^{d+1}_{\varepsilon,P}$.

Iterating the above construction, we obtain an a.e. defined map

$$H(g) = H^N(H^{N-1}(\cdots(H^{d+1}(g))\cdots)) \colon \mathbb{R}^N \to \mathbb{R}^\nu.$$

We invite the reader to compare this construction to the *thickening* procedure that we presented in Chapter 2 (Section 2.5). Here, the construction by mere homogeneous extensions is much easier to define, since we do not need to put additional efforts to work with a smooth construction when working with low-regularity maps in the range $0 < s < 1$.

We next introduce a $W^{s,p}$-type seminorm adapted to skeletons. Given a map $g \colon \mathscr{K}^l_{\varepsilon,P} \to \mathbb{R}^\nu$, we let

$$|g|^p_{W^{s,p}(\mathscr{K}^l_{\varepsilon,P})} = \iint\limits_{\mathscr{K}^l_{\varepsilon,P} \times \mathscr{K}^l_{\varepsilon,P}} \frac{|g(x) - g(y)|^p}{|x-y|^{l+sp}_\infty} \,\mathrm{d}\mathscr{H}^l(x) \,\mathrm{d}\mathscr{H}^l(y)$$

and define

$$\begin{aligned} W^{s,p}_1(\mathscr{K}^l_{\varepsilon,P}) = \{g \colon \mathscr{K}^l_{\varepsilon,P} \to \mathbb{R}^\nu \colon |g|_{W^{s,p}(\mathscr{K}^l_{\varepsilon,P})} < +\infty \\ \text{and there exists } c_g \in \mathbb{R}^\nu \text{ such that } \operatorname{supp}(g - c_g) \subset \mathbb{B}^N\}. \end{aligned}$$

We have the following result.

Lemma 6.6.16. *If $u \in W^{s,p}_1(\mathbb{R}^N)$, then, for every $\varepsilon > 0$ and almost every $P \in Q^N_\varepsilon$, we have*

$$u_{|\mathscr{K}^l_{\varepsilon,P}} \in W^{s,p}_1(\mathscr{K}^l_{\varepsilon,P}), \quad \text{for every } 0 \le l \le N. \tag{6.6.74}$$

Proof. Using, when $|x-y|_\infty \ge 2\varepsilon$, the inequality

$$|u(x) - u(y)| \le |u(x) - c_u| + |u(y) - c_u|,$$

we find that

$$|u|^p_{W^{s,p}(\mathscr{K}^l_{\varepsilon,P})} \lesssim \iint\limits_{\substack{\mathscr{K}^l_{\varepsilon,P} \times \mathscr{K}^l_{\varepsilon,P} \\ |x-y|_\infty < 2\varepsilon}} \frac{|u(x) - u(y)|^p}{|x-y|^{l+sp}_\infty} \,\mathrm{d}\mathscr{H}^l(x) \,\mathrm{d}\mathscr{H}^l(y) + \int_{\mathscr{K}^l_{\varepsilon,P}} |u(x) - c_u|^p \,\mathrm{d}\mathscr{H}^l(x),$$

and therefore

$$\begin{aligned} \int_{Q^N_\varepsilon} |u|^p_{W^{s,p}(\mathscr{K}^l_{\varepsilon,P})} \,\mathrm{d}P \lesssim \int_{Q_\varepsilon} \iint\limits_{\substack{\mathscr{K}^l_{\varepsilon,P} \times \mathscr{K}^l_{\varepsilon,P} \\ |x-y|_\infty < 2\varepsilon}} \frac{|u(x) - u(y)|^p}{|x-y|^{l+sp}_\infty} \,\mathrm{d}\mathscr{H}^l(x) \,\mathrm{d}\mathscr{H}^l(y) \,\mathrm{d}P \\ + \|u - c_u\|^p_{L^p}. \end{aligned} \tag{6.6.75}$$

By [BM15, Lemma 6.1], the integral on the right-hand side of (6.6.75) is dominated up to a constant by $|u|^p_{W^{s,p}(\mathbb{R}^N)}$. Combining this with the fact that $u = c_u$ outside $\mathbb{B}^N$, we obtain

$$\int_{Q_\varepsilon^N} |u|^p_{W^{s,p}(\mathscr{K}^l_{\varepsilon,P})}\,\mathrm{d}P \lesssim |u|^p_{W^{s,p}(\mathbb{R}^N)} + \|u - c_u\|^p_{L^p(\mathbb{R}^N)} \lesssim |u|^p_{W^{s,p}(\mathbb{R}^N)},$$

which implies (6.6.74). □

We will call a mesh $\mathscr{K}^N_{\varepsilon,P}$ such that (6.6.74) holds a *good mesh*.

Proof of Theorem 6.6.15. With no loss of generality, we may assume that $c_u = 0$, and thus $u \in W^{s,p}(\mathbb{R}^N)$. We divide the proof into 4 steps.

Step 1. — If $sp \geq 1$ then, for any $u \in W^{s,p}_1(\mathbb{R}^N;\mathbb{R}^\nu)$ (not necessarily $\mathscr{N}$-valued), there exist sets $D_\varepsilon \subset Q_\varepsilon^N$ such that (j) $|D_\varepsilon|/|Q_\varepsilon^N| \to 1$ as $\varepsilon \to 0$; and (jj) for every $P_\varepsilon \in D_\varepsilon$, we have

$$H(u_{|\mathscr{K}^d_{\varepsilon,P_\varepsilon}}) \to u \text{ in } W^{s,p}(\mathbb{R}^N) \text{ as } \varepsilon \to 0.$$

Indeed, define

$$D_\varepsilon = \left\{P \in Q_\varepsilon^N : \left\|u - H(u_{|\mathscr{K}^d_{\varepsilon,P}})\right\|^p_{W^{s,p}(\mathbb{R}^N)} \leq \left(\fint_{Q_\varepsilon^N} \left\|u - H(u_{|\mathscr{K}^d_{\varepsilon,P}})\right\|^p_{W^{s,p}(\mathbb{R}^N)}\,\mathrm{d}P\right)^{1/2}\right\}.$$

We then have

$$\int_{Q_\varepsilon^N} \left\|u - H(u_{|\mathscr{K}^d_{\varepsilon,P}})\right\|^p_{W^{s,p}(\mathbb{R}^N)}\,\mathrm{d}P \geq (|Q_\varepsilon^N| - |D_\varepsilon|)\left(\fint_{Q_\varepsilon^N} \left\|u - H(u_{|\mathscr{K}^d_{\varepsilon,P}})\right\|^p_{W^{s,p}(\mathbb{R}^N)}\,\mathrm{d}P\right)^{1/2},$$

which implies that

$$|D_\varepsilon|/|Q_\varepsilon^N| \geq 1 - \left(\fint_{Q_\varepsilon^N} \left\|u - H(u_{|\mathscr{K}^d_{\varepsilon,P}})\right\|^p_{W^{s,p}(\mathbb{R}^N)}\,\mathrm{d}P\right)^{1/2}. \tag{6.6.76}$$

On the other hand, we have the following result [BM15, (5.54)].

Lemma 6.6.17. *If $sp \geq 1$ then, for every $u \in W^{s,p}(\mathbb{R}^N;\mathbb{R}^\nu)$,*

$$\lim_{\varepsilon\to 0} \fint_{Q_\varepsilon^N} \left\|u - H(u_{|\mathscr{K}^d_{\varepsilon,P}})\right\|^p_{W^{s,p}(\mathbb{R}^N)}\,\mathrm{d}P = 0. \tag{6.6.77}$$

We complete Step 1 via (6.6.76) and (6.6.77).

Step 2. — Under the assumptions of the theorem, there exists a sequence $(\mathscr{K}_{\varepsilon_n,P_n})_{n\in\mathbb{N}}$

of good meshes such that

$$H(u_{|\mathscr{K}^d_{\varepsilon_n,P_n}}) \to u \text{ in } W^{s,p}(\mathbb{R}^N;\mathscr{N}) \tag{6.6.78}$$

and

$$u_{|\partial C^{d+1}} \text{ is nullhomotopic, for each } C^{d+1} \in K^{d+1}_{\varepsilon_n,P_n}. \tag{6.6.79}$$

Indeed, since $|A_n|/|Q^N_{\varepsilon_n}| > c_0$ for every n, then, for n sufficiently large, we have $|A_n \cap D_{\varepsilon_n}| > 0$. We complete Step 2 by choosing, for large n, $P_n \in A_n \cap D_{\varepsilon_n}$.

From now on, we fix n sufficiently large such that

$$\operatorname{supp} u \subset B^N_{1-9\sqrt{N}\varepsilon_n}(0). \tag{6.6.80}$$

By a standard smoothing argument, it suffices to prove that, under the assumptions (6.6.78) and (6.6.79), $H(u_{|\mathscr{K}^d_{\varepsilon_n,P_n}})$ can be approximated with Lipschitz maps from $\mathbb{R}^N$ to $\mathscr{N}$ supported in $\mathbb{B}^N$. This will be proved for each fixed n. In order to lighten the notation, we write ε, respectively P, instead of ε_n, respectively P_n.

Step 3. — If (6.6.79) and (6.6.80) hold, then $H(u_{|\mathscr{K}^d_{\varepsilon,P}})$ can be approximated with $H(v)$ for some Lipschitz map $v\colon \mathscr{K}_{\varepsilon,P} \to \mathscr{N}$ satisfying

$$\operatorname{supp} v \subset B^N_{1-7\sqrt{N}\varepsilon}(0). \tag{6.6.81}$$

For this purpose, we first approximate $u_{|\mathscr{K}^d_{\varepsilon,P}}$ with Lipschitz maps on $\mathscr{K}^d_{\varepsilon,P}$ by the means of the following lemma.

Lemma 6.6.18. *There exists a sequence of Lipschitz maps $(v_n)_{n\in\mathbb{N}} \subset \operatorname{Lip}(\mathscr{K}^d_{\varepsilon,P};\mathscr{N})$ such that*

$$v_n \to u_{|\mathscr{K}^d_{\varepsilon,P}} \textit{ in } W^{s,p}_1(\mathscr{K}^d_{\varepsilon,P};\mathscr{N}) \textit{ as } n \to +\infty$$

and, for any cube $C^d \in K^d_{\varepsilon,P}$, if $u = 0$ in C^d, then $v_n = 0$ in C^d for all n.

Granted Lemma 6.6.18, we complete Step 3 by the following continuity property of H.

Lemma 6.6.19. *For maps v and $(v_n)_{n\in\mathbb{N}}$ in $W^{s,p}(\mathscr{K}^d_{\varepsilon,P};\mathbb{R}^\nu)$ and supported in $B^N_{1-7\sqrt{N}\varepsilon}(0)$, it holds that*

$$[v_n \to v \textit{ in } W^{s,p}_1(\mathscr{K}^d_{\varepsilon,P};\mathbb{R}^\nu)] \implies [H(v_n) \to H(v) \textit{ in } W^{s,p}(\mathbb{R}^N;\mathbb{R}^\nu)].$$

In view of Steps 2 and 3, we complete the proof of Theorem 6.6.15 via the following step.

Step 4. — Under the assumptions (6.6.78)–(6.6.80), for large n, the map $H(v_n)$ can be approximated in $W^{s,p}$ with Lipschitz maps with support in $\mathbb{B}^N$.

For this purpose, we first notice that for all cubes $C^{d+1} \in K^{d+1}_{\varepsilon,P}$, we have $W^{s,p}(\partial C^{d+1}) \hookrightarrow (\mathrm{VMO} \cap L^1)(\partial C^{d+1})$. Combining this with Lemma 6.2.35, and the fact that u takes non-zero values only on finitely many cubes, we get that for n sufficiently large and for all cubes $C^{d+1} \in K^{d+1}_{\varepsilon,P}$, $v_{n|\partial C^{d+1}} \sim u_{|\partial C^{d+1}}$, and thus $v_{n|\partial C^{d+1}}$ is nullhomotopic. From now on, we consider such n's.

We next adapt to our setting an approximation result initially obtained by F. Bethuel [Bet91, Section II]. This is the content of the following lemma. The reader may also compare with the *shrinking* construction presented in Chapter 2 (Section 2.7).

Lemma 6.6.20. *Let $u \in \mathrm{Lip}(\mathscr{K}^d_{\varepsilon,P}; \mathscr{N})$ be such that $u_{|\partial C^{d+1}}$ is nullhomotopic for all cubes $C^{d+1} \in \mathscr{K}^{d+1}_{\varepsilon,P}$ and $\operatorname{supp} u \subset B^N_{1-7\sqrt{N}\varepsilon}(0)$. For $1 \leq q < d+1$, the map $H(v)$ is a strong limit in $W^{1,q}$ of maps in $\mathrm{Lip}(\mathbb{R}^N; \mathscr{N})$ with value 0 outside of a compact subset of $\mathbb{B}^N$.*

We complete Step 4 (and the proof of Theorem 6.6.15) by combining Lemma 6.6.20 with the Gagliardo–Nirenberg embedding

$$W^{1,q} \cap L^\infty \hookrightarrow W^{s,p}, \quad \text{for every } sp < q < d+1. \qquad \square$$

We now justify Lemmas 6.6.18 and 6.6.20 used in the proof of Theorem 6.6.15. They are variants of [BM15, Lemma 7.1] and [Mir23, Proposition 2.8]. (However, in [Mir23] the topological setting is different.) We adapt here the local arguments in [BM15, Mir23] to the case of maps defined in the full space and constant at infinity.

Proof of Lemma 6.6.18. It suffices to repeat the proof of Lemma 7.1 in [BM15]. There, the maps are defined only on a cube. However, applied to our situation, the construction in [BM15] yields a map v satisfying (6.6.81). $\square$

Proof of Lemma 6.6.20. It suffices to repeat the proof of Proposition 2.8 in [Mir23] with two changes: (i) in the first step of the proof, we obtain the existence of a Lipschitz extension $w\colon \mathscr{K}^{d+1}_{\varepsilon,P} \to \mathscr{N}$ of v using the assumption that $v_{|\partial C^{d+1}}$ is nullhomotopic (in [Mir23, Proposition 2.8], the assumption is $\pi_d(\mathscr{N}) = \{0\}$); and (ii) in the second step of the proof, we consider a different homotopy G, designed to preserve the property that we approximate with compactly supported Lipschitz maps. More specifically, instead of requiring, as in [Mir23, proof of Proposition 2.8] that, when θ close to 1, $G(x,\theta) = a$ for some $a \in \mathscr{K}^{d+1}$, we require that $G(x,\theta)$ stays outside $\operatorname{supp} g$. For this purpose, we consider the map G defined in Lemma 6.6.21 below (with $l = d$ and, in (4), $r = 1 - 5\sqrt{N}\varepsilon$). For this G and each $P \in Q^N_\varepsilon$, we have $B^N_{1-7\sqrt{N}\varepsilon}(0) \subset B^N_{1-5\sqrt{N}\varepsilon}(P) \subset B^N_{1-3\sqrt{N}\varepsilon}(0)$. Using this fact, it is straightforward that the approximating sequence considered in the third step of the proof of [Mir23, Proposition 2.8] is supported in $B^N_{1-\sqrt{N}\varepsilon}(0)$. ◻

The following lemma relies only on the topological structure of a bounded mesh, so that, for simplicity, we assume that $\varepsilon = 1$ and $P = 0$. Let K^N_M be the collection of cubes $2K + Q_1$, with $K \in \{-M, \dots, M\}^N$, and let K^l_M for $0 \le l \le N-1$ be the corresponding l-skeleton.

Lemma 6.6.21. *Let $0 \le l \le N-1$. There exists a Lipschitz homotopy $G = G(x,\theta)\colon \mathscr{K}^N_M \times [0,1] \to \mathscr{K}^N_M$ such that*

(1) *$G(x,0) = x$, for every $x \in \mathscr{K}^N_M$;*
(2) *$G(x,\theta) \in \partial Q^N_{2M+1}$, for every $x \in \mathscr{K}^N_M \setminus Q^N_{1/2}$, for every $\theta \ge 1/2$;*
(3) *$G(x,\theta) \in \mathscr{K}^{l+1}_M$, for every $x \in \mathscr{K}^l_M$, for every θ;*
(4) *for every $r > 0$ and each cube $C \in K^l_M$, if $C \cap B^N_r = \varnothing$, then $G(x,\theta) \notin B^N_r$, for every $x \in C$, for every θ.*

Proof. Consider the Lipschitz map $g\colon [-(2M+1), 2M+1] \times [0,1] \to [-(2M+1), 2M+1]$ given by

$$g(a,\theta) = \operatorname{sgn}(a)\min\{(4M\theta + \theta + 1)|a|, 2M+1\}.$$

For $1 \le i \le N$, set

$$G_i(x^1, \dots, x^{i-1}, x^i, x^{i+1}, \dots, x^N, \theta) = (x^1, \dots, x^{i-1}, g(x^i,\theta), x^{i+1}, \dots, x^N).$$

Clearly, G_i satisfies

(j) $G_i(x,0) = x$, for every $x \in \mathscr{K}^N_M$;
(jj) $G_i(x,1) \in \mathscr{K}^l_M \cap \{x^i = 2M+1\}$, for every $x \in \mathscr{K}^l_M$ with $x^i \ge 1/2$;
(jjj) $G_i(x,1) \in \mathscr{K}^l_M \cap \{x^i = -2M-1\}$, for every $x \in \mathscr{K}^l_M$ with $x^i \le -1/2$;
(jjjj) $G_i(x,\theta) \in \mathscr{K}^{l+1}_M$, for every $x \in \mathscr{K}^l_M$, for every θ.

Let

$$G(x,\theta) = \begin{cases} G_1(x, 2N\theta), & \text{if } \theta \le 1/(2N), \\ G_2(G(x,1/(2N)), 2N\theta - 1), & \text{if } 1/(2N) < \theta \le 1/N, \\ \dots \\ G_N(G(x,(N-1)/(2N)), 2N\theta - N + 1), & \text{if } (N-1)/(2N) < \theta \le 1/2, \\ G(x,1/2), & \text{if } 1/2 < \theta \le 1. \end{cases}$$

Using (j)–(jjjj), we easily find that $G(x, \theta)$ satisfies all the required properties. $\square$

6.6.6 Approximation with smooth maps to $\mathscr{N}$

Recall that we consider: (a) $0 < s < 1$ and $1 < p < +\infty$ such that $sp = d$; (b) an integer $N > d$.

Only in this section, we make the extra assumption that *the cohomology of $\mathscr{N}$ sees its homotopy*. More specifically, we assume that $\mathscr{N}$ has the following property:

$$\text{for each } f \in C^\infty(\mathbb{S}^d; \mathscr{N}), \text{ we have}$$
$$\left[\int_{\mathbb{S}^d} f^\sharp \omega = 0, \quad \text{for every smooth closed } d\text{-form } \omega \text{ on } \mathscr{N}\right] \implies f \text{ is nullhomotopic.} \tag{6.6.82}$$

Standard results in algebraic topology provide sufficient conditions for the validity of (6.6.82). We briefly discuss this in Section 6.7. To give a flavor of that discussion, we note here that $\mathscr{N} = \mathbb{S}^d$ satisfies (6.6.82).

In this section, we prove the following theorem.

Theorem 6.6.22. *Assume that* (6.6.82) *holds. Let $u \in W^{s,p}_1(\mathbb{R}^N; \mathscr{N})$ and let $T = T_\omega$ be defined as in Section* 6.6.2. *Then,*

$$u \in \overline{C^\infty_1(\mathbb{R}^N; \mathscr{N})}^{W^{s,p}_1} \quad \text{if and only if} \quad \text{for every smooth closed } d\text{-form } \omega \text{ on } \mathscr{N}, T_\omega u = 0.$$

Before proving Theorem 6.6.22, we present a (equivalent) form of (6.6.82) adapted to our context.

Lemma 6.6.23. *Assume that* (6.6.82) *holds. Let C be a cube in $\mathbb{R}^{d+1}$. Then*

$$\textit{for each } f \in \mathrm{VMO}(\partial C; \mathscr{N}), \textit{ we have}$$
$$\left[\mathscr{I}_{\partial C, \omega}(f) = 0, \quad \textit{for every smooth closed } d\textit{-form } \omega \textit{ on } \mathscr{N}\right] \implies f \textit{ is nullhomotopic.}$$

Proof. First, let us note that, if (6.6.82) holds, then it also holds for continuous maps. (This follows from a standard smoothing argument and Corollary 6.3.28.)

By Corollary 6.3.29, there exists ε_1 such that

$$\mathscr{I}_{\partial C, \omega}(f) = \mathscr{I}_{\partial C, \omega}(f^\varepsilon), \quad \text{for every smooth closed } d\text{-form } \omega \text{ on } \mathscr{N}, \text{ for every } \varepsilon < \varepsilon_1. \tag{6.6.83}$$

Let $\Psi\colon \mathbb{S}^d \to \partial C$ be a bi-Lipschitz orientation preserving map. By Corollary 6.3.31, we have

$$\mathscr{I}_{\partial C,\omega}(g) = \mathscr{I}_{\mathbb{S}^d,\omega}(g \circ \Psi), \quad \text{for every } g \in C(\partial C;\mathscr{N}). \tag{6.6.84}$$

The conclusion of the lemma follows from (6.6.83), (6.6.84) (with $g = f^\varepsilon$), and the validity of (6.6.82) for continuous maps. □

Proof of Theorem 6.6.22. The direct implication does not rely on (6.6.82), and follows from Theorem 6.6.1 when $d \geq 2$, respectively Theorem 6.6.4 when $d = 1$, since for $u \in C^\infty_1(\mathbb{R}^N;\mathscr{N})$ and $\zeta \in \mathrm{Lip}(\mathbb{R}^N;\Lambda^{N-d-1})$, we have

$$\langle Tu, \zeta\rangle = \int_{\mathbb{R}^N} u^\sharp\omega \wedge \mathrm{d}\zeta = (-1)^d \int_{\mathbb{R}^N} \mathrm{d}(u^\sharp\omega \wedge \zeta) = 0.$$

We divide the proof of the reverse implication into three steps: dimensional reduction, proof in the special case where $N = d+1$, and a cohomology argument.

For simplicity, in the proof we denote points in $\mathbb{R}^N$ as (x, y), with $x \in \mathbb{R}^{d+1}$ and $y \in \mathbb{R}^{N-d-1}$. We note that, if $u \in W^{s,p}_1(\mathbb{R}^N;\mathscr{N})$ then, for a.e. $y_0 \in \mathbb{R}^{N-d-1}$, we have $u(\cdot, y_0) \in W^{s,p}_1(\mathbb{R}^{d+1};\mathscr{N})$.

Step 1. — Fix ω. By Proposition 6.6.10 (applied with $\ell = d+1$ and $l = N-d-1$), for a.e. $y_0 \in \mathbb{R}^{N-d-1}$, we have $T_\omega u(\cdot, y_0) = 0$.

Step 2. — Let $N = d+1$ and fix ω. Let $u \in W^{s,p}_1(\mathbb{R}^{d+1};\mathscr{N})$ satisfy $T_\omega u = 0$. Then, by Proposition 6.6.14, for a.e. $\varepsilon > 0$, we have

$$\mathscr{I}_{P+\partial Q_\varepsilon,\omega}(u_{|P+\partial Q_\varepsilon}) = 0 \quad \text{for a.e. } P \in \mathbb{R}^{d+1}. \tag{6.6.85}$$

Step 3. — We complete the proof of Theorem 6.6.22 via Remark 6.6.11, (6.6.85), (6.6.82), Lemma 6.6.23, and Theorem 6.6.15. □

6.7 Reading homotopy from integral invariants

In this section, we study some necessary conditions that ensure the validity of the assumption (6.6.82), which plays a crucial role in Section 6.6.6.

We first recall that, given a smooth Riemannian manifold $\mathscr{N}$, there exists a map $\mathfrak{hur}\colon \pi_j(\mathscr{N}) \to H_j(\mathscr{N};\mathbb{Z})$, called the *Hurewicz homomorphism*, that maps a homotopy class $[f] \in \pi_j(\mathscr{N})$ to the cycle $f_\sharp[\mathbb{S}^j]$.

The following proposition characterizes the validity of (6.6.82) (and even slightly more) in terms of the Hurewicz map.

Proposition 6.7.1. *Assume that $H_j(\mathcal{N};\mathbb{Z})$ is torsionfree. Then,*

$$\text{for each } f, g \in C^\infty(\mathbb{S}^j;\mathcal{N}), \text{ we have}$$
$$\left[\int_{\mathbb{S}^j} f^\sharp\omega = \int_{\mathbb{S}^j} g^\sharp\omega, \quad \text{for every smooth closed } j\text{-form } \omega \text{ on } \mathcal{N}\right] \implies f \sim g \tag{6.7.1}$$

if and only if $\mathfrak{hur}$ *is injective.*

In particular, (6.7.1) implies (6.6.82), specializing to g being a constant map. Proposition 6.7.1 is well-known to experts, but for the sake of completeness we present here an argument, using as little technology as possible.

Proof. By the de Rham theorem, there exists an identification $\mathcal{J}_{\mathcal{N}} \colon H^j_{\mathrm{dR}}(\mathcal{N}) \to H^j(\mathcal{N};\mathbb{R})$ between the de Rham cohomology and the singular cohomology. If a cycle σ is associated with a sufficiently smooth domain of $\mathcal{N}$, then

$$\langle \mathcal{J}_{\mathcal{N}}(\alpha), \sigma\rangle = \int_\sigma \alpha,$$

with the integral being defined as in Section 6.3.5.

We start with the reverse implication, and assume that $\mathfrak{hur}$ is injective. Let $f, g \in C^\infty(\mathbb{S}^j;\mathcal{N})$. Our proof is in two steps: we first prove that

$$\int_{\mathbb{S}^k} f^\sharp\omega = \langle \mathcal{J}_{\mathcal{N}}(\omega), f_\sharp[\mathbb{S}^j]\rangle, \quad \text{for every closed } j\text{-form } \omega \text{ on } \mathcal{N}, \tag{6.7.2}$$

then find, using (6.7.2) and the assumptions on $\mathcal{N}$, that $f \sim g$.

Step 1. — To prove (6.7.2), we start from the fact that $\mathcal{J}_{\mathcal{N}}$ is a natural transformation between the de Rham cohomology functor and the singular cohomology functor, that is, $\mathcal{J}_{\mathbb{S}^j} \circ f^\sharp = f^\sharp \circ \mathcal{J}_{\mathcal{N}}$ [Lee13, Proposition 18.13]. Hence, we find that

$$\int_{\mathbb{S}^j} f^\sharp\omega = \langle \mathcal{J}_{\mathbb{S}^j}(f^\sharp\omega), [\mathbb{S}^j]\rangle = \langle f^\sharp\mathcal{J}_{\mathcal{N}}(\omega), [\mathbb{S}^j]\rangle,$$
$$\text{for every smooth closed } j\text{-form } \omega \text{ on } \mathcal{N}.$$

Now, we recall that, thanks to the universal coefficients theorem for cohomology, see e.g. [Hat02, Theorem 3.2], we have

$$H^j(\mathcal{N};\mathbb{R}) \cong \mathrm{Hom}(H_j(\mathcal{N};\mathbb{Z});\mathbb{R}).$$

Moreover, this correspondence is natural, meaning that the map $f^\sharp$ induced in cohomology by f is dual to the map $f_\sharp$ induced in homology; see e.g. [Hat02, Page 196]. Therefore,

$$\langle f^\sharp \mathcal{J}_{\mathcal{N}}(\omega), [\mathbb{S}^j]\rangle = \langle \mathcal{J}_{\mathcal{N}}(\omega), f_\sharp[\mathbb{S}^j]\rangle, \quad \text{for every smooth closed } j\text{-form } \omega \text{ on } \mathcal{N}.$$

This concludes the proof of (6.7.2).

Step 2. — The de Rham homomorphism being an isomorphism, (6.7.2) and the fact that

$$\int_{\mathbb{S}^j} f^\sharp\omega = \int_{\mathbb{S}^j} g^\sharp\omega, \quad \text{for every smooth closed } j\text{-form } \omega \text{ on } \mathcal{N} \tag{6.7.3}$$

imply that $f_\sharp[\mathbb{S}^j]$ and $g_\sharp[\mathbb{S}^j]$ coincide when evaluated against any homomorphism from $H_j(\mathcal{N};\mathbb{Z})$ to $\mathbb{R}$. But, since $H_j(\mathcal{N};\mathbb{Z})$ is torsionfree, it is isomorphic to $\mathbb{Z}^l$ for some $l \in \mathbb{N}$. Hence, $f_\sharp[\mathbb{S}^j] = g_\sharp[\mathbb{S}^j]$.

Therefore, if $\mathfrak{hur}$ is injective, then (6.7.3) implies that $[f] = [g]$ in $\pi_j(\mathcal{N})$, showing that (6.7.1) holds.

We now turn to the direct implication. We have to prove that, if $f \in C^\infty(\mathbb{S}^j;\mathcal{N})$ is such that $f_\sharp[\mathbb{S}^j] = 0$, then f is nullhomotopic. By (6.7.2), we find that

$$\int_{\mathbb{S}^j} f^\sharp\omega = 0, \quad \text{for every smooth closed } j\text{-form } \omega \text{ on } \mathcal{N},$$

and hence (6.7.1) applied with g a constant map implies that f is nullhomotopic. We observe that the proof of this implication does not rely on the fact that $H_j(\mathcal{N};\mathbb{Z})$ is torsionfree. □

Combining Proposition 6.7.1 with the Hurewicz theorem, see e.g. [Hat02, Theorem 4.37], which asserts that $\mathfrak{hur}$ is an isomorphism whenever either $j \geq 2$ and $\mathcal{N}$ is $(j-1)$-connected, or $j = 1$ and $\pi_1(\mathcal{N})$ is abelian, we obtain the following, more readable, sufficient condition for (6.7.1) to hold.

Proposition 6.7.2. *Assume that $\pi_j(\mathcal{N})$ is torsionfree, and that either $j \geq 2$ and $\mathcal{N}$ is $(j-1)$-connected, or $j = 1$ and $\pi_1(\mathcal{N})$ is abelian. Then,* (6.7.1) *holds.*

Let us give examples of some typical situations that illustrate the various assumptions above.

Example 6.7.3. Let $\mathcal{N} = \mathbb{T}^2$ be the 2-dimensional torus and $j = 2$. Since $\pi_2(\mathbb{T}^2) = \{0\}$, every map $f : \mathbb{S}^2 \to \mathbb{T}^2$ is nullhomotopic. Therefore, (6.7.1) trivially holds.

On the other hand, $\pi_1(\mathbb{T}^2) = \mathbb{Z}^2$ is nontrivial, whence Proposition 6.7.2 does not apply. Actually, $H_2(\mathbb{T}^2;\mathbb{Z}) = \mathbb{Z}$, so that the Hurewicz homomorphism is not an isomorphism. It is nevertheless injective, as it is nothing else but the zero map $\{0\} = \pi_2(\mathbb{T}^2) \to H_2(\mathbb{T}^2;\mathbb{Z})$.

This highlights the fact that the assumptions in Proposition 6.7.2 are more stringent than the ones of Proposition 6.7.1, and that only the *injectivity* of $\mathfrak{hur}$ matters.

One can obtain a less trivial example, where *there* actually *is* some topology to be detected, by taking for instance $\mathbb{T}^2 \times \mathbb{S}^2$. □

Example 6.7.4. Let $\mathscr{N} = \mathbb{RP}^2$ be the 2-dimensional projective plane and $j = 1$. Since $\pi_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$, there is a homotopically nontrivial smooth map $f\colon \mathbb{S}^1 \to \mathbb{RP}^2$. On the other hand, since $H^1_{\mathrm{dR}}(\mathbb{RP}^2) = \{0\}$, all smooth closed 1-forms ω are exact. Therefore, (6.7.3) trivially holds true for any pair of maps, and hence (6.7.1) fails.

The issue here is that $H_1(\mathbb{RP}^2;\mathbb{Z}) = \pi_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$ has torsion. This is actually a more general phenomenon, since the de Rham cohomology does not see torsion. This highlights why it is crucial to assume, in Proposition 6.7.1, that the relevant homology group is torsionfree. □

Example 6.7.5. Let $\mathscr{N} = \mathbb{S}^1 \vee \mathbb{S}^1$ be a bouquet of two circles and $j = 1$. Strictly speaking, this is not a manifold, but one can easily work instead with a manifold with the same relevant properties by considering for instance a torus with two holes.

In this case, we have $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1) = \mathbb{Z} * \mathbb{Z} \neq \mathbb{Z}^2 = H_1(\mathbb{S}^1 \vee \mathbb{S}^1;\mathbb{Z})$. But there is no injective group morphism $\mathbb{Z} * \mathbb{Z} \to \mathbb{Z}^2$. Indeed, if a and b are generators of $\mathbb{Z} * \mathbb{Z}$ and $g\colon \mathbb{Z} * \mathbb{Z} \to \mathbb{Z}^2$ is a morphism, then $g(aba^{-1}b^{-1}) = g(a) + g(b) - g(a) - g(b) = 0$ and thus g is not injective. In particular, the Hurewicz homomorphism is not injective. The issue here is that $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1)$ is not abelian, while homology groups are always abelian. We note that this may only arise when $j = 1$, as π_j is always abelian when $j \geq 2$.

On the other hand, if $f\colon \mathbb{S}^1 \to \mathbb{S}^1 \vee \mathbb{S}^1$ realizes the commutator $[a, b] = aba^{-1}b^{-1}$ in $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1)$, one has

$$\int_{\mathbb{S}^1} f^\sharp \omega = 0, \quad \text{for every smooth closed 1-form } \omega \text{ on } \mathbb{S}^1 \vee \mathbb{S}^1,$$

showing that (6.7.1) fails in this situation. This highlights the importance of the abelian assumption when $j = 1$. □

6.8 Further results

This section is devoted to the proof of Lemma 6.3.36 and an improvement of Theorem 6.4.1, in the spirit of J. Bourgain, H. Brezis, and Nguyen H.-M. [BBN05].

Proof of Lemma 6.3.36. We use the same notation as in Sections 6.3.1 and 6.3.5. Let $\overline{f}_{i,\varepsilon}$ and $\overline{f}_{\varepsilon}$ be as in the proof of Lemma 6.3.34. Then,

$$\overline{f}_{i,\varepsilon} \circ \varphi_i = (\xi_i \circ \varphi_i)(\rho_\varepsilon * \overline{f}_i) \to (\xi_i \circ \varphi_i)\overline{f}_i = (\xi_i \circ \varphi_i)(f \circ \varphi_i) \text{ in } W^{s,p}(V_i)$$

as $\varepsilon \to 0$. Since φ_i is bi-Lipschitz, this implies that $\overline{f}_{i,\varepsilon} \to \xi_i f$ in $W^{s,p}(U_i)$. Convergence also holds in $W^{s,p}(\mathscr{M})$, since

$$\begin{aligned}
&\int_{\mathscr{M}}\int_{\mathscr{M}} \frac{|(\overline{f}_{i,\varepsilon}(x) - \xi_i(x)f(x)) - (\overline{f}_{i,\varepsilon}(y) - \xi_i(y)f(y))|^p}{\operatorname{dist}(x,y)^{d+sp}} \,\mathrm{d}\mathscr{H}^d(x)\,\mathrm{d}\mathscr{H}^d(y)\\
&\le 2\int_{\mathscr{M}\setminus U_i}\int_{U_i} \frac{|\overline{f}_{i,\varepsilon}(x) - \xi_i(x)f(x)|^p}{\operatorname{dist}(x,y)^{d+sp}} \,\mathrm{d}\mathscr{H}^d(x)\,\mathrm{d}\mathscr{H}^d(y) + |\overline{f}_{i,\varepsilon} - \xi_i f|^p_{W^{s,p}(U_i)}\\
&\le C(\varepsilon_1)\|\overline{f}_{i,\varepsilon} - \xi_i f\|_{\mathscr{L}^p(U_i)} + |\overline{f}_{i,\varepsilon} - \xi_i f|^p_{W^{s,p}(U_i)} \to 0 \text{ when } \varepsilon \to 0.
\end{aligned}$$

(In the last inequality, we have used the fact that for ε small and any i, $\operatorname{supp}\overline{f}_{i,\varepsilon}$ is contained in a fixed compact subset of U_i'.)

Therefore, when $\varepsilon \to 0$, we have $\overline{f}_{\varepsilon} \to f$ in $W^{s,p}$. By (6.4.1), this implies that $\overline{f}_{\varepsilon} \to f$ in $\mathrm{BMO} \cap L^1$. We are now in position to repeat the proof of Lemma 6.3.34 and find that, for small ε, one can define $\Pi \circ \overline{f}_{\varepsilon}$, which is Lipschitz and $\mathscr{N}$-valued. By Lemma 6.5.11, we have $\Pi \circ \overline{f}_{\varepsilon} \to f$ in $W^{s,p}$. □

We next present an improvement, inspired by [BBN05], of estimate (6.4.2) in Theorem 6.4.1. The setting is the one of Section 6.4: (a) $\mathscr{M}$ is a compact d-dimensional Lipschitz manifold oriented by a finite chart structure $\{(U_i, V_i, \varphi_i)\}_{i\in I}$; (b) $\mathscr{N}$ is a closed manifold; (c) ω is a smooth *closed* d-form on $\mathscr{N}$; (d) $0 < s < 1$ and $1 < p < +\infty$ are such that $sp = d$; (e) $\mathscr{I}(f)$ is the invariant whose existence is proved in Theorem 6.4.1; (f) ι is as in Proposition 1.4.2.

Theorem 6.8.1. *For $0 < \eta < \iota$, there exists a finite constant $C_\eta = C(\mathscr{M}, \mathscr{N}, \omega, s, p, \eta)$ such that*

$$|\mathscr{I}(f)| \le C_\eta \iint_{\{(x,y)\in\mathscr{M}\times\mathscr{M} : |f(y)-f(x)|>\eta\}} \frac{1}{[\operatorname{dist}(x,y)]^{2d}} \,\mathrm{d}\mathscr{H}^d(x)\,\mathrm{d}\mathscr{H}^d(y), \quad \text{for every } f \in W^{s,p}(\mathscr{M};\mathscr{N}). \tag{6.8.1}$$

In order to see that (6.8.1) is indeed a refinement of (6.4.2), it suffices to note the obvious inequalities

$$\begin{aligned}&\iint_{\{(x,y)\in\mathcal{M}\times\mathcal{M}:|f(y)-f(x)|>\eta\}} \frac{1}{[\operatorname{dist}(x,y)]^{2d}}\,\mathrm{d}\mathcal{H}^d(x)\,\mathrm{d}\mathcal{H}^d(y)\\ &\quad\leq \frac{1}{\eta^p}\iint_{\{(x,y)\in\mathcal{M}\times\mathcal{M}:|f(y)-f(x)|>\eta\}} \frac{|f(x)-f(y)|^p}{[\operatorname{dist}(x,y)]^{2d}}\,\mathrm{d}\mathcal{H}^d(x)\,\mathrm{d}\mathcal{H}^d(y) \leq \frac{1}{\eta^p}|f|^p_{W^{s,p}}.\end{aligned}$$

When $d \geq 2$, estimate (6.8.1) is due to J. Van Schaftingen [VS20]. For more subtle questions as the range of the η's such that (6.8.1) holds and the optimal dependence of C_η on η, see Nguyen H.-M. [Ngu07, Ngu17] and [VS20].

Proof. Let $F = F(x,\varepsilon)$ be as in (6.4.4), with $f_\varepsilon = f_\varepsilon(x)$ as in (6.2.9).

Let $0 < \beta < \iota - \eta$ and set

$$h_\beta(x) = \inf\{\varepsilon > 0 \colon \operatorname{dist}(F(x,\varepsilon),\mathcal{N}) \geq \iota - \beta\}.$$

Let $\widetilde{\Pi}_\beta \in C^\infty_c(\mathbb{R}^\nu;\mathbb{R}^\nu)$ be such that $\widetilde{\Pi}_\beta(z) = \Pi(z)$, for every $z \in \mathcal{N}_{\iota-\beta}$. By repeating the proof of Theorem 6.4.1 (see the proof of (6.4.24)), we have

$$|\mathcal{I}(f)| \lesssim \int_{\mathcal{M}} \frac{1}{[h_\beta(x)]^d}\,\mathrm{d}\mathcal{H}^d(x). \tag{6.8.2}$$

By the proof of (6.4.30), for a.e. $x \in \mathcal{M}$ we have

$$\begin{aligned}\iota - \beta &\leq |F(x,h_\beta(x)) - f(x)|\\ &\leq \int_{\{y\in\mathcal{M}:|f(y)-f(x)|>\eta\}} \widetilde{\rho}(x,h_\beta(x),y)|f(y)-f(x)|\,\mathrm{d}\mathcal{H}^d(y) + \eta.\end{aligned}\tag{6.8.3}$$

Combining (6.8.3), (6.4.5), and (6.2.13), we have

$$(\iota-\beta-\eta)\mathcal{H}^d(B_{h_\beta(x)}(x)) \lesssim \mathcal{H}^d(\{y \in B_{h_\beta(x)}(x) \colon |f(y)-f(x)| > \eta\}).$$

This implies that

$$\begin{aligned}&\int_{\{y\in\mathcal{M}:|f(y)-f(x)|>\eta\}} \frac{1}{[\operatorname{dist}(x,y)]^{2d}}\,\mathrm{d}\mathcal{H}^d(y)\\ &\gtrsim \frac{\mathcal{H}^d(B_{h_\beta(x)}(x))}{[\min\{h_\beta(x),\operatorname{diam}(\mathcal{M})\}]^{2d}}(\iota-\beta-\eta).\end{aligned}\tag{6.8.4}$$

On the other hand, (6.3.4) implies that

$$\frac{1}{[h_\beta(x)]^d} \lesssim \frac{\mathscr{H}^d(B_{h_\beta(x)}(x))}{[\min\{h_\beta(x), \operatorname{diam}(\mathscr{M})\}]^{2d}}. \tag{6.8.5}$$

We obtain (6.8.1) from (6.8.2), (6.8.4) (integrated in x), and (6.8.5). □

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Notation

General notation

(N1) $\mathscr{M}$ and $\mathscr{N}$ are compact Riemannian manifolds, isometrically embedded into $\mathbb{R}^N$ and $\mathbb{R}^\nu$, respectively, that serve as the domain and the target space of mappings;

(N2) Ω is an open domain in $\mathbb{R}^m$, bounded and sufficiently smooth unless otherwise stated;

(N3) $m \in \mathbb{N}_*$ generally stands for the dimension of the domain, either Ω or $\mathscr{M}$;

(N4) $1 \leq p < +\infty$ and $0 < s < +\infty$ stand for the integrability and regularity parameters of Sobolev spaces, and we decompose $s = k + \sigma$, with $k \in \mathbb{N}$ and $\sigma \in [0, 1)$;

(N5) d stands for a dimensional parameter, in general $d = \lfloor sp \rfloor$;

(N6) Sequences are denoted as $(x_n)_{n\in\mathbb{N}}$, the index being most of the time n;

(N7) $B^m_r(a)$ stands for the open ball with center a and radius r in $\mathbb{R}^m$ (with respect to the 2-norm), the center may be omitted if it is 0, and $\mathbb{B}^m = B^m_1 = B^m_1(0)$ stands for the unit ball (see also (N29) for geodesic balls);

(N8) $Q^m_r(a)$ stands for the open cube with center a and inradius r in $\mathbb{R}^m$ (that is, the ∞-ball with radius r), the center may be omitted if it is 0;

(N9) Closed balls and cubes are denoted by $\overline{B^m_r}(a)$ and $\overline{Q^m_r}(a)$, respectively;

(N10) For estimates, we avoid the use of a constant $C > 0$ that changes of value from one line to another. In statements, we use $C > 0$ and make explicit the parameters of the statement on which it depends. In proofs, we either use $A \lesssim B$ to mean that $A \leq CB$ for some constant C depending on the same parameters as in the corresponding statement, or we use numbered constants if they need to be referred to.

Functions

(N11) We work in general with the differential of a map u, which is denoted by $\mathrm{D}u$, and maps a point x to the linear map $h \mapsto \mathrm{D}u(x)[h]$, rather than with the gradient ∇u, which is a matrix;

(N12) χ_X stands for the indicator function of the set X, which takes value 1 on X and 0 outside;

(N13) A mollifier is a smooth nonnegative map compactly supported in $\mathbb{B}^m$ and with integral equal to one, they are denoted either by φ or by ρ;

(N14) If φ is a mollifier, then φ_η stands for the map defined by $\varphi_\eta(x) = \eta^{-m}\varphi(x/\eta)$;

(N15) $\operatorname{supp} u$ stands for the (analytic) support of the function u defined as the closure of the set where u does not vanish;

(N16) $\operatorname{Supp} \Phi$ stands for the geometric support of the function Φ defined as the closure of the set where Φ does not coincide with the identity, see Section 2.1;

(N17) $\mathcal{J}u = |\det \mathrm{D}u|$ stands for the Jacobian determinant of u.

Functions spaces

(N18) For a domain X and a target Y, we use the general notation $A(X;Y)$ for functions spaces consisting of maps from X to Y. The codomain of a functions space is sometimes omitted either when it is $\mathbb{R}$ or when it is obvious from the context;

(N19) $A_{\mathrm{loc}}(X;Y)$ denotes the space of functions from X to Y that belong to $A(K;Y)$ for any compact subset K of X;

(N20) $A_{\mathrm{c}}(X;Y)$ denotes the space of functions in $A(X;Y)$ with compact support in X;

(N21) $A_1(\mathbb{R}^N;Y)$ denotes the space of functions in $A(\mathbb{R}^N;Y)$ that are constant outside of $\mathbb{B}^N$;

(N22) L^p stands for the Lebesgue space of p-integrable functions. With a slight abuse of notation, we do not distinguish functions and equivalence classes up to almost everywhere equality;

(N23) $W^{s,p}$ denotes the Sobolev space with regularity parameter s and integrability parameter p, see Section 1.2;

(N24) $H^{s,p}_S$ stands for the strong closure of smooth maps in $W^{s,p}$, see Section 1.4.2;

(N25) $H^{s,p}_W$ stands for the weak sequential closure of smooth maps in $W^{s,p}$, see Section 1.4.6;

(N26) C stands for the space of continuous functions, and C^k for the space of k times continuously differentiable functions;

(N27) BMO and VMO stand for the spaces of functions with bounded and vanishing mean oscillation, see Section 6.2.1;

(N28) $\mathscr{R}_i$ stands for the class of maps that are smooth outside a singular set of dimension i, see Section 1.4.4, and $\mathscr{R}_i^{\mathrm{rig}}$ and $\mathscr{R}_i^{\mathrm{uncr}}$ are variants where the singular set is made of affine spaces or cannot exhibit crossings, respectively, see Section 3.1.

Differential geometry

(N29) On a manifold, $B_r(a)$ denotes the geodesic ball with center a and radius r, and in this case, the dimension is omitted in the superscript;

(N30) $\mathscr{N}_\iota$ denotes the tubular neighborhood of $\mathscr{N}$ of radius ι, defined as $\mathscr{N}_\iota = \mathscr{N} + B^\nu_\iota$;

(N31) $\Pi\colon \mathscr{N}_\iota \to \mathscr{N}$ denotes a smooth retraction onto $\mathscr{N}$, and we always assume that $\iota > 0$ is sufficiently small so that such a map is well-defined and smooth, see Proposition 1.4.2;

(N32) $\mathrm{d}_x f$ stands for the differential of the map f at x, viewed as a 1-form, see e.g. Section 6.3.3;

(N33) $T_x\mathscr{M}$ stands for the tangent plane to $\mathscr{M}$ at x, see e.g. Section 6.3.2;

(N34) $\alpha \wedge \beta$ stands for the wedge product of the forms α and β;

(N35) $H^j_{\mathrm{dR}}(Y)$ denotes the de Rham cohomology group of Y of order j.

Topology

(N36) Two maps f and g being homotopic by denoted as $f \sim g$, see Section 1.4.2;

(N37) Two topological spaces X and Y being homotopically equivalent is denoted by $X \simeq Y$;

(N38) $\pi_j(Y)$ denotes the homotopy group of Y of order j, see Section 1.4.3;

(N39) $H_j(Y; G)$ and $H^j(Y; G)$ denote the homology and cohomology groups of Y of order j with values into the coefficient group G, most of the time G is either $\mathbb{Z}$ or $\mathbb{R}$;

(N40) $X \vee Y$ denotes the wedge sum of the spaces X and Y, obtained by gluing X and Y along one point;

(N41) $X \# Y$ denotes the connected sum of X and Y, obtained by removing a ball to X and Y and gluing the remaining spaces along the boundaries of the balls that have been removed.

Cubications

(N42) K^m_η denotes a general cubication of radius η in $\mathbb{R}^m$, see Section 2.2;

(N43) K^l_η denotes the l-skeleton of K^m_η, and $\mathscr{K}^l_\eta$ denotes its underlying set (we always distinguish families of cubes and the set formed by their union);

(N44) T^{l^*} denotes a dual skeleton, where $l^* = m - l - 1$, see Section 2.2;

(N45) In Chapter 2, E^m_η stands in general for the set of bad cubes, and U^m_η for the set of cubes that intersect a bad cube, see Section 2.6.

Sobolev spaces of mappings into manifolds

Abstract: This thesis is concerned with several properties of Sobolev spaces of mappings between manifolds. An important amount of research has been carried out on these spaces since the beginning of the 80's, notably motivated by their strong connection with problems arising from geometry, from physics, or from numerical methods. Although they are defined as metric subspaces of classical Sobolev spaces of vector-valued mappings, spaces of mappings with values into a manifold exhibit striking qualitative differences with the former ones. A typical instance of such a difference is the fact that smooth maps into a given manifold need not be dense among the Sobolev mappings with values into the same target, in strong contrast with the classical density result in real-valued functions spaces. Following this observation, a whole area of research was initiated, focused notably on the four following questions: (i) characterize those values of the parameters s and p, the domains, and the targets for which strong density of smooth maps does occur; (ii) find a suitable class of almost smooth maps which is always dense among Sobolev mappings; (iii) when strong density fails, characterize the closure of smooth maps; (iv) determine what happens if strong convergence is replaced by a weaker notion. This thesis aims at presenting a contribution in the study of each of these questions.
First, we solve the missing case $s > 1$ noninteger in the first two questions, where the main difficulty is the combination of the rigidity of higher order spaces with the nonlocal character in fractional order, hence concluding the complete answer to the strong density problem. We also push further the study of the second question by establishing the strong density of an improved class of almost smooth maps. Then, we construct two families of analytical obstructions to the weak approximation property, showing that, for any $p \in \mathbb{N} \setminus \{0, 1\}$ — the only case to be left open — there exist targets so that the weak approximation property fails.
Finally, we construct integral invariants allowing to characterize the closure of smooth maps for a large class of target manifolds in the range $0 < s < 1$, corresponding to the situation where already the construction of such invariants is a delicate task.

Keywords: Sobolev spaces; mappings into manifolds; strong density; weak density; analytical and topological obstructions; analytical and topological invariants; pullbacks of differential forms; harmonic maps.

Espaces de Sobolev à valeurs variétés

Résumé : Cette thèse s'intéresse à certaines propriétés des espaces de Sobolev d'applications entre variétés. Ces espaces ont fait l'objet d'une recherche intensive depuis le début des années 1980, notamment motivée par leur forte connection avec des problèmes issus de la géométrie, de la physique, ou encore du calcul numérique. Bien que naturellement définis comme des sous-espaces métriques d'espaces de Sobolev classiques de fonctions à valeurs vectorielles, les espaces de fonctions à valeurs dans une variété exhibent des différences qualitatives frappantes avec ces derniers. Un exemple typique d'une telle situation est le fait que les applications lisses à valeurs dans une variété donnée ne sont pas nécessairement denses parmi les applications de Sobolev à valeurs dans la même cible, en contraste flagrant avec le résultat de densité classique dans les espaces de fonctions à valeurs réelles.
Suite à cette observation, un pan de recherche entier s'ouvrit, se focalisant notamment sur les quatre questions suivantes : (i) caractériser les valeurs des paramètres s et p, les domaines, et les cibles pour lesquels il y a densité des applications lisses ; (ii) trouver une classe convenable d'applications presque lisses qui est toujours dense parmi les applications de Sobolev à valeurs dans une variété ; (iii) lorsque la densité forte échoue, caractériser la clôture des applications lisses ; (iv) étudier ce qu'il advient lorsqu'on considère une notion plus faible de convergence. Cette thèse a pour but de présenter une contribution à l'étude de chacune de ces questions.
Dans un premier temps, on résout le cas manquant $s > 1$ non entier des deux premières questions, où la difficulté principale est la combinaison de la rigidité des espaces d'ordre supérieur et du caractère non local propre à l'ordre fractionnaire, finalisant ainsi la réponse complète au problème de la densité forte. Par ailleurs, on pousse plus avant l'étude de la deuxième question en établissant la densité forte d'une classe améliorée d'applications presque lisses.
Ensuite, on construit deux familles d'obstructions analytiques à la propriété d'approximation faible, montrant que pour tout $p \in \mathbb{N} \setminus \{0, 1\}$ — le seul cas encore ouvert — il existe des cibles pour lesquelles la propriété d'approximation faible échoue.
Enfin, on construit des invariants intégraux permettant de caractériser la clôture des fonctions lisses pour une grande famille de variétés cibles dans la gamme $0 < s < 1$, correspondant au cas où la construction même de tels invariants est une tâche délicate.

Mots clés : Espaces de Sobolev ; applications entre variétés ; densité forte ; densité faible ; obstructions analytiques et topologiques ; invariants analytiques et topologiques ; tiré en arrière des formes différentielles ; applications harmoniques.

Image en couverture : Version revisitée par l'auteur de la figure 5.1 réalisée par Jean Van Schaftingen. Produit avec Asymptote.

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