Nonlinear Tools for Geometric Analysis

Antoine Detaille

Eidgenössische Technische Hochschule Zürich
Herbstsemester 2026

– September 9, 2026 – 20:32

Contents

1 Introduction

The goal of this course is to introduce a selection of problems from geometric analysis, and mainly, to cover some tools that turn out to be useful in the study of several of them. This course may be thought of as a continuation to Functional Analysis I and II. The main characteristic feature of this course is the nonlinear aspect of the problems that we will encounter, which is responsible for most of the challenge they present. For this reason, many of the tools that one is familiar with from functional analysis are not applicable as such, and one needs to either resort to twists to make them applicable in a nonlinear context, or to develop completely new techniques.

Geometric analysis being a broad topic, it is impossible to cover the whole of it, not even an anyhow significant part of it, over one course. For this reason, the problems and tools presented here remain a *selection*, which is necessarily biased by the experience and personal tastes of the author. It is nevertheless believed (and hoped) that the tools presented here have a strong potential to be reused in many other context in geometric analysis, and that the selected problem feature some of the typical difficulties that one faces in the field, hence giving some good preparation to start studying in an independent way other problems from geometric analysis.

Handling of versions. Those notes will be updated until at least the end of the semester, with the aim of hopefully improving them, adding extra material I could think of, and fixing typos. Any remark or suggestion (including a list of typos!) is more than welcome! Refer to the footer to known if you own the latest version.

Use of footnotes. I have gathered in footnotes (and they will be exclusively devoted to that) some comments, anecdotes, or sometimes jokes I thought of when writing those notes. They should therefore not be taken too seriously, and can definitely be ignored if your goal is exclusively to learn the material from the class.

Use of AI. No AI was used at any point to prepare these notes. The conception of the program of the course, its practical implementation, and all redaction and pedagogy choices are exclusively the outcome of the work of the (human) author of this text.

1.1 Our main object of study: Sobolev spaces into manifolds, and applications

In this course, we focus on the study of problems involving *Sobolev mappings taking their values into a compact manifold*, and even more specifically, on the study of the space of such mappings *per se*. Otherwise stated, Sobolev mappings will not be a tool to study, for instance, partial differential equations, but the object of study itself. Already here, there is a choice being made, and at least in two regards. First, manifold valued Sobolev mappings are not the only objects to arise in geometric analysis. And second, even working with those objects, there would be plenty of problems applying them to study. We refer to Section 1.3 for a brief review of other problems. Nevertheless, anyone having some introductory knowledge in functional analysis as well as in partial differential equations and calculus of variation is probably well aware of the importance of Sobolev spaces as a highly convenient framework to formulate many problems in analysis, notably due to their nice properties as function spaces, which makes already a good motivation for studying them.

To become more concrete, in this course, we consider a target space $\mathscr{N}$, which we assume to be a smooth closed (i.e., compact and without boundary) Riemannian manifold. We will always, unless otherwise stated, make the additional assumption that $\mathscr{N} \subset \mathbb{R}^{\nu}$ for some $\nu \in \mathbb{N}$, that is, $\mathscr{N}$ is viewed as a submanifold of some Euclidean space, with the Riemannian metric being induced by the ambient metric. This is not a loss of generality, because of the celebrated *Nash isometric embedding theorem*[1] [Nas54, Nas56], which asserts that any (abstract) Riemannian manifold can always be isometrically embedded into some Euclidean space, provided that one takes the dimension of the ambient space sufficiently large.

Our domain will be denoted Ω, and will always be taken to be a smooth bounded

[1] This theorem has an interesting story. In the 50's, the isometric embedding problem was among the most challenging problems in differential geometry. In 1953, John Nash was a very talented young mathematician at MIT. But not only was he good, he was also very aware of it and often boasting, which did not fail to irritate his colleagues. One day, fed up with Nash's behaviour, his office mate, Warren Ambrose, challenged him, hoping to teach him some humility: "Gnash, if you are so good, why don't you solve the isometric embedding problem?". J. Nash, to make sure that this problem was worth of his attention, went for a crazy bet: He publicly announced that he solved the problem, before even starting to work with it, to gauge the reaction of the community. Then, in the space of only a few years, he managed to solve the problem, developing on the way tools that were at this time completely unknown, and would turn out to be fruitful even in completely unrelated areas only several decades later (notably giving birth to *convex integration*). To understand to reach of Nash's work, let us conclude by quoting Mikhail Gromov's opinion about it: "At first, I found this proof as convincing as lifting oneself by the hair. But in the end, Nash, miraculously, did lift you in the air by the hair!"

A nice telling of this story is available in this article of the blog *Image des Mathématiques* (in French): `https://images.math.cnrs.fr/freeze/Gnash-un-tore-plat.html`. Nice depictions of the embedding process invented by Nash have been devised by the members of the *Hévéa project*: `https://hevea-project.fr/ENIndexHevea.html`.

open subset of $\mathbb{R}^m$. All the discussions in this course can be extended to the case where the domain is a compact Riemannian manifold $\mathscr{M}$ of dimension m, with more or less technical difficulties. To avoid an excess of technicality, starting with how to even define Sobolev spaces of mappings *on* manifolds, we restrict here to the more familiar case where the domain is an open subset of $\mathbb{R}^m$ — most of the time, we will further restrict to the case where Ω is the unit ball, $\Omega = \mathbb{B}^m$. The target manifold being embedded, the definition of $\mathscr{N}$-valued Sobolev mappings is rather straightforward: For $1 \leq p < +\infty$ and $0 < s < +\infty$, we define

$$W^{s,p}(\Omega; \mathscr{N}) = \{u \in W^{s,p}(\Omega; \mathbb{R}^\nu) \colon u(x) \in \mathscr{N} \text{ for every } x \in \Omega\}.$$

The definition of real-valued Sobolev spaces, as well as their main properties, will be recalled in the next chapter, in Section 2.1. This definition should really be understood as imposing a constraint on the values permitted for our maps: Sobolev mappings with values into $\mathscr{N}$ are nothing else than classical, vector-valued, Sobolev functions, which are additionally constrained to take their values into the target $\mathscr{N}$ everywhere. Sobolev functions, as L^p functions, being only defined almost everywhere, it would somehow make more sense to ask that $u(x) \in \mathscr{N}$ *almost everywhere*. However, we can always modify u on a null set to fit the definition that we gave.

Precisely because of the geometric constraint imposed on their values, manifold-valued Sobolev mappings are a well-suited tool in the modelling of many problems with a geometric flavour. Let us conclude this section by giving a few examples, as a motivation.

Perhaps the most well-known example is related to the *physics of ordered media*. Assume for instance being willing to describe the configuration adopted by a field of liquid crystals — as the ones used in LCD screens — located in the unit ball $\mathbb{B}^3$, and which are forced to adopt a given configuration on the boundary $\partial\mathbb{B}^3$, described by a map g. The feature that we wish to describe is the direction of the crystals at each point, which suggests to consider *vector-valued* mappings $u \colon \mathbb{B}^3 \to \mathbb{R}^3$, so that $u(x)$ is a vector indicating the direction taken by the field at the point x. However, this choice is not totally accurate. Indeed, a vector in $\mathbb{R}^3$ carries at the same time a *direction* and a *length* information, the latter being irrelevant to our situation. To obtain only a direction information, one is therefore led to work with *sphere-valued* mappings $u \colon \mathbb{B}^3 \to \mathbb{S}^2$.

The corresponding least action principle then dictates that the effective configuration should be a solution of the problem

$$\min\left\{\int_{\mathbb{B}^3} |\mathrm{D}u|^p \colon u \in W^{1,p}(\mathbb{B}^3; \mathbb{S}^2),\ \operatorname{tr} u = g\right\}, \tag{1.1.1}$$

where g is some fixed $W^{1-1/p,p}(\partial\mathbb{B}^3;\mathbb{S}^2)$ map. As a matter of fact, this model corresponds exactly to a particular case of the so-called *Oseen–Frank* model, which is in use to describe the physics of liquid crystals. We refer the reader to [HK88] for a nice presentation of this topic. Here, working with Sobolev spaces allows to use all their good properties as a function space to implement techniques from the calculus of variations, or from partial differential equations once having written the corresponding Euler–Lagrange equation.

More generally, the field of ordered media physics features many other natural targets that arise in applications. Again when modelling liquid crystals, one might be willing to make the description even more accurate by taking into account the fact that the crystals are not oriented, that is, $-u(x)$ describes the same configuration as $u(x)$. For this purpose, it is natural to work instead with the target being the 2-dimensional projective plane $\mathbb{RP}^2$ which is precisely constructed from $\mathbb{S}^2$ by identifying all pairs of antipodal points together. Other physical models featuring more involved target manifolds comprise *biaxial liquid crystals*, having two preferred directions, and which may be described by mappings with values into $\mathbb{S}^3/H$, with H being the finite group of quaternions; and also several phases of superfluid helium, involving targets built upon groups of rotations. We refer the reader to [Mer79] for a complete survey of this topic, and to [BC07] for a short presentation.

Another similar direction of research that has attracted a lot of interest recently is concerned with the study of *heterogeneous materials*, namely, composite bodies formed by several different materials, typically two with one of them arranged in the other one according to a periodic structure. One may then study the physical properties of the material.

For instance, if the materials have ferromagnetic properties, one may then try to describe the state of the composite when subjected to an external magnetic field. In this situation, physical models assert that the appropriate quantity to study is the so-called *magnetization field*, which is a vector field M encoding the relevant magnetic properties of the body. But, a principle from micromagnetics, called the *fundamental constraint of micromagnetics theory*, asserts that the magnetization field M is saturated, which means that $|M| = M_0$, for some positive constant M_0 depending on the temperature of the body. To summarize, up to normalization, the relevant quantity to study such a model is again a mapping with values into the sphere $\mathbb{S}^2$.

Once more, by virtue of a least action principle postulated by the physical models, the actual configuration that will be observed may be obtained as a minimizer of a calculus of variations problem similar to (1.1.1). More precisely, the quantity in (1.1.1) would account for some part of the energy to be minimized, called the *internal energy*, and one would need to complete the functional with additional terms to account for the contribution of the other physically relevant quantities.

We refer the reader to [ADF15] for one typical instance of the type of research which is performed in this direction, as well as to [GHP24, DH24] for very recent developments that build a bridge between the theory of Sobolev mappings into manifolds and the study of heterogeneous materials, relying on several theoretical tools that will show up later on in this course, such as the *method of singular projection*.

We conclude this short exposition with an example from a totally different area, namely, numerical methods. Also in this discipline, mappings to manifolds have proved their usefulness, notably for their ability to encode an orientation when the target is chosen to be a sphere, a projective plane, or a group of rotations. We refer the reader to [HTWB11] for such an application to the description of the attitudes of a cube for computer graphics, or to the *Hextreme* project `https://www.hextreme.eu/` for applications to the generation of meshes for finite elements methods. In the latter project, nice meshes were notably generated through solving a suitable Ginzburg–Landau-type problem, which is once again a calculus of variations problem with a geometric constraint in the spirit of (1.1.1).

We do not claim here to give an exhaustive survey of possible applications of mappings to manifolds, but only a few, hopefully representative, examples. Other instances would include the study of deformations in *Cosserat materials* [ET58], involving mappings into $\mathbb{R}^3 \times SO(3)$, the study of the Ginzburg–Landau functional, which constitutes a whole area of research on its own, and involves notably the study of $\mathbb{S}^1$-valued maps, or gauge theory, where mappings with values into a *Lie group* G encode the corresponding gauge changes.

1.2 A case study: The strong approximation problem

This course will focus on the *strong approximation problem* of Sobolev mappings taking their values into a manifold. However, this should not be thought as an objective *per se*, but rather as an excuse to explain some tools from geometric analysis that have been developed to tackle this problem or related ones. For this reason, we will make several digressions with respect to this problem during the lecture, and not go directly to the most recent existing proofs.

In this section, we introduce the problem, and summarize what is going to be discussed in the lecture about it, before going to more concrete results and arguments in the next chapters.

The starting point for the the strong approximation is the following fundamental *density theorem* for real-valued Sobolev functions, that we recall below.

Theorem 1.2.1. *The space $C^\infty(\overline{\Omega})$ is dense in $W^{s,p}(\Omega)$. That is, for every $W^{s,p}$ map $u\colon \Omega \to \mathbb{R}$, there exists a sequence of smooth maps $u_n\colon \overline{\Omega} \to \mathbb{R}$ such that $u_n \to u$ as $n \to +\infty$, strongly with respect to the $W^{s,p}$ norm.*

In the above, $C^\infty(\overline{\Omega})$ denotes those functions on Ω that are smooth *up to the boundary*. More precisely, elements in $C^\infty(\overline{\Omega})$ are exactly restrictions to Ω of functions in $C^\infty_c(\mathbb{R}^m)$. In particular, some caution is needed, as $C^\infty(\overline{\mathbb{R}^m}) \neq C^\infty(\mathbb{R}^m)$, despite $\overline{\mathbb{R}^m} = \mathbb{R}^m$. This will however ultimately not matter here, as we always consider bounded domains.

This theorem is classical in the theory of Sobolev spaces, and its proof can be found in many references. We refer the reader to [Bre11, Corollary 9.8] or [Wil22, Theorem 6.3.2] for a proof on smooth domains, relying on extension operators, when $s = 1$, to [Ada75, 3.18 Theorem] for a proof when Ω merely has the segment condition, when $s = k \in \mathbb{N}_*$, and to [Leo23, Theorem 6.70] for the fractional case $s \notin \mathbb{N}$.

The standard proof of Theorem 1.2.1 relies on regularisation by convolution. Namely, one considers a mollifying kernel $\varphi \in C^\infty_c(\mathbb{B}^m)$ such that $\varphi \geq 0$ and $\int_{\mathbb{R}^m} \rho = 1$, defines φ_ε by $\varphi_\varepsilon(x) = \varepsilon^{-m}\varphi\left(\frac{x}{\varepsilon}\right)$, and considers $u_\varepsilon = \varphi_\varepsilon * u$. Then, it holds that u_ε is smooth, and that $u_\varepsilon \to u$ in $W^{s,p}$ as $\varepsilon \to 0$. One has to face technicalities when working on a domain, as the convolution is not defined near the boundary, and an extension procedure is therefore required, however, the core of the argument is the above sketch.

Given the importance and utility of this approximation theorem when manipulating Sobolev functions, a natural question that arises is whether or not the same can be done for manifold-valued maps, *preserving the manifold constraint*. Another way to phrase this problem is as follows. Let us define $H^{s,p}_{\mathrm{S}}(\Omega;\mathscr{N})$ as the completion of $C^\infty(\overline{\Omega};\mathscr{N})$ with respect to the $W^{s,p}$ norm. Clearly, we have $H^{s,p}_{\mathrm{S}}(\Omega;\mathscr{N}) \subset W^{s,p}(\Omega;\mathscr{N})$. The strong approximation problem asks whether or not one gets the whole space $W^{s,p}(\Omega;\mathscr{N})$ when doing so.

Question 1.2.2 (Strong approximation problem). Do we have $H^{s,p}_{\mathrm{S}}(\Omega;\mathscr{N}) = W^{s,p}(\Omega;\mathscr{N})$? Otherwise stated, can any map $u \in W^{s,p}(\Omega;\mathscr{N})$ be approximated with a sequence of smooth $\mathscr{N}$-valued maps, strongly with respect to the $W^{s,p}$ norm?

Let us already emphasise that one cannot answer positively to this question by applying Theorem 1.2.1, nor (a straightforward modification of) its proof. Indeed, Theorem 1.2.1 — applied component by component — would only provide an approximating sequence of $\mathbb{R}^\nu$-valued maps, that need not satisfy the constraint of being $\mathscr{N}$-valued. Moreover, the proof of Theorem 1.2.1 relies on regularisation by convolution, which is morally nothing else than a convex combination procedure, as one take the integral of u with a function which has integral equal to 1. But, convex combinations of points on a manifold have *no reason* to still belong to the manifold. This is our first example in this lecture of a *linear technique* not being applicable, because of the nonlinearity of the problem under study.

One could think that this is just an issue with the proof, that can be bypassed by resorting to another technique, but that in the end density of smooth maps should hold. However, in a *striking contrast* with the situation for real-valued Sobolev functions, smooth maps *may fail* to be dense in Sobolev spaces of mappings with values into a manifold. The first counterexample was provided by R. Schoen and K. Uhlenbeck in 1983 [SU83], and reads as follows.

Proposition 1.2.3. *Let* $2 \leq p < 3$. *Then,* $x/|x| \in W^{1,p}(\mathbb{B}^3; \mathbb{S}^2) \setminus H^{1,p}_{\mathrm{S}}(\mathbb{B}^3; \mathbb{S}^2)$.

Proving Proposition 1.2.3 will be, after some reminders on Sobolev spaces and topology, the main objective of Chapter 2. Let us however already mention the core mechanism at work. The obstruction to the map $u_0(x) = x/|x|$ being approximated with smooth maps taking their values into the sphere is of *topological* nature. More specifically, the regularity $W^{1,p}$ with $2 \leq p < 3$ in dimension 3 is sufficiently mild to allow for u_0 to have a singularity at the origin, around which it realises scaled copies of the identity map $\mathbb{S}^2 \to \mathbb{S}^2$ — which would not be possible, for instance, for continuous maps, by virtue of Brouwer's theorem — but sufficiently strong to preserve the degree on spheres, hence forbidding approximation with smooth maps, which cannot display this first property.

The general phenomenon at work was later on identified by F. Bethuel and Zheng X. [BZ88], leading to an obstruction to the strong approximation property phrased in terms of the *homotopy groups* of the target.

Proposition 1.2.4. *Assume that* $sp < m$. *If* $\pi_{\lfloor sp \rfloor}(\mathscr{N}) \neq \{0\}$, *then* $H^{s,p}_{\mathrm{S}}(\Omega; \mathscr{N}) \subsetneq W^{s,p}(\Omega; \mathscr{N})$.

Here, $\lfloor \cdot \rfloor$ denotes the floor function, that is, the biggest integer less or equal than a number.

Let us pause to comment an important assumption in Proposition 1.2.4, namely, that $sp < m$. This assumption is crucial for obstructions to show up. Indeed, when $sp \geq m$, which we will call the subcritical dimension, then it *always holds* $H^{s,p}_{\mathrm{S}}(\Omega; \mathscr{N}) = W^{s,p}(\Omega; \mathscr{N})$. This is what is sometimes called the *easy case*, where the higher regularity of Sobolev mappings allows to return to the real-valued case, and will be the main topic of Chapter 3. At the end, in this regime, one is able to show that, applying the real-valued theory componentwise, for a suitably chosen approximating sequence of smooth maps u_n with values into the ambient space $\mathbb{R}^\nu$, u_n takes its values close to the manifold for n sufficiently large. This allows to reproject back onto $\mathscr{N}$ and conclude the argument. However, this involves a step which is less trivial than it looks like, namely, composing a Sobolev map with a smooth map. This will be an opportunity to see how an excursion through a refined scale of spaces may lead to exploitable regularity improvements, even when the final purpose is to work with Sobolev spaces.

Let us return to the case $sp < m$, and resume our story where we left it, that is, at Proposition 1.2.4. Phrased otherwise, this proposition provides a *necessary* condition for smooth maps to be dense in Sobolev spaces of mappings. The natural question that arises at this stage is whether or not this condition is also sufficient, or if additional obstructions may arise. This was answered by F. Bethuel in his seminal 1991 paper [Bet91] for $s = 1$ — although some partial results were already known from the work with Zheng X. [BZ88] — and then extended to other values of the regularity parameter by H. Brezis and P. Mironescu [BM15] (2015, $0 < s < 1$), P. Bousquet, A. Ponce, and J. Van Schaftingen [BPVS15] (2015, $s \in \mathbb{N}_*$), and in [Det26] (2026, $s > 1$ noninteger), leading to the following theorem.

Theorem 1.2.5. *Assume that $sp < m$. Then, $H^{s,p}_{\mathrm{S}}(\mathbb{B}^m; \mathscr{N}) = W^{s,p}(\mathbb{B}^m; \mathscr{N})$ if and only if $\pi_{\lfloor sp \rfloor}(\mathscr{N}) = \{0\}$.*

Despite Theorem 1.2.5 providing a beautiful answer to Question 1.2.2, elegantly linking an analytical question to a topological property of the target manifold, one could legitimately remain somehow unsatisfied. Indeed, the density of smooth maps is a desirable property when working with Sobolev spaces, and Theorem 1.2.5 gives no positive information in case the $\lfloor sp \rfloor$-th homotopy group of the target is nontrivial. To make matters worse, homotopy groups are notoriously hard to compute, and have the annoying tendency of not being often trivial — it suffices to open the table of homotopy groups of spheres in Hatcher [Hat02, p. 339], which are arguably the simplest non trivial manifolds, to realise this.

Faced with this observation, there are essentially two main paths to try making progress. A first possibility, that will not be covered in this lecture, but which is briefly explained and documented in the next section, is to try to characterise *which maps are strongly approximable with smooth maps*, that is, give tractable characterisations of the space $H^{s,p}_{\mathrm{S}}(\Omega; \mathscr{N})$. The second one is to look for a nice class of "almost smooth maps" — in a sense to be made precise — that would be dense in $W^{s,p}(\Omega; \mathscr{N})$. That is, we would like to give up a bit of smoothness to recover density, while still obtaining a nice class of maps that is convenient to manipulate and to make computations and constructions with.

To this endeavour, it seems relevant to go back to Schoen and Uhlenbeck's seminal counterexample. Another important feature of the map $u_0(x) = x/|x|$ that they used is that, despite not being smooth, it is smooth *outside of a point singularity*. An examination of other counterexamples built upon this one reveals that typically, non approximable maps feature a topological singular set of dimension $m - \lfloor sp \rfloor - 1$, around which they realise copies of nontrivial elements of $\pi_{\lfloor sp \rfloor}(\mathscr{N})$. In $W^{s,p}(\mathbb{B}^4; \mathbb{S}^2)$ for instance, a typical counterexample would be build as a cylinder over u_0, hence displaying a full line of singularities.

This suggests to introduce the following class of mappings. Given $\ell \in \{0, \ldots, m-1\}$, we define $\mathscr{R}_\ell(\Omega; \mathscr{N})$ as the set of all maps $C^\infty(\overline{\Omega} \setminus \mathscr{S}; \mathscr{N})$ such that

$$|\mathrm{D}^j u(x)| \leq \frac{C}{\operatorname{dist}(x, \mathscr{S})^j} \quad \text{for every } j \in \mathbb{N}_*, \tag{1.2.1}$$

where $C > 0$ is a constant depending on u and j, and where $\mathscr{S} = \mathscr{S}_u \subset \mathbb{R}^m$ is a finite union of ℓ-dimensional affine spaces, also depending on u. Here, $C^\infty(\overline{\Omega} \setminus \mathscr{S}; \mathscr{N})$ means that u is the restriction to Ω of a map in $C^\infty_c(\mathbb{R}^m \setminus \mathscr{S}; \mathscr{N})$. In particular, the set $\mathscr{S}$ extends outside of Ω — and parts of it outside of Ω may be used to compute the distance in (1.2.1). That is, elements of $\mathscr{R}_\ell(\Omega; \mathscr{N})$ are maps that are smooth outside of some singular set of dimension ℓ, with a controlled rate of blow-up of the derivatives when approaching the singular set. This latter condition ensures that $\mathscr{R}_{m-\ell-1}(\Omega; \mathscr{N}) \subset W^{s,p}(\Omega; \mathbb{N})$ when $sp < \ell$. A prototypical example of a mapping in the class $\mathscr{R}$ is the hedgehog map $u_0(x) = x/|x|$ — which should be no surprise, as this class was constructed specifically for that reason — as we have $u_0 \in \mathscr{R}_0(\mathbb{B}^3; \mathbb{S}^2)$. The following theorem shows that, in some sense, singularities of the type $x/|x|$ are the worse that can happen.

Theorem 1.2.6. *Assume that $sp < m$. Then, $\mathscr{R}_{m-\lfloor sp \rfloor-1}(\mathbb{B}^m; \mathscr{N})$ is dense in $W^{s,p}(\mathbb{B}^m; \mathscr{N})$.*

As for Theorem 1.2.5, the main breakthrough regarding Theorem 1.2.6, that is, the model case $s = 1$, was accomplished by F. Bethuel in his seminal 1991 paper [Bet91] — with also some partial results obtained with Zheng X. [BZ88] for the special case of $\mathscr{R}_0$ — and then extended to other values of s in [BM15, BPVS15, Det26].

Aside from its interest *per se* — in practice, Theorem 1.2.6 is applied way more often than Theorem 1.2.5, which suffers from its topological assumption which is difficult to verify in practice as we already explained — Theorem 1.2.6 is a key ingredient in the *proof* of Theorem 1.2.5. Indeed, all currently known proofs to derive Theorem 1.2.5 that work for a general target go first by proving the density of the class $\mathscr{R}_{m-\lfloor sp \rfloor-1}$, and then obtaining the density of smooth maps under the extra topological assumption, by explaining how to exploit the latter to remove the singularities of mappings in the class $\mathscr{R}$.

So far, the attentive reader may have noticed that the domain in the statements of Theorems 1.2.5 and 1.2.6 is the unit ball $\mathbb{B}^m$, and not a general domain Ω. This is not an innocuous fact, and stems from the fact that, when the domain Ω is allowed to have nontrivial topology, new obstructions to the strong approximation property may arise. Those obstructions are called *global*, because they involve the interplay between the topology of the domain and the target, and are not to be seen locally, on a ball. This was understood by Hang F. and Lin F. [HL03a] in 2003 in the case

$s = 1$. Theorem 1.2.6 remains valid as such on a general domain, and the necessary and sufficient condition $\pi_{\lfloor sp \rfloor}(\mathscr{N}) = \{0\}$ from Theorem 1.2.5 has to be modified to take into account also the topology of the domain, leading to a generalised condition, which encompasses $\pi_{\lfloor sp \rfloor}(\mathscr{N}) = \{0\}$ and reduces to it if the domain has trivial topology. However, this condition being more involved to state, we will not delve into this matter in this lecture.

The standard proof of Theorems 1.2.5 and 1.2.6 relies on the *method of good and bad cubes*, and will be the main topic of Chapter 6. Somehow, this method can be viewed as inspired from the Calderón–Zygmund decomposition, with the important conceptual difference that, in the Calderón–Zygmund decomposition, one obtains a collection of cubes of different sizes, while in the method of good and bad cubes, one works with a grid of cubes all having the same size. This method also happened to be used, in an independent fashion, in other problems from geometric analysis, notably in the study of *Sobolev homeomorphisms*, as well as *weak Yang–Mills connections*. This will be documented more precisely in the next section.

However, before we present the general proof of Theorems 1.2.5 and 1.2.6, somehow following the historical order, we will present an alternative argument, which features a conceptually simpler approach, at the price of not being applicable to the general case of an arbitrary target manifold, as it requires stronger assumptions on the topology of the target. This argument is based on the *method of singular projection*, which was devised by R. Hardt and Lin F. [HL87], taking roots in the work of H. Federer and W. Fleming on *normal and integral currents* [FF60].

The idea is to somehow find a way to still apply the linear theory. If there would exist a smooth retraction $P\colon \mathbb{R}^\nu \to \mathscr{N}$, that is, such that $P = \mathrm{id}$ on $\mathscr{N}$, then one would be able to apply the standard theorems for vector-valued maps, and conclude by reprojecting everything onto $\mathscr{N}$ by P. However, for a compact manifold $\mathscr{N}$, such a map *never exists* (it is actually a nice exercise in topology to prove it!). Nevertheless, one can in some cases obtain a *singular retraction*. More specifically, under suitable assumptions, one can construct a retraction $P\colon \mathbb{R}^\nu \setminus \mathscr{S} \to \mathscr{N}$, where $\mathscr{S}$ is some singular set whose dimension depends on the topology of $\mathscr{N}$. Two challenges arise however when willing to bring back to $\mathscr{N}$ by composing with the singular retraction P. First, there is no guarantee that the map that we wish to compose with P does not take its values into the singular set $\mathscr{S}$ on some set of positive measure, making it ill-defined. Second, even if we would manage to avoid the previous issue, it is not clear how to obtain estimates or any kind of continuity for the above construction, as the singularities of P prevent us from applying to it standard composition results with smooth maps. Dealing with those two issues is at the core of the method of singular projection, that we will cover in Chapter 4.

After that, we will take the method of projection as a excuse to study a further tool in geometric analysis, namely the *nonlinear uniform boundedness principle*. As the name suggests, this tool can be thought of as a nonlinear version of the Banach–Steinhaus theorem, and it has proved to be very powerful in constructing counterexamples. More specifically, this principle asserts that, to construct a counterexample to a *qualitative* property, such as approximability, or existence of some objects, it is sometimes sufficient to contradict the *quantitative* version of the property, showing that some estimate fails to be satisfied by constructing a sequence along which some energy blows up superlinearly with respect to the Sobolev energy of the sequence. In Chapter 5, we will explore the history of this principle, state a general version of it, and use the method of projection as an excuse to show an application of it.

In our last chapter, we will wander a bit further apart from the strong approximation problem, by having a glimpse at one of the possible next natural problems to study, that is, the *weak approximation problem*. More specifically, once faced with the failure of the strong approximation property for some targets $\mathscr{N}$, another possible direction of research to explore is to try relaxing the notion of convergence under use, and looking for approximations with smooth maps with respect, for instance, to the *weak convergence*. As opposed to the strong approximation problem, which by now has what can be qualified of a full solution, with a pretty clear set of conditions being involved, the weak approximation problem is still broadly open, and there is not even a clear conjectural condition that seems to emerge from the already known cases. This problem is highly challenging and combines many different techniques, to the extent that it could be the topic of a lecture on its own. In Chapter 7, we will offer a brief introduction to this problem, and study in a more detailed way a model example of application of one technique related to this problem, namely, the *dipole insertion procedure*. Apart from its use in the weak approximation problem, this technique has also shown its usefulness in order related problems, notably finding applications in the construction of harmonic maps — a striking example being the construction of everywhere discontinuous harmonic maps by T. Rivière [Riv95] — or more recently in problems related to Yang–Mills connections [MR], which is our motivation for presenting it in Chapter 7.

1.3 Broadening our horizon: A selection of other problems from geometric analysis

We conclude this introduction with a brief review of other problems from geometric analysis, along with references for the curiosity of the reader. Just as the main content of this lecture, this section is necessarily biased by the experience and knowledge of the author, and the topics proposed, as well as the references given for them, are necessarily far from being exhaustive. Moreover, we limit ourselves to a very sketchy exposition, and prefer to refer to a few recent papers, ideally surveys, rather than listing a huge quantity of relevant references.

To start with, staying in the topic of Sobolev mappings into manifolds, one encounters other problems in addition to the strong and weak approximation.

The extension problem. Gagliardo's trace extension theorem asserts that a function $g \in W^{1-1/p,p}(\partial\Omega)$ is the trace on $\partial\Omega$ of a map $u \in W^{1,p}(\Omega)$; see [Gag57]. The extension problem for Sobolev mappings with values into manifolds asks whether or not the same can be done for mappings taking their values into a compact manifold. Here, we refer to J. Van Schaftingen [VS24], which provides the final solution to the problem, and to the references therein for partial results that were known before.

The lifting problem. A classical result in topology asserts that any continuous map u from a simply connected domain into the circle $\mathbb{S}^1$ can be written as $u = e^{\imath\theta}$ for some continuous real-valued function θ, which is called a *lifting* of u. The same can be done more generally for a *Riemannian covering*. The lifting problem for Sobolev mappings asks if the same can be done for Sobolev regularity instead, that is, if a Sobolev map into a given manifold can be lifted as a Sobolev map into one of its Riemannian coverings. The last missing case for a complete solution to this problem was solved by P. Mironescu and J. Van Schaftingen [MVS21a], which also review the history of the problem with all the contributions for the other cases.

The homotopy problem. Here, one considers $W^{s,p}(\Omega; \mathscr{N})$ viewed as a topological space, where the topology arises from the $W^{s,p}$ norm, and ask for its topological properties, notably its connected components. We refer to Hang. F and Lin F. [HL03a], and the references therein, notably to the work of H. Brezis, Li Y., and B. White.

Degree problems and related. Every continuous map $\mathbb{S}^1 \to \mathbb{S}^1$ has a *topological degree*, counting the number of times that the map winds around $\mathbb{S}^1$, with orientation. In particular, $W^{s,p}(\mathbb{S}^1; \mathbb{S}^1)$ maps have a degree. Starting from that, several questions arise. Can we define a robust notion of degree for lower regularity maps? Can we estimate the degree of a map with its Sobolev energy, or other related quantities? Can we minimise functionals — e.g., the Dirichlet energy — with degree-type constraints, like prescribed degree on the boundary, or singularities with prescribed degree? Since here the main object of study is circle-valued maps, a very complete reference is the book [BM21], which contains everything you want to know about Sobolev maps to the circle[2].

[2] And even everything you do not want to know, or rather, you do not know that you want to know!

Sobolev homeomorphisms. Sobolev homeomorphisms, that is, homeomorphisms $f : \Omega \subset \mathbb{R}^m \to f(\Omega) \subset \mathbb{R}^m$ such that $f \in W^{1,p}(\Omega)$, are of special importance in nonlinear elasticity. Indeed, the homeomorphism constraint models the non-interpenetrability of the material. An important question in this field is the Ball–Evans problem, asking about approximability of Sobolev homeomorphisms with piecewise affine homeomorphisms or with diffeomorphisms. The solution to this problem in the planar case $m = 2$ for $p = 1$ was given by S. Hencl and A. Pratelli [HP18], using a good and bad cubes-type approach in the spirit of what we will present in Chapter 6. We refer to [Hen26] and the references therein for a survey of the recent progress on this topic.

Apart from their study per se, Sobolev mappings also provide a convenient framework for a lot of problems with a geometric flavour, involving notably partial differential equations or the calculus of variations, and sometimes inspired from problems from physics.

Harmonic maps. Harmonic maps are an immense topic on their own. The starting point may be seen as minimising the Dirichlet energy

$$E(u) = \int_{\Omega} |\mathrm{D}u|^2,$$

but among $\mathscr{N}$-valued maps. The nonlinearity of the constraint translates as a nonlinearity in the Euler–Lagrange equation associated with this problem: While the Euler–Lagrange equation for the Dirichlet energy for real-valued maps is the Laplace equation $\Delta u = 0$, in presence of a nonlinear target constraint, it features a nonlinear term involving the second fundamental form of the target. For sphere-valued maps for instance, it becomes $\Delta u = -|\mathrm{D}u|^2 u$. In turn, solutions of this equation are also studied. Because of the loss of linearity and convexity, solutions need not be unique, and do not coincide with minimisers of the corresponding variational problem. They can for instance also be obtained as mere stationary points. Typical questions include, but not only, existence for various constraints (prescribed boundary values, prescribed homotopy class, and so on), uniqueness, regularity, study of the singular set. Listing even a small part of the literature would be an arduous task. We refer to [HW08] and the references therein for a survey[3].

There is a huge jungle of other variational problems in geometric analysis, some of them were already mentioned in Section 1.1. This includes, but not only, homogeni-

[3] For more recent developments, there are many experts of this topic here at ETH (not me!), in groups 5 and 6, do not hesitate to ask around!

sation, Ginzburg–Landau theories, the Allen–Cahn problem, nonlinear elasticity, and many more.

Leaving now the world of Sobolev mappings, let us conclude with some other geometric objects and quantities studied by geometric analysts.

The Willmore functional. The Willmore functional, originally introduced by Sophie Germain and Siméon Denis Poisson to model elastic membranes, is the L^2 norm of the second fundamental form of an immersion $\Phi\colon \Sigma \to \mathbb{R}^m$, where Σ is a two-dimensional surface. Here, the nonlinearity does not come from a target constraint — after all, the immersions are nice $\mathbb{R}^m$-valued maps — but rather from the functional under use, which involves the second fundamental form of the immersion, and which, in turns, makes the corresponding Euler–Lagrange equation nonlinear. A first question is to find a suitable class of *weak immersions* in which the minimisation problem can be studied. One a suitable framework has been defined, typical questions include, but not only, existence, uniqueness, and regularity of minimisers, or the evolution under the flow of the Willmore functional. We refer to [LMR26] and the references therein for a survey.

The Yang–Mills functional. The Yang–Mills functional is at the interface between geometry and mathematical physics, and has notably led to the construction of "fake" $\mathbb{R}^4$. It is defined as the L^2 norm of a connection A with values into the Lie algebra $\mathfrak{g}$ of a Lie group G. Here again, the nonlinearity does not come from any kind of constraint, as connections are brave vector-valued maps, but they come from the nonlinearity of the quantity used to define the corresponding functional — the curvature of a connection A is defined as $F_A = \mathrm{d}A + A \wedge A$, which mixes a linear and a quadratic term. Once again, one is first led to find a suitable class of weak connections to state the minimisation problem, and one can then consider notably questions like existence, uniqueness, and regularity of solutions, as well as finer of their properties. We refer to [Riv14] and the references therein for a survey. This reference also contains a survey of problems related to the Willmore functional that was discussed above, and of the more general class of *conformally invariant problem* they belong to. We also mention the recent paper [CR24], as it features the method of good and bad cubes in yet another context.

Minimal surfaces. Here, we could say that the starting point is the classical *Plateau problem*: Find the surface of least area whose boundary is prescribed. (This is for instance a model of soap films supported by a wire.) In this area, we are going even further from Sobolev mappings. Indeed, the natural framework to study this kind of problem relies on the language of geometric measure theory, and uses objects such as *currents* or *varifolds*. However, let us mention that they have strong connections[4] with the study of Sobolev mappings, and especially with problems like defining their singular set, weak approximation, and relaxed energy. Instead of giving a reference this time, we limit ourselves to mentioning that this was the topic of a master course taught last year here at ETH, by A. Skorobogatova.

Even after this enumeration, we are far from having covered all topics from geometric analysis. Moreover, what exactly enters in the scope of *geometric analysis* is already a hard question. The question notably arises about when do we leave geometric analysis in both directions, towards analysis or towards geometry. Problems related for instance to Sobolev maps *on manifolds* (notably involving the spectrum of manifolds, and the "can one hear the shape of a drum" problem) or free boundary problems on the one side, and problems related to symplectic, Kähler, or complex manifolds but relying on tools from analysis on the other side, could arguably be classified as geometric analysis. And the list goes on.

[4]Unintended pun, but the language of geometric measure theory is used to define *connections* for the singular set of Sobolev mappings.

2 Towards our first topological obstruction: Reminders from analysis and topology

In this chapter, our main objective is to explain and prove Schoen and Uhlenbeck's seminal counterexample, that we announced in the introduction as Proposition 1.2.3. On the way, we start by providing some reminders about fundamental notions regarding Sobolev spaces and that will be useful to us all along the course. In the process, we spend some more time on the notion of *genericity*, which will play a fundamental role in the sequel. We also briefly recall some notions of homotopy theory that will be useful to us in this chapter and the rest of the course.

2.1 Reminders and complements on Sobolev spaces

As has become evident from the introduction, the central object in this lecture is the *Sobolev space* $W^{s,p}$, where $0 < s < +\infty$ is the regularity parameter and $1 \leq p < +\infty$ the integrability parameter — we do not consider $p = +\infty$ here. Let us recall the definition of this space. We start with $s = k \in \mathbb{N}_*$. In this case, the Sobolev space $W^{k,p}(\Omega)$ is the space of all elements $u \in L^p(\Omega)$ whose derivatives (in the sense of distributions) up to order k also belong to $L^p(\Omega)$. That is, for every multi-index $\alpha \in \mathbb{N}^m$ with $|\alpha| \leq k$, there exists a map $g_\alpha \in L^p(\Omega)$ such that

$$\int_\Omega u\partial^\alpha\varphi = (-1)^{|\alpha|}\int_\Omega g_\alpha\varphi \quad \text{for every } \varphi \in C^\infty_{\mathrm{c}}(\Omega).$$

We write $g_\alpha = \partial^\alpha u$. The $W^{k,p}(\Omega)$ norm is then defined as

$$\|u\|^p_{W^{k,p}(\Omega)} = \sum_{|\alpha|\leq k} \|\partial^\alpha u\|^p_{L^p(\Omega)},$$

with the convention that $\partial^{(0,\ldots,0)}u = u$.

We now introduce fractional Sobolev spaces of order $s = \sigma \in (0,1)$. For this purpose, we introduce the *Gagliardo seminorm* of a measurable map $u\colon \Omega \to \mathbb{R}$ as

$$|u|^p_{W^{\sigma,p}(\Omega)} = \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{m+\sigma p}}\,\mathrm{d}x\,\mathrm{d}y.$$

The space $W^{\sigma,p}(\Omega)$ is then defined as the set of those $L^p(\Omega)$ maps u such that $|u|_{W^{\sigma,p}(\Omega)} <$

$+\infty$, endowed with the norm defined as

$$\|u\|^p_{W^{\sigma,p}(\Omega)} = \|u\|^p_{L^p(\Omega)} + |u|^p_{W^{\sigma,p}(\Omega)}.$$

Finally, we can define higher order fractional Sobolev spaces. For $s = k + \sigma$ with $k \in \mathbb{N}_*$ and $\sigma \in (0,1)$ — in the sequel, we shall always denote by k the integer part and by σ the fractional part of the regularity exponent — we define $W^{s,p}(\Omega)$ as those maps $u \in W^{k,p}(\Omega)$ whose k-th order derivative belongs to $W^{\sigma,p}(\Omega)$, that we endow with the norm defined as

$$\|u\|^p_{W^{s,p}(\Omega)} = \|u\|^p_{W^{k,p}(\Omega)} + |\mathrm{D}^k u|^p_{W^{\sigma,p}(\Omega)}.$$

Vector-valued spaces are defined analogously, component by component.

Fractional Sobolev spaces may really be thought of as a continuous scale of regularity between two integer order Sobolev spaces, in a way that can be rigorously formalised via *interpolation theory*. However, even letting aside those, say, more theoretical considerations, they appear naturally notably as the trace spaces of integer order spaces, as we shall see later on.

We now recall, without proof, some fundamental properties of Sobolev spaces. Basic properties of integer order spaces can be found, for instance, in [Bre11, Leo17, Wil22]. More advanced properties are to be found, e.g., in [Ada75, Maz11]. For fractional Sobolev spaces, we recommend the book [Leo23], and also the survey [DNPV12]. For function spaces in general, good and complete references are notably [RS96, Tri83, Tri92, Tri06].

We start with some basic properties of the function space $W^{s,p}(\Omega)$.

Theorem 2.1.1. *The space $W^{s,p}(\Omega)$ is a separable Banach space for $1 \leq p < +\infty$ and $0 < s < +\infty$. When $1 < p < +\infty$, it is reflexive and uniformly convex.*

When discussing Sobolev spaces, one of their key features are *embeddings*, which can in some sense be interpreted as saying that Sobolev functions are more regular than expected from their definition. We present below the general form of the Sobolev embeddings, covering the full range of the regularity parameter s. We use as a shortcut the convention $W^{0,p} = L^p$ in order to have unified statements, and we also include the case where the integrability exponent can be infinite, to allow notably for L^∞.

Theorem 2.1.2 (Sobolev's embedding)**.** *Let $0 \leq r \leq s < +\infty$ and $1 \leq p \leq q \leq +\infty$ satisfy*

$$s - \frac{m}{p} \geq r - \frac{m}{q}. \tag{2.1.1}$$

Then, we have the continuous embedding

$$W^{s,p}(\Omega) \hookrightarrow W^{r,q}(\Omega),$$

unless we are in one of the two exceptional cases below, where the embedding fails:

1. $m = 1$, $s \in \mathbb{N}_*$, $p = 1$, $1 < q < +\infty$, *and* $r = s - 1 + \frac{1}{q}$;
2. $1 < p < +\infty$, $q = +\infty$, *and* $s - \frac{m}{p} = r \in \mathbb{N}$.

Theorem 2.1.3 (Morrey's embedding)**.** *Let* $0 < s \leq 1$ *and* $sp > m$. *Let* $\alpha = s - \frac{m}{p}$. *Then,*

$$W^{s,p}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega}).$$

Of course, one may combine Morrey and Sobolev's embedding theorems, or apply the Morrey embedding to higher order derivatives, to deduce version of the Morrey embedding for $s > 1$. What will mostly be useful to us is the fact that, when $sp > m$, then $W^{s,p}(\Omega)$ embeds into $C^0(\overline{\Omega})$. This embedding is even compact (Rellich–Kondrashov's theorem).

In the limiting case $sp = m$, this embedding fails. Nevertheless, and this will be very important for us in the sequel, we have an embedding into a limiting space, very close to C^0, that is, the space VMO of functions with *vanishing mean oscillation*. We first recall some definitions.

We say that $u \in L^1(\Omega)$ belongs to the space BMO(Ω) of functions with *bounded mean oscillation* whenever

$$|u|_{\mathrm{BMO}(\Omega)} = \sup_{\substack{0<r<r_0\\ B_r(x)\subset\Omega}} \fint_{B_r(x)} \fint_{B_r(x)} |u(z) - u(y)| \,\mathrm{d}z \,\mathrm{d}y < +\infty.$$

Here, $\fint$ denotes the average integral, that is, the integral divided by the mass of the domain of integration.

The space BMO(Ω) is then endowed with the norm $\|u\|_{\mathrm{BMO}(\Omega)} = \|u\|_{L^1(\Omega)} + |u|_{\mathrm{BMO}(\Omega)}$. Of course, this norm depends on the choice of the maximal size of balls r_0, but it can be shown that two different choices of r_0 lead two norms that are comparable up to equivalent constants, so that the space BMO(Ω) itself is independent on the choice of r_0.

We say that $u \in \mathrm{BMO}(\Omega)$ has *vanishing mean oscillation* whenever

$$\lim_{r_0\to 0} \sup_{\substack{0<r<r_0\\ B_r(x)\subset\Omega}} \fint_{B_r(x)} \fint_{B_r(x)} |u(z) - u(y)| \,\mathrm{d}z \,\mathrm{d}y = 0.$$

The subspace $\mathrm{VMO}(\Omega) \subset \mathrm{BMO}(\Omega)$ of functions with vanishing mean oscillation is a closed subset of $\mathrm{BMO}(\Omega)$. By Sarason's lemma [Sar75], it coincides with the closure of $C^0(\Omega)$ in $\mathrm{BMO}(\Omega)$.

Theorem 2.1.4 (Limiting embedding)**.** *Let $sp = m$. Then,*

$$W^{s,p}(\Omega) \hookrightarrow \mathrm{VMO}(\Omega).$$

Let us emphasize that this embedding is *not* compact. Even before starting to look for a counterexample, this can be guessed by the fact that both the $W^{s,p}$ and BMO seminorms are scale invariant in critical dimension. A counterexample can then be constructed by scaling copies of a fixed map; it will converge (pointwise, or in L^1) to 0, but have constant positive $W^{s,p}$ and BMO seminorms.

We need one last, perhaps less classical, type of inequalities involving Sobolev functions, namely, Gagliardo–Nirenberg interpolation inequalities[1]. As the name suggests, they involve estimating a norm of *intermediate* order by a norm of higher order and one of lower order.

Theorem 2.1.5. *Let $0 \leq s_1 < s < s_2 < +\infty$, $1 \leq p_1, p_2, p \leq +\infty$, and $\theta \in (0,1)$ be such that*

$$s = (1-\theta)s_1 + \theta s_2 \quad \textit{and} \quad \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}.$$

Then, there exists a constant $C > 0$ depending only on $s_1, s_2, s, p_1, p_2, p, m$, and Ω such that

$$\|u\|_{W^{s,p}(\Omega)} \leq \|u\|^{1-\theta}_{W^{s_1,p_1}(\Omega)} \|u\|^{\theta}_{W^{s_2,p_2}(\Omega)} \quad \textit{for every } u \in W^{s_1,p_1}(\Omega) \cap W^{s_2,p_2}(\Omega),$$

unless if

$$s_2 \geq 1 \textit{ is an integer, } p_2 = 1, \textit{ and } s_2 - s_1 \leq 1 - \frac{1}{p_1}. \tag{2.1.2}$$

Let us make it very precise that (2.1.2) does not correspond to open cases: Each time

[1] Those inequalities definitely do not have a "straightforward" story. The model case, involving only integer order spaces, was obtained independently by Emilio Gagliardo [Gag59] and Louis Nirenberg [Nir59]. It seems that they both went to the same congress and gave a lecture featuring notably those inequalities, hence discovering that they had both obtained essentially the same results at the same time. It was later on extended to the fractional setting, in several partial contributions. Let us notably mention a contribution by Frédérique Oru, which was however never published, and has therefore been reproduced by several authors in subsequent works. For some time, it remained somehow folklore to know for which cases the Gagliardo–Nirenberg inequality would hold or not, the exact dividing line between both being somehow vague (there was notably a case stated in Nirenberg's paper which was actually wrong, as noticed later on by several experts in the community). One had to wait for the work of Haïm Brezis and Petru Mironescu [BM18] (in 2018!) to have a clearly presented picture of which cases lead to a valid Gagliardo–Nirenberg inequality or not.

(2.1.2) holds, there is an example of a function that belongs to $W^{s_1,p_1}(\Omega) \cap W^{s_2,p_2}(\Omega)$ but not to $W^{s,p}(\Omega)$.

We conclude by recalling the *theory of traces*, which also gives a further motivation to consider fractional Sobolev spaces as well.

Theorem 2.1.6. *Let $s \notin \mathbb{N}_*$. There is a (unique) continuous linear operator $\operatorname{tr}\colon W^{s+\frac{1}{p},p}(\Omega) \to W^{s,p}(\partial\Omega)$ extending by density the restriction to the boundary operator $C^0(\overline{\Omega}) \to C^0(\partial\Omega)$. Moreover, this operator has a linear continuous right inverse: For every $u \in W^{s,p}(\partial\Omega)$, there exists $U \in W^{s+\frac{1}{p},p}(\Omega)$, that can be chosen linearly with respect to u, such that $\operatorname{tr} U = u$ and*

$$\|U\|_{W^{s+\frac{1}{p},p}(\Omega)} \leq C\|u\|_{W^{s,p}(\partial\Omega)},$$

for some constant $C > 0$ depending on s, p, and Ω.

The philosophy behind Theorem 2.1.6 is as follows. Functions modelled on Lebesgue spaces, such as Sobolev functions, are only defined almost everywhere. In particular, $\partial\Omega$ being a null set, it makes no sense to speak about the restriction of a Sobolev map on this set. Of course, one can always fix a representative, but then, the restriction is going to depend on this choice, and will therefore not provide a stable notion. The theory of traces says that the extra regularity of Sobolev functions compared to mere L^p functions *indeed allows* to define a robust notion of restriction to the boundary, with the price that one looses some regularity — namely, one looses exactly $\frac{1}{p}$ derivative — when taking the trace. The inverse part of the trace theorem asserts that, in some sense, this loss of regularity is optimal. For instance, the fractional Sobolev space $W^{1-\frac{1}{p},p}$ is the optimal trace space of the integer order Sobolev space $W^{1,p}$, as any function in $W^{1-\frac{1}{p},p}$ may arise as the trace of some function in $W^{1,p}$.

Let us just comment that the somehow unnatural assumption $s \notin \mathbb{N}_*$ can be interpreted as the fact that Sobolev spaces are not the right spaces to work with trace theory. If one works instead with Besov spaces — that shall be mentioned later on in Chapter 3 — then there is no such restriction on the regularity parameter. From this point of view, the assumption $s \notin \mathbb{N}_*$ arises from the fact that the correspondence between Sobolev and Besov spaces fails for integer order spaces.

2.2 Slicing: The notion of genericity

In this section, we explore the notion of *genericity*, which allows to speak about the restriction of a Sobolev function over "many" subsets of lower dimension. We do not attempt to formulate a general statement of genericity, but rather illustrate the principle on some model examples. Later on in the lecture, we will simply invoke *a genericity* *argument*, and leave to the reader the care of checking that it indeed applies, and how.

Example 2.2.1 (Convergence)**.** Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $L^p((0,1)^2)$, such that $u_n \to u$ in L^p. We claim that there exists a subsequence $(u_{n_j})_{j\in\mathbb{N}}$ of $(u_n)_{n\in\mathbb{N}}$ such that, for almost every $y \in (0,1)$, $u_n(\cdot, y) \to u(\cdot, y)$ in $L^p((0,1))$. Let us emphasise importantly that, here, we are working with functions and not with equivalence classes. In particular, changing the representative of an equivalence class will change the null set on which the above convergence may fail.

To prove the claim, we argue as follows. We choose a subsequence in such a way that

$$\int_{(0,1)^2} |u_{n_j} - u|^p \leq 2^{-j} \quad \text{for every } j \in \mathbb{N}.$$

The monotone convergence theorem ensures that

$$\int_{(0,1)^2} \sum_{j\in\mathbb{N}} |u_{n_j} - u|^p < +\infty.$$

By Tonelli's theorem, we deduce that

$$\int_{(0,1)} \sum_{j\in\mathbb{N}} |u_{n_j}(\cdot, y) - u(\cdot, y)|^p < +\infty \quad \text{for almost every } y \in (0,1).$$

But, for any y as above, applying again the monotone convergence theorem to pull the sum out of the integral and then the fact that a summable sequence converges to 0, we have $u_n(\cdot, y) \to u(\cdot, y)$ in $L^p((0,1))$, and the conclusion follows. □

It is possible to have a more abstract point of view on this example. Indeed, Tonelli's theorem implies that we can identify $L^p((0,1)^2)$ with $L^p((0,1); L^p((0,1)))$. In this case, the above example is nothing else but the partial converse of the dominated convergence theorem, applied to a Lebesgue space of Banach space-valued functions.

The next example shows that, generically, Sobolev functions still have Sobolev regularity on slices.

Example 2.2.2 (Regularity)**.** We claim that, if $u \in W^{1,p}((0,1)^2)$, then $u(\cdot, y) \in W^{1,p}((0,1))$ for almost every $y \in (0,1)$, with $u(\cdot, y)' = \partial_x u(\cdot, y)$.

Let us start with mentioning that the delicate point is to justify the weak differentiability with the claimed formula. Indeed, once this is established, the integrability readily follows via Tonelli's theorem. We somehow avoid the difficulty by invoking the previous example combined with an approximation argument.

More specifically, we consider $(u_n)_{n\in\mathbb{N}}$ to be a sequence in $C^\infty(\overline{(0,1)^2})$ strongly converging to u in $W^{1,p}((0,1)^2)$. Clearly, we have $u_n(\cdot, y)' = \partial_x u_n(\cdot, y)$. Invoking the previous example, up to extracting a subsequence, we may assume that $u_n(\cdot, y) \to u(\cdot, y)$ and $\partial_x u_n(\cdot, y) \to \partial_x u(\cdot, y)$ in $L^p((0,1))$ for almost every $y \in (0,1)$. But then, the closure lemma implies that, for such y, $u(\cdot, y) \in W^{1,p}((0,1))$ with $u(\cdot, y)' = \partial_x u(\cdot, y)$. □

As our last example, we come back to the notion of traces. As we explained after Theorem 2.1.6, the notion of restriction of a Sobolev function is an ill-defined notion, which typically depends on the choice of representative, whence the motivation for introducing the concept of trace. However, resorting to the notion of genericity, it can be shown that restrictions and traces coincide "almost always".

Example 2.2.3. Let $u \in W^{1,p}((0,1)^2)$. We claim that $\operatorname{tr}_{(0,1)\times\{y\}} u = u(\cdot, y)$ for almost every $y \in (0,1)$. (Of course, this almost everywhere depends on the choice of the representative.)

Indeed, let $\{u_n : n \in \mathbb{N}\}$ be a sequence in $C^\infty(\overline{(0,1)^2})$ such that $u_n \to u$ in $W^{1,p}((0,1)^2)$. By Example 2.2.1, up to extracting a subsequence, we have $u_n(\cdot, y) \to u(\cdot, y)$ in $L^p((0,1))$ for almost every $y \in (0,1)$. But, by the continuity of the trace operator — here, we are not using the optimal space for the regularity of the trace — we have $u_n(\cdot, y) \to \operatorname{tr}_{(0,1)\times\{y\}} u$ in $L^p((0,1))$ for *every* $y \in (0,1)$. The uniqueness of the limit yields the conclusion. □

We finish this section by presenting an alternative, more systematic, point of view on genericity, which was recently introduced by P. Bousquet, A. Ponce, and J. Van Schaftingen [BPVS25]. Once again, we only illustrate the principle on an example, revisiting Example 2.2.1.

Proposition 2.2.4. *Let $u \in L^p(\Omega)$, and let $(u_n)_{n\in\mathbb{N}}$ be a sequence that converges in $L^p(\Omega)$ towards u. There exists a subsequence $(u_{n_j})_{j\in\mathbb{N}}$ and a summable map $w\colon \Omega \to [0, +\infty]$ such that, for every measure space (X, μ) and every measurable map $\gamma\colon X \to \Omega$ such that $w \circ \gamma$ is summable, it holds that $u_{n_j} \circ \gamma$ and $u \circ \gamma$ belong to $L^p(X, \mu)$ and*

$$u_{n_j} \circ \gamma \to u \circ \gamma \quad \text{in } L^p(X, \mu).$$

In the vocabulary introduce by Bousquet, Ponce, and Van Schaftingen, the map w is called a *detector*, and maps γ such that $w \circ \gamma$ is summable are called *Fuglede maps*, which is denoted by $\gamma \in \operatorname{Fug}_w(X; \Omega)$. Proposition 2.2.4 explain how to construct detectors for L^p convergence, but this can be adapted to many other contexts; see [BPVS25, Chapter 2]. For instance, one can also construct detectors for Sobolev regularity on slices.

Before giving its proof, let us pause briefly to comment on Proposition 2.2.4. It is in the same spirit as Example 2.2.1, asserting that, up to extraction, L^p convergence still occurs on generic slices. However, the flexibility given by the choice of the space X and the map γ allows to treat at once all possible types of slices — with a common subsequence working for all of them. To recover the conclusion of Example 2.2.1, it suffices to take

$X = (0,1)$ and $\gamma_y(x) = (x, y)$. Then, $v \circ \gamma_y = v(\cdot, y)$, and by Tonelli's theorem, $w \circ \gamma_y$ is summable for almost every $y \in (0,1)$.

Let us also importantly emphasise that the choice of the map w *depends* on the choice of the representative of u and the elements of the sequence $(u_n)_{n\in\mathbb{N}}$.

Proof of Proposition 2.2.4. Let $(\kappa_j)_{j\in\mathbb{N}}$ be a sequence of numbers such that $\kappa_j \geq 1$ and $\kappa_j \to +\infty$. We choose our subsequence in such a way that

$$\sum_{j\in\mathbb{N}} \kappa_j \|u_{n_j} - u\|_{L^p(\Omega)} < +\infty.$$

Define

$$w = \Big(|u| + \sum_{j\in\mathbb{N}} \kappa_j |u_{n_j} - u|\Big)^p.$$

By Fatou's lemma and Minkowski's inequality, we have

$$\left(\int_\Omega w\right)^{\frac{1}{p}} \leq \|u\|_{L^p(\Omega)} + \sum_{j\in\mathbb{N}} \kappa_j \|u_{n_j} - u\|_{L^p(\Omega)} < +\infty.$$

Clearly, $|u|^p \leq w$, and similarly, as $\kappa_j \geq 1$, we have

$$|u_{n_j}| \leq (|u| + \kappa_j |u_{n_j} - u|)^p \leq w,$$

so that, if $w \circ \gamma$ is summable, then $u \circ \gamma$ and $u_{n_j} \circ \gamma$ belong to $L^p(X, \mu)$. Moreover, we have

$$\int_X |u_{n_j} \circ \gamma - u \circ \gamma|^p \, \mathrm{d}\mu \leq \kappa_j^{-p} \int_X w \circ \gamma \, \mathrm{d}\mu,$$

and as $\kappa_j \to +\infty$, we conclude that $u_{n_j} \circ \gamma \to u \circ \gamma$ in $L^p(X, \mu)$. □

We end this section with the conclusion that, after this brief exposition, we have at hand essentially two approaches to genericity. The first one is more elementary — in the sense that it does not require to define objects such as detectors and Fuglede maps — and "by hand", as illustrated in the first three examples. The second one is more systematic, relying on the concept of Fuglede maps and detectors. One is often free to choose between both, depending on the context and the needs of the situation.

2.3 A survival kit in homotopy theory

In this section, we recall or introduce the notions of homotopy theory that will be needed in the rest of the course. For readers having a taste for algebraic topology, let us mention that quite an important amount of it can be used in geometric analysis. Already in the very specific field of Sobolev mappings into manifolds, notions from algebraic topology that have found important uses include homotopy groups, homology and cohomology, and even rational homotopy. However, in this lecture, we will essentially only need the notion of homotopy.

We recall that two continuous maps f and $g\colon X \to Y$ between topological spaces are *homotopic* whenever there exists a continuous map $H\colon X \times [0,1] \to Y$ such that $H(\cdot,0) = f$ and $H(\cdot,1) = g$. Intuitively, this means that f can be deformed continuously into g. Being homotopic is an equivalence relation, and we write $f \simeq g$ when f and g are homotopic.

Fundamental to us is the fact that, when the target is a closed smooth manifold, uniformly close maps are homotopic.

Proposition 2.3.1. *Let $f, g\colon X \to \mathscr{N}$ be two continuous maps, where $\mathscr{N}$ is a compact manifold. There exists $\delta > 0$ depending only on $\mathscr{N}$ such that, if $\sup_X |f - g| \leq \delta$, then $f \simeq g$.*

The crucial ingredient in the proof of Proposition 2.3.1 is the existence of a *uniform tubular neighbourhood*, on which there is a well-defined *smooth retraction onto $\mathscr{N}$*.

Proposition 2.3.2. *There exists $\iota > 0$ depending only on $\mathscr{N}$ such that, if we let $\mathscr{N}_\iota = \mathscr{N} + B^\nu_\iota = \{x \in \mathbb{R}^\nu \colon \operatorname{dist}(x, \mathscr{N}) < \iota\}$, then the nearest point projection $\Pi\colon \mathscr{N}_\iota \to \mathscr{N}$ is well-defined and smooth.*

Such a result is well-known, but is seems difficult to find a proof of it in the literature. We refer the reader to R. Foote [Foo84]. We also mention that, for essentially all we do in the sequel, the only property of $\mathscr{N}$ as a Riemannian manifold that we use is the existence of such a smooth retraction $\Pi\colon \mathscr{N}_\iota \to \mathscr{N}$. Hence, the assumption that $\mathscr{N}$ is a smooth embedded closed Riemannian manifold could be relaxed to the one that $\mathscr{N}$ is a closed subset of $\mathbb{R}^\nu$ having a uniform neighbourhood on which there is a well-defined smooth retraction onto $\mathscr{N}$. Also, even smoothness could be relaxed to C^k, with k depending on the order of regularity of the Sobolev spaces under study.

Proof of Proposition 2.3.1. Fix $\delta = \iota/2$, with ι being the radius of a tubular neighbourhood of $\mathscr{N}$. We define

$$H(x,t) = \Pi((1-t)f(x) + tg(x)).$$

Since $|f(x)-(1-t)f(x)-tg(x)| \leq t|f(x)-g(x)| \leq \delta$ and $f(x) \in \mathcal{N}$, we have $(1-t)f(x)+tg(x) \in \mathcal{N}_\iota$, so that H is well-defined. It is then clear that H is a continuous homotopy connecting f to g, and the conclusion follows. □

It is an interesting exercise to find counterexamples to the conclusion of Proposition 2.3.1 when one assumption, for instance compactness, is dropped.

Among all possible domains for homotopy classes, one which will be of key importance to us is the case of spheres. Namely, we define the *ℓ-th homotopy group of $\mathcal{N}$*, that we denote $\pi_\ell(\mathcal{N})$, as the set of all homotopy classes of continuous maps $\mathbb{S}^\ell \to \mathcal{N}$. That is,

$$\pi_\ell(\mathcal{N}) = C^0(\mathbb{S}^\ell;\mathcal{N})/\simeq\,,$$

where $\simeq$ denotes the homotopy equivalence relation. Let us already mention that, by a straightforward regularisation and reprojection argument relying on Proposition 2.3.1, it can be shown that any homotopy class contains a *smooth* representative.

As suggested by the name, $\pi_\ell(\mathcal{N})$, which is merely a set from the definition we gave, can be endowed with a group structure. The resulting group is even abelian when $\ell \geq 2$. However, we shall not need this group structure in this lecture, and shall therefore only consider $\pi_\ell(\mathcal{N})$ as being as set. Actually, for our purposes, it would even suffice to define $\pi_\ell(\mathcal{N}) = \{0\}$, which happens whenever any continuous map $f\colon \mathbb{S}^\ell \to \mathcal{N}$ is homotopic to a constant map. Equivalently, this holds whenever any continuous map $f\colon \mathbb{S}^\ell \to \mathcal{N}$ can be extended as a continuous map $g\colon \overline{\mathbb{B}^{\ell+1}} \to \mathcal{N}$. The relation between this extension and the corresponding homotopy to a constant is given by

$$H(x,t) = g(tx).$$

So far, we have presented a homotopy theory for continuous maps. However, when studying low-regularity maps, it is a natural question to ask whether or not such theory can be adapted to less regular mappings. We conclude this section by a short discussion about how homotopy theory can be adapted to the slightly larger space VMO, which will be of crucial importance to us when dealing with critical Sobolev spaces that do not embed into C^0 but do in VMO. This follows the work by H. Brezis and L. Nirenberg [BN95].

Given a map $f \in L^1(\mathbb{S}^\ell;\mathcal{N})$, we define

$$f_r(x) = \fint_{B_r(x)} f,$$

where here $B_r(x) \subset \mathbb{S}^\ell$ is a geodesic ball. (Let us mention that we restrict here to $\mathbb{S}^\ell$

for simplicity, but the whole discussion carries on to a more general setting mutatis mutandis.) We observe that f_r is continuous, but need not be $\mathscr{N}$-valued, because of the use of a convolution procedure. We remedy this issue via the following proposition.

Proposition 2.3.3. *It holds*

$$\operatorname{dist}(f_r(x),\mathscr{N}) \leq M_r f = \sup_{\substack{0<s\leq r\\ x\in\mathbb{S}^\ell}} \fint_{B_s(x)}\fint_{B_s(x)} |f(z)-f(y)|\,\mathrm{d}z\,\mathrm{d}y.$$

Proof. Since $f(y)\in\mathscr{N}$ for every $y\in\mathbb{S}^\ell$, we have

$$\operatorname{dist}(f_r(x),\mathscr{N}) \leq |f_r(x)-f(y)| \leq \fint_{B_r(x)} |f(z)-f(y)|\,\mathrm{d}z.$$

Taking the average over all $y\in B_r(x)$ yields the conclusion. □

In particular, this means that if $M_r f$ is sufficiently small, then $f^r = \Pi\circ f_r$ is well-defined, $\mathscr{N}$-valued, and continuous, where Π denotes the nearest point projection onto $\mathscr{N}$. But, if $f\in\mathrm{VMO}(\mathbb{S}^\ell;\mathscr{N})$, then $M_r f\to 0$ as $r\to 0$, implying that the maps f^r are well-defined for r sufficiently small. Moreover, those maps are clearly homotopic to each other, as they form a continuous family of continuous maps with respect to the parameter r. This implies that, for $f\in\mathrm{VMO}(\mathbb{S}^\ell;\mathscr{N})$, we can define its homotopy class as

$$[f]=[f^r] \quad \text{for any } r>0 \text{ sufficiently small,}$$

and this does not depend on the choice of r.

The following proposition shows that this is a good notion, in the sense that it is robust with respect to the VMO convergence.

Proposition 2.3.4. *Let $f\in\mathrm{VMO}(\mathbb{S}^\ell;\mathscr{N})$ and let $r_0>0$ be sufficiently small. There exists a constant C_{r_0} depending only on ℓ and r_0 such that, if $g\in\mathrm{VMO}(\mathbb{S}^\ell;\mathscr{N})$ satisfies*

$$|f-g|_{\mathrm{VMO}(\mathbb{S}^\ell)} \leq \frac{\iota}{2} \quad \textit{and} \|f-g\|_{L^1(\mathbb{S}^\ell)} \leq C_{r_0}\iota,$$

then $f^r\simeq g^r$ for every $0<r\leq r_0$. In particular, homotopy classes of VMO *maps are stable under* BMO *convergence.*

Proof. By the triangle inequality, we have

$$M_{r_0}g \leq M_{r_0}f + M_{r_0}(g-f) \leq M_{r_0}f + |g-f|_{\mathrm{BMO}(\mathbb{S}^\ell)}.$$

Taking r_0 sufficiently large, we therefore have

$$M_{r_0} f < \iota \quad \text{and} \quad M_{r_0} g < \iota,$$

so that f^r and g^r are well-defined for $0 < r \leq r_0$ by Proposition 2.3.3.

Meanwhile, we estimate

$$|f_{r_0}(x) - g_{r_0}(x)| \leq \fint_{B_{r_0}(x)} |f(y) - g(y)| \, \mathrm{d}y \lesssim \|f - g\|_{L^1(\mathbb{S}^\ell)},$$

with a hidden constant depending on r_0 because of the volume normalisation in the average integral. Taking the constant C_{r_0} sufficiently small, we therefore deduce from the same proof as in Proposition 2.3.1 that $f^{r_0} \simeq g^{r_0}$, and therefore, by transitivity of the homotopy relation, $f^r \simeq g^r$ for $0 < r \leq r_0$. □

2.4 The topological obstruction to the strong approximation property

With all the reminders from the previous sections at hand, we are now in position to prove the seminal topological obstruction to the strong approximation property exhibited by Schoen and Uhlenbeck [SU83]. More precisely, we prove the general version for an arbitrary target, which is due to Bethuel and Zheng when $s = 1$ [BZ88], and that reads as follows.

Theorem 2.4.1. *Let $f \colon \mathbb{S}^\ell \to \mathscr{N}$ be a smooth map. Define $u_0(x) = f\left(\frac{x}{|x|}\right)$. If f is not nullhomotopic, then*

$$u_0 \in W^{s,p}(\mathbb{B}^{\ell+1}; \mathscr{N}) \setminus H^{s,p}_{\mathrm{S}}(\mathbb{B}^{\ell+1}; \mathscr{N})$$

whenever $\lfloor sp \rfloor = \ell$. In particular, if $\pi_{\lfloor sp \rfloor}(\mathscr{N}) \neq \{0\}$, then

$$H^{s,p}_{\mathrm{S}}(\mathbb{B}^{\ell+1}; \mathscr{N}) \subsetneq W^{s,p}(\mathbb{B}^{\ell+1}; \mathscr{N}).$$

Proof. We first show that $u_0 \in W^{s,p}(\mathbb{B}^{\ell+1})$ when $sp < \ell + 1$. If $s = k \in \mathbb{N}_*$, this is a straightforward computation in polar coordinates, using the estimate

$$|\mathrm{D}^j u_0(x)| \lesssim |x|^{-j}.$$

For the fractional case, we rely on the Gagliardo–Nirenberg inequality — although a direct proof is also possible. Namely, write $s = k + \sigma$ with $0 < \sigma < 1$. By the integer order case just above, we have $u_0 \in W^{k+1,\frac{sp}{k+1}}(\mathbb{B}^{\ell+1})$. But now, we write

$$s = (k+1)\frac{s}{k+1} = (k+1)\frac{s}{k+1} + 0 \cdot \frac{k+1-s}{k+1}$$

and

$$\frac{1}{p} = \frac{k+1}{sp}\frac{s}{k+1} = \frac{k+1}{sp}\frac{s}{k+1} + \frac{1}{\infty}\frac{k+1-s}{k+1}.$$

Since clearly $u_0 \in L^\infty$, applying the Gagliardo–Nirenberg interpolation inequality with $\theta = \frac{s}{k+1}$ yields $u_0 \in W^{s,p}(\mathbb{B}^{\ell+1})$.

We now turn to the proof that, if in addition $sp \geq \ell$, then $u_0 \notin H^{s,p}_{\mathrm{S}}(\mathbb{B}^{\ell+1};\mathscr{N})$. The argument is by contradiction, assuming that there would be a sequence $(u_n)_{n\in\mathbb{N}_*}$ in $C^\infty(\overline{\mathbb{B}^{\ell+1}};\mathscr{N})$ converging towards u_0 in $W^{s,p}(\mathbb{B}^{\ell+1})$. By a genericity argument, we deduce that, up to extracting a subsequence,

$$u_{n|\partial B_r} \to u_{0|\partial B_r} \quad \text{in } W^{s,p}(\partial B_r), \text{ for almost every } 0 < r < 1.$$

But now, having $sp \geq \ell$, on ∂B_r, we have either the Morrey embedding $W^{s,p}(\partial B_r) \hookrightarrow C^0(\partial B_r)$, or the limiting embedding $W^{s,p}(\partial B_r) \hookrightarrow \mathrm{VMO}(\partial B_r)$. Relying on either Proposition 2.3.1 or Proposition 2.3.4, we deduce that $u_{n|\partial B_r} \simeq u_{0|\partial B_r}$ for n sufficiently large. However, by construction, $u_{n|\partial B_r}$ is homotopic to a constant — it is the restriction to ∂B_r of the continuous map u_n defined on the whole $\overline{B_r}$ — while $u_{0|\partial B_r}$ is not homotopic to a constant — it is just a scaled version of f, which was assumed not to be nullhomotopic. This concludes the proof. □

Here, we stated Theorem 2.4.1 when the domain is $\mathbb{B}^{\ell+1}$. However, it is not too difficult to extend this obstruction, first to higher dimensions by taking a cylinder over the map u_0 that we constructed, adding extra dummy variables, and then to more general domains, inserting a copy of u_0 in a small ball contained in the domain — this latter step requires some technical work, but is a good exercise to do in full details.

2.5 Comments and perspectives

2.5.1 Detecting topological singularities: The Jacobian, and other friends

Section 2.4 highlighted the important role played by *topological singularities*. A natural question that arises then is whether or not there is a way to detect them, to define them in a robust way for general Sobolev mappings.

Let us consider the model case of $W^{1,\ell}(\mathbb{B}^{\ell+1};\mathbb{S}^\ell)$. In this case, a natural candidate for doing so is the *Jacobian*. More precisely, given $u \in W^{1,\ell}(\mathbb{B}^{\ell+1};\mathbb{S}^\ell)$, we define its *distributional Jacobian* Ju as

$$\langle Ju, \alpha \rangle = -\int_{\mathbb{B}^{\ell+1}} \mathrm{d}\alpha \wedge u^\sharp \omega_{\mathbb{S}^\ell} \quad \text{for every } \alpha \in C^\infty_c(\mathbb{B}^{\ell+1}),$$

where $\omega_{\mathbb{S}^\ell}$ is the volume form on $\mathbb{S}^\ell$ and $u^\sharp\omega_{\mathbb{S}^\ell}$ denotes the pullback of $\omega_{\mathbb{S}^\ell}$ by u. This is well-defined, as $u^\sharp\omega_{\mathbb{S}^\ell}$ involves a product of ℓ derivatives of u, and hence belongs to L^1. In other words, we are defining $Ju = \mathrm{d}(u^\sharp\omega_{\mathbb{S}^\ell})$ in the sense of distributions. If u is smooth, then we can commute the pullback and the exterior differential and find $Ju = u^\sharp(\mathrm{d}\omega_{\mathbb{S}^\ell}) = 0$, as $\omega_{\mathbb{S}^\ell}$ is closed. We insist that *this computation is only justified when u is smooth*. In the following example, we compute the Jacobian of the hedgehog map $u_0 = \frac{x}{|x|}$.

Example 2.5.1. Let $\alpha \in C^\infty_c(\mathbb{B}^{\ell+1})$. By the dominated convergence theorem,

$$\langle Ju_0, \alpha \rangle = \lim_{\varepsilon \to 0} -\int_{\mathbb{B}^{\ell+1} \setminus B_\varepsilon} \mathrm{d}\alpha \wedge u_0^\sharp \omega_{\mathbb{S}^\ell}.$$

We are now in position to apply Stokes' formula, and find

$$\langle Ju_0, \alpha \rangle = \lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon} \alpha \wedge u_0^\sharp \omega_{\mathbb{S}^\ell}.$$

By continuity of α, we get

$$\langle Ju_0, \alpha \rangle = \lim_{\varepsilon \to 0} \alpha(0) \int_{\partial B_\varepsilon} u_0^\sharp \omega_{\mathbb{S}^\ell}.$$

Now, u_0 being homogeneous, a change of variable gives

$$\int_{\partial B_\varepsilon} u_0^\sharp \omega_{\mathbb{S}^\ell} = \int_{\mathbb{S}^\ell} \omega_{\mathbb{S}^\ell} = |\mathbb{S}^\ell|.$$

(This can also be viewed as a consequence of Brouwer's integral formula for the degree, as u_0 is a degree 1 map on B_ε.) Hence, we conclude that $\langle Ju_0, \alpha \rangle = |\mathbb{S}^k|\alpha(0)$, that is, $Ju_0 = |\mathbb{S}^k|\delta_0$. □

As Example 2.5.1 shows, the Jacobian detects the topological singularity of the hedgehog at the origin. More generally, the same computation would show that the Jacobian of a map in the class $\mathscr{R}_0(\mathbb{B}^{\ell+1}; \mathbb{S}^\ell)$ is a weighted sum of Dirac masses, one for each topological singularity, and the weight is the topological degree around the singularity. As the Jacobian is clearly continuous with respect to the $W^{1,\ell}$ topology, it can be extended by density to the whole space $W^{1,\ell}(\mathbb{B}^{\ell+1}; \mathbb{S}^\ell)$, and used as a definition of the singular set of a map in this case.

The fact that it really encodes the notion of topological singularities of a map is further confirmed by the following theorem, due to F. Bethuel [Bet90].

Theorem 2.5.2. *A map $u \in W^{1,\ell}(\mathbb{B}^{\ell+1};\mathbb{S}^\ell)$ belongs to $H^{1,\ell}_{\mathrm{S}}(\mathbb{B}^{\ell+1};\mathbb{S}^\ell)$ if and only if $Ju = 0$.*

On a historical note, let us mention that the idea of considering the Jacobian in the sense of distributions comes from the work of J. Ball [Bal76]. For a survey of the rich history of the Jacobian, we refer to H. Brezis, J. Mahwin, and P. Mironescu [BMM24].

This was only the starting point of a long series of works directed towards characterising the singular set of a general Sobolev mappings[2]. Let us just mention some related work, with a brief explanation and references.

- In 1991, F. Bethuel, J.-M. Coron, F. Demengel, and F. Hélein [BCDH91] extended the construction of the Jacobian to $W^{1,\ell}(\mathbb{B}^m;\mathscr{N})$ where $\mathscr{N}$ has the property that "its ℓ-cohomology detects its ℓ-homotopy", using pullbacks of arbitrary forms. This was generalised to higher-order integer Sobolev spaces by P. Bousquet, A. Ponce, and J. Van Schaftingen [BPVS25]. Other similar characterisations in a more geometric measure theoretic language, relying on *Cartesian currents*, were established by M. Giaquinta, G. Modica, J. Souček, and collaborators, see notably the monograph [GMS98a, GMS98b].

- In 2003 and 2008, R. Hardt and T. Rivière [HR03, HR08] developed the notion of *scans* to treat the more general case of singularities coming from *rational homotopy*.

- In 2003, M. R. Pakzad and T. Rivière [PR03] defined the singular set as a *flat chain with values into the $\pi_\ell(\mathscr{N})$*, using the language of geometric measure theory. This allows to cover some cases where $\pi_\ell(\mathscr{N})$ has torsion, which cannot be seen by Jacobian-like objects. This was extended by G. Canevari and G. Orlandi [CO19, CO26].

- In parallel, an important amount of research was devoted to extend Jacobian-like objects to fractional Sobolev spaces with $0 < s < 1$. Here, the first challenge is that $u^\sharp\omega_{\mathbb{S}^\ell}$ need not even make sense as an L^1 function, as taking the pullback requires one full derivative of the map. Hence, even the definition of the Jacobian requires some work. The nice idea to overcome this difficulty was proposed by J. Bourgain, H. Brezis, and P. Mironescu [BBM05], relying on an extension of traces. The properties of this Jacobian were further explored by P. Bousquet [Bou07], also with P. Mironescu [BM14]. For sphere-valued maps, a characterisation of the

[2] Which is not over yet, if you are looking for nice problems to work on!

singular set was stated by D. Mucci [Muc24], and a counterpart of Bethuel, Coron, Demengel, and Hélein's result was proved in [DMX25].

3 The subcritical case: Using the linear theory in the non-linear setting

In this chapter, we explain what happens in the range $sp \geq m$. As we already mentioned in the introduction, in this case, the situation of the strong approximation problem essentially boils down to the standard linear theory, and the strong approximation property holds. Before we move to more precise matters, let us make a general qualitative remark. This dichotomy between $sp \geq m$ and $sp < m$ is not specific to this specific problem, but is a more general feature of most geometric problems involving Sobolev regularity. The general principle, loosely stated and of course not free of exceptions, is the following. When $sp > m$, the availability of the Morrey embedding implies that things essentially go like in the corresponding linear theory, and there is usually no major additional difficulty other than technical caused by the geometry of the problem. When $sp < m$, the nonlinear problem behaves qualitatively differently from the corresponding linear one. Because Sobolev mappings can have wild singularities and are very far away from being continuous, new phenomena are to be expected, and proofs require new nontrivial ideas, that do not come from the linear theory. The case $sp = m$ is often critical. It is not uncommon that most of, if not all, the linear theory can be retrieved, but the failure of the Morrey embedding makes matters much harder, and even when the results coincide with the corresponding ones from the linear theory, they often require new nontrivial ideas, somehow exploiting that Sobolev maps in this regime are "almost" continuous.

3.1 The strong approximation property for $sp \geq m$

The objective of this (rather short) section is to prove the following theorem, which completely solves the strong approximation problem for $sp \geq m$.

Theorem 3.1.1. *Assume that $sp \geq m$. Then,*

$$H^{s,p}_{\mathrm{S}}(\Omega; \mathscr{N}) = W^{s,p}(\Omega; \mathscr{N}).$$

In order not to betray our reader, let us say a word of warning. At some point of the proof, we are going to hide an important detail under the rug, only to reveal this after the proof. This detail will be a good opportunity for a digression about another useful functional analysis technique, that will be the main concern of the rest of the chapter. The attentive reader can try to spot what is the aforementioned detail before we reveal it.

Proof. For the sake of simplicity, let us assume that $\Omega = \mathbb{B}^m$. In this case, replacing u with u_λ defined as $u_\lambda(x) = u(\lambda x)$ with $\lambda < 1$, letting $\lambda \to 1$, and using a diagonal argument, we may assume the existence of an open set $\omega \Subset \mathbb{R}^m$ such that $\Omega \Subset \omega$ — here, we simply take $\omega = B_{\lambda^{-1}}$ — on which u is well-defined and $W^{s,p}$. This twist is just to ensure we can perform a regularisation by convolution procedure, and can be done, with the price of more technicality, on any sufficiently smooth domain — actually, the *segment condition* suffices.

Let us define $u_\varepsilon = \varphi_\varepsilon * u$, where φ is a standard mollifier and $\varphi_\varepsilon(x) = \varepsilon^{-m}\varphi\left(\frac{x}{\varepsilon}\right)$. By our discussion above, u_ε is well-defined for ε sufficiently small, and moreover, $u_\varepsilon \in C^\infty(\overline{\Omega})$ and $u_\varepsilon \to u$ in $W^{s,p}(\Omega)$ as $\varepsilon \to 0$. The only issue is that, because of the convolution procedure, u_ε need not take its values into $\mathscr{N}$. Therefore, we wish to estimate the distance between u_ε and $\mathscr{N}$.

As $u(y) \in \mathscr{N}$ for every $y \in \Omega$, we have

$$\operatorname{dist}(u_\varepsilon(x), \mathscr{N}) \leq |u_\varepsilon(x) - u(y)| \lesssim \fint_{B_\varepsilon(x)} |u(z) - u(y)| \,\mathrm{d}z.$$

Taking the average with respect to $y \in B_\varepsilon(x)$, we reach

$$\operatorname{dist}(u_\varepsilon(x), \mathscr{N}) \lesssim \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |u(z) - u(y)| \,\mathrm{d}z \,\mathrm{d}y.$$

But now, since $sp \geq m$, we in particular have $u \in \mathrm{VMO}(\Omega)$, so that

$$\lim_{\varepsilon \to 0} \sup_{x \in \Omega} \operatorname{dist}(u_\varepsilon(x), \mathscr{N}) = 0.$$

(Strictly speaking, we have only stated the embedding into VMO for $sp = m$. However, it clearly also holds when $sp > m$, either by using the injection into a Sobolev space of lower order for which the exponents have the critical value and the the embedding into VMO, or by using the injection into C^0, itself contained in VMO.) This implies that, for $\varepsilon > 0$ sufficiently small, we have $u_\varepsilon \in \mathscr{N}_\iota$, where $\iota > 0$ is the radius of a tubular neighbourhood. Composing with the nearest point projection Π, we finally deduce that $\Pi \circ u_\varepsilon \to \Pi \circ u = u$ in $W^{s,p}(\Omega)$, and since $\Pi \circ u_\varepsilon \in C^\infty(\overline{\Omega})$, the conclusion follows. □

The last step of our above proof was a bit sloppy. Indeed, our claim is that, since $u_\varepsilon \to u$ in $W^{s,p}(\Omega)$, then $\Pi \circ u_\varepsilon \to \Pi \circ u = u$ in $W^{s,p}(\Omega)$. This is indeed true, as stated in Theorem 3.1.2 below. However, this theorem is far from being trivial, at least when $s > 1$ is not an integer, whence our last claim being quite sloppy as written.

Theorem 3.1.2. *Let $F\colon \mathbb{R}^\nu \to \mathbb{R}^{\nu'}$ be a $C^{\lceil k \rceil}$ map such that $\mathrm{D}^j F \in L^\infty$ for every $j \in \{0, \dots, \lceil k \rceil\}$.*

1. *If $0 < s \leq 1$, then the operator $W^{s,p}(\Omega; \mathbb{R}^\nu) \to W^{s,p}(\Omega; \mathbb{R}^{\nu'})$ defined by $u \mapsto F \circ u$ is well-defined and continuous.*

2. *If $s > 1$, then the operator $W^{s,p}(\Omega; \mathbb{R}^\nu) \cap W^{1,sp}(\Omega; \mathbb{R}^\nu) \to W^{s,p}(\Omega; \mathbb{R}^{\nu'})$ defined by $u \mapsto F \circ u$ is well-defined and continuous.*

We will call the above defined operator the *composition operator*, in a rather self-explanatory fashion. It is also sometimes to be encountered under the name *Nemytskij operator*[1].

Let us emphasise importantly that, when $s > 1$, mere $W^{s,p}$ regularity is not enough to allow to take the composition, one needs to impose some extra lower-order regularity. When saying that the composition operator is continuous $W^{s,p}(\Omega; \mathbb{R}^\nu) \cap W^{1,sp}(\Omega; \mathbb{R}^\nu) \to W^{s,p}(\Omega; \mathbb{R}^{\nu'})$, we mean that, for every sequence $(u_n)_{n \in \mathbb{N}}$ that converges in $W^{s,p}$ and remains uniformly bounded in $W^{1,sp}$, $(F \circ u_n)_{n \in \mathbb{N}}$ converges in $W^{s,p}$. By virtue of the Gagliardo–Nirenberg inequality, one reaches the same conclusion assuming rather a uniform bound in L^∞. Indeed, interpolating then provides the required $W^{1,sp}$ bound to apply the version of the theorem above.

The rest of this chapter is devoted to explaining the proof of Theorem 3.1.2 when $s > 1$ is not an integer. We follow the original approach by H. Brezis and P. Mironescu [BM01a], which has the interesting feature of resorting to an excursion through refined scales of spaces, even though the final result is about Sobolev spaces. Let us mention that, since then, a proof using only Sobolev spaces techniques has been proposed by V. Maz'ya and T. Shaposhnikova [MS02]; see also [BPVS13]. We nevertheless follow the approach via refined scales of spaces, precisely because our point here is not that much the final theorem, but more to illustrate how introducing new scales of spaces can yield to microscopic improvement that translate into a gain in the original scale of spaces one was interested in.

[1] This is notably the terminology used by Runst and Sickel [RS96].

As often the case for Russian names, there are many possible transliterations. If you read Cyrillic, the original spelling is Виктор Владимирович Немыцкий, you can then choose your favourite transliteration. Another interesting, language-related, feature about Nemytskij is that, if one opens its list of publications on Zentralblatt for instance, it appears that he started his career by working simultaneously on topology and on dynamical systems related problems, and was quite consistently publishing about the first topic in German, and about the second one in French.

On an other note, at least Runst and Sickel use this terminology without quoting any paper from Nemytskij. We are not aware whether or not this is one more instance of Stigler's law.

3.2 A new scale of refined function spaces: Triebel–Lizorkin spaces

In this section, we introduce the Triebel–Lizorkin spaces $F^s_{p,q}$, for $-\infty < s < +\infty$, $0 < p \leq +\infty$, and $0 < q \leq +\infty$, and we state some of their basic properties. This relies on the Littlewood–Paley decomposition, and can therefore be seen as a motivation and continuation of the last chapter of Functional Analysis II. A standard reference for Triebel–Lizorkin spaces is [Tri83].

We start by recalling the Littlewood–Paley decomposition. Let $\psi_0 \in C^\infty_c(\mathbb{R}^m)$ be a cut-off function such that $0 \leq \psi_0 \leq 1$, $\psi_0(\xi) = 1$ for $|\xi| \leq 1$, and $\psi_0(\xi) = 0$ for $|\xi| \geq 2$. For $j \geq 1$, we define $\psi_j(\xi) = \psi_0(2^{-j}\xi) - \psi_0(2^{-j+1}\xi)$. Finally, we let $\varphi_j = \mathcal{F}^{-1}(\psi_j)$, where $\mathcal{F}^{-1}_j$ denotes the inverse Fourier transform. This way, we have

$$\varphi_j(x) = 2^{jm}\varphi_0(2^j x) - 2^{(j-1)m}\varphi_0(2^{j-1}x).$$

Moreover, by telescoping sum,

$$\sum_{i \leq j} \varphi_i(x) = 2^{jm}\varphi_0(2^j x).$$

Given a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^m)$, we define $f_j = f * \varphi_j$. Hence, it holds

$$f = \sum_{j \in \mathbb{N}} f_j \quad \text{in } \mathcal{S}'(\mathbb{R}^m).$$

This is called the *Littlewood–Paley decomposition* of f.

We now define the $F^s_{p,q}$ norm of a tempered distribution $f \in \mathcal{S}'$ as

$$\|f\|_{F^s_{p,q}} = \big\| \|2^{sj} f_j\|_{\ell^q} \big\|_{L^p(\mathbb{R}^m)}.$$

The Triebel–Lizorkin space $F^s_{p,q}$ is then nothing else but the space of those tempered distributions having finite $F^s_{p,q}$ norm.

The reader should be aware that, when one but not both of the exponents p and q is infinite, this does not coincide with the standard definition of those spaces. We shall nevertheless stick to this definition for the purposes of these notes, even in those exceptional cases.

The following proposition encodes the fundamental properties of the space $F^s_{p,q}$.

Proposition 3.2.1. *The space $F^s_{p,q}$ is a quasi-Banach space, Banach if both $p \geq 1$ and $q \geq 1$. Moreover, except in the exceptional case above where one but not both of the exponents p and q is infinite, two different choices of ψ_0 yield two equivalent $F^s_{p,q}$ norms, so that the space $F^s_{p,q}$ itself does not depend on the choice of ψ_0.*

Next, we present some cases where Triebel–Lizorkin spaces coincide with spaces we have already encountered.

Proposition 3.2.2. 1. *If* $1 < p < +\infty$, *then* $F^0_{p,2} = L^p$.

2. *If* $1 < p < +\infty$, *then* $F^k_{p,2} = W^{k,p}$ *for* $k \in \mathbb{N}_*$.

3. *If* $1 \leq p < +\infty$ *and* $0 < s < +\infty$ *is not an integer, then* $F^s_{p,p} = W^{s,p}$.

The first case of the above proposition is nothing else but the L^p theorem for the Littlewood–Paley decomposition that was covered in Functional Analysis II.

Just as Sobolev spaces, Triebel–Lizorkin spaces satisfy a whole family of embedding theorems, and also have a trace theory. This could be the topic of a whole course in itself, and we shall not delve into such matters. We refer the interested reader to [RS96, Tri83]. In the sequel, we will just need the elementary embedding $F^s_{p,q} \hookrightarrow L^p$ for $s > 0$, that we shall admit without proof.

We finish with the following lemma, that we admit, and that gives a control on the $F^s_{p,q}$ norm of sums of functions satisfying a suitable support condition on their Fourier transform.

Lemma 3.2.3. *Let* $0 < s < +\infty$, $1 < p < +\infty$, *and* $1 < q < +\infty$. *For every* $j \in \mathbb{N}$, *let* $f^j \in \mathcal{S}'$ *be a tempered distribution satisfying* $\operatorname{supp} \mathcal{F}(f^j) \subset B_{2^{j+1}}$. *Then,*

$$\left\| \sum_{j \in \mathbb{N}} f^j \right\|_{F^s_{p,q}} \leq C \left\| \|2^{sj} f^j\|_{\ell^q} \right\|_{L^p(\mathbb{R}^m)},$$

for some constant $C > 0$ *depending on* m, s, p, *and* q.

Let us observe that the right-hand side of the above estimate is nothing else but what would be the $F^s_{p,q}$ norm of the function in the left-hand side if the f^j were a Littlewood–Paley decomposition. The ingredients of the proof of Lemma 3.2.3 are nothing else but some clever decomposition and Hölder's inequalities, along with a vector-valued maximal inequality.

3.3 Product and composition theorems: When a microscopic gain at the Triebel–Lizorkin scale saves the game

We now move towards our proof of the composition theorem stated as Theorem 3.1.2. We start with the following lemma, due to F. Oru, and that can be thought of as a microscopic improvement of the Gagliardo–Nirenberg inequality in the Triebel–Lizorkin scale.

Lemma 3.3.1. *Let* $-\infty < s_1 < s_2 < +\infty$, $0 < p_1, p_2 \leq +\infty$, $0 < q_1, q_2 \leq +\infty$, *and* $0 < \theta < 1$. *Define*

$$s = \theta s_1 + (1-\theta)s_2 \quad \textit{and} \quad \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}.$$

Then, for every $0 < q \leq +\infty$, *we have*

$$\|f\|_{F^s_{p,q}} \leq C\|f\|^{\theta}_{F^{s_1}_{p_1,q_1}} \|f\|^{1-\theta}_{F^{s_2}_{p_2,q_2}} \quad \textit{for every } f \in \mathcal{S}',$$

for some constant $C > 0$ *depending on* m, *the* s_i, *the* p_i, *the* q_i, *and* θ.

Here, a comment is in order. Choosing suitably the q_i and q — to be equal to 2 for integer order spaces, and to the corresponding p_i or p otherwise — one recovers the standard Gagliardo–Nirenberg inequality stated as Theorem 2.1.5. However, Oru's lemma states that we can gain a bit, and choose those microregularity parameters *as we please*. That is what we call *a microscopic improvement at the Triebel–Lizorkin scale*.

The proof of Oru's lemma relies on the following algebraic lemma.

Lemma 3.3.2. *Let* $-\infty < s_1 < s_2 < +\infty$, $0 < q < +\infty$, $0 < \theta < 1$, *and set* $s = \theta s_1 + (1-\theta)s_2$. *Then, for every sequence* $(a_j)_{j\in\mathbb{N}}$, *it holds*

$$\|2^{sj}a_j\|_{\ell^q} \leq C\|2^{s_1 j}a_j\|^{\theta}_{\ell^\infty}\|2^{s_2 j}a_j\|^{1-\theta}_{\ell^\infty},$$

for some constant $C > 0$ *depending on the* s_i, q, *and* θ.

Proof. We let $C_1 = \sup_j 2^{s_1 j}|a_j|$, $C_2 = \sup_j 2^{s_2 j}|a_j|$, so that $C_1 \leq C_2$. We may assume that $C_1 > 0$, otherwise all the a_j vanish and there is nothing to prove. As $s_1 < s_2$, there exists a largest $j_0 \in \mathbb{N}$ such that $C_1 2^{-s_1 j_0} \leq C_2 2^{-s_2 j_0}$, and we therefore have

$$C_2 \approx C_1 2^{(s_2-s_1)j}.$$

This implies that

$$\|2^{s_1 j}a_j\|^{\theta}_{\ell^\infty}\|2^{s_2 j}a_j\|^{1-\theta}_{\ell^\infty} = C_1^{\theta}C_2^{1-\theta} \approx C_1 2^{(s_2-s_1)j_0(1-\theta)}. \tag{3.3.1}$$

Meanwhile, we write

$$|a_j| \leq \frac{C_1}{2^{s_1 j}} \quad \text{for } j \leq j_0 \quad \text{and} \quad |a_j| \leq \frac{C_2}{2^{s_2 j}} \quad \text{for } j > j_0.$$

This yields

$$
\begin{aligned}
\|2^{sj}a_j\|_{\ell^q} &\le \Big(\sum_{j\le j_0} C_1^q 2^{(s-s_1)qj} + \sum_{j>j_0} C_2^q 2^{(s-s_2)qj}\Big)^{\frac{1}{q}} \\
&\lesssim \Big(\sum_{j\le j_0} C_1^q 2^{(s-s_1)qj} + \sum_{j>j_0} C_1^q 2^{-\theta(s_2-s_1)jq+(s_2-s_1)qj_0}\Big)^{\frac{1}{q}} \\
&= C_1 2^{(s_2-s_1)j_0(1-\theta)}\Big(\sum_{j\le j_0} 2^{-(1-\theta)(s_2-s_1)(j_0-j)q} + \sum_{j>j_0} 2^{-\theta(s_2-s_1)(j-j_0)q}\Big)^{\frac{1}{q}} \\
&\lesssim C_1 2^{(s_2-s_1)j_0(1-\theta)}.
\end{aligned}
\tag{3.3.2}
$$

The conclusion follows by combining (3.3.1) and (3.3.2). □

Proof of Lemma 3.3.1. By comparison between the ℓ^∞ norm and the ℓ^r norms with $0 < r < +\infty$, we have

$$\|f\|^{\theta}_{F^{s_1}_{p_1,q_1}}\|f\|^{1-\theta}_{F^{s_2}_{p_2,q_2}} \ge \|f\|^{\theta}_{F^{s_1}_{p_1,\infty}}\|f\|^{1-\theta}_{F^{s_2}_{p_2,\infty}} \quad \text{and} \quad \|f\|_{F^s_{p,\infty}} \le \|f\|_{F^s_{p,q}}.$$

Hence, it suffices to assume that $0 < q < +\infty$ and $q_1 = +\infty = q_2$.

But in this case, applying Lemma 3.3.2 pointwise yields

$$\|f\|_{F^s_{p,q}} \lesssim \Big\|\|2^{s_1j}f_j\|^{\theta}_{\ell^\infty}\|2^{s_2j}f_j\|^{1-\theta}_{\ell^\infty}\Big\|_{L^p(\mathbb{R}^m)}.$$

From Hölder, we deduce that

$$\Big\|\|2^{s_1j}f_j\|^{\theta}_{\ell^\infty}\|2^{s_2j}f_j\|^{1-\theta}_{\ell^\infty}\Big\|_{L^p(\mathbb{R}^m)} \le \Big\|\|2^{s_1j}f_j\|_{\ell^\infty}\Big\|^{\theta}_{L^{p_1}(\mathbb{R}^m)}\Big\|\|2^{s_2j}f_j\|_{\ell^\infty}\Big\|^{1-\theta}_{L^{p_2}(\mathbb{R}^m)} = \|f\|^{\theta}_{F^{s_1}_{p_1,\infty}}\|f\|^{1-\theta}_{F^{s_2}_{p_2,\infty}}.$$

This yields the conclusion. □

We now present another result featuring Triebel–Lizorkin spaces, this time concerned with products, namely, the Runst–Sickel lemma [RS96]. We split the statement into two parts. The first one is an estimate for products of functions.

Lemma 3.3.3. *Let* $0 < s < +\infty$, $1 < q < +\infty$, $1 < p_1, p_2 \le +\infty$, *and* $1 < r_1, r_2 \le +\infty$ *be such that*

$$0 < \frac{1}{p} = \frac{1}{p_1} + \frac{1}{r_2} = \frac{1}{p_2} + \frac{1}{r_1} < 1.$$

If $f \in F^s_{p_1,q} \cap L^{r_1}$ and $g \in F^s_{p_2,q} \cap L^{r_2}$, then

$$\|fg\|_{F^s_{p,q}} \leq C\Big(\big\|Mf\|2^{js}g_j\|_{\ell^q}\big\|_{L^p(\mathbb{R}^m)} + \big\|Mg\|2^{js}f_j\|_{\ell^q}\big\|_{L^p(\mathbb{R}^m)}\Big)$$

and

$$\|fg\|_{F^s_{p,q}} \leq C\big(\|f\|_{F^s_{p_1,q}}\|g\|_{L^{r_2}} + \|g\|_{F^s_{p_2,q}}\|f\|_{L^{r_2}}\big),$$

for some constant $C > 0$ depending on m, s, q, p_i, and r_i.

In the above, Mf and Mg denote the Hardy–Littlewood maximal functions of respectively f and g. Also, the assumption $s > 0$ has some importance that is quite hidden in the above statement: At least when $s < 0$, elements of $F^s_{p,q}$ need not be actually functions, but can be wilder types of distributions. Therefore, it would not even be clear in first instance how to make sense of the product fg in this setting. On the other hand, when $s > 0$, we have $F^s_{p,q} \hookrightarrow L^p$, so we can define the product almost everywhere as the pointwise multiplication.

Proof. Let us first note that the second estimate follows from the first one, as a direct consequence of Hölder's inequality and the Hardy–Littlewood maximal inequality. It therefore remains to prove the first estimate.

For this purpose, we start by writing the decomposition

$$fg = \sum_{i\in\mathbb{N}} G_i + \sum_{j\in\mathbb{N}} F_j,$$

with

$$G_i = \Big(\sum_{j\leq i} f_j\Big)g_i \quad \text{and} \quad F_j = \Big(\sum_{i<j} g_i\Big)f_j.$$

We observe that $\operatorname{supp}\mathcal{F}(G_i) \subset B_{2^{i+2}}$ and $\operatorname{supp}\mathcal{F}(F_j) \subset B_{2^{j+2}}$, as a consequence of the definition of the Littlewood–Paley decomposition and the formula relating product and convolutions through the Fourier transform. Therefore, and since $1 < p, q < +\infty$, Lemma 3.2.3 applies and yields

$$\|fg\|_{F^s_{p,q}} \lesssim \big\|\|2^{si}G_i\|_{\ell^q}\big\|_{L^p(\mathbb{R}^m)} + \big\|\|2^{sj}F_j\|_{\ell^q}\big\|_{L^p(\mathbb{R}^m)}.$$

Now, we recall that

$$\sum_{j\leq i} f_j = 2^{im}\varphi(2^i\cdot) * f,$$

whence, by definition of the convolution product and of the maximal function,

$$\left|\sum_{j\le i} f_j\right| \lesssim Mf.$$

Therefore, we find

$$\left\|\|2^{si}G_i\|_{\ell^q}\right\|_{L^p(\mathbb{R}^m)} \lesssim \left\|Mf\|2^{si}g_i\|_{\ell^q}\right\|_{L^p(\mathbb{R}^m)}.$$

Similarly,

$$\left\|\|2^{sj}F_j\|_{\ell^q}\right\|_{L^p(\mathbb{R}^m)} \lesssim \left\|Mg\|2^{sj}f_j\|_{\ell^q}\right\|_{L^p(\mathbb{R}^m)}.$$

The conclusion follows. □

The next lemma contains the continuity of the product.

Lemma 3.3.4. 1. *Let* $0 < s < +\infty$, $1 < q < +\infty$, $1 < p_1, p_2 \le +\infty$, *and* $1 < r_1, r_2 \le +\infty$ *be such that*

$$0 < \frac{1}{p} = \frac{1}{p_1} + \frac{1}{r_2} = \frac{1}{p_2} + \frac{1}{r_1} < 1.$$

The map $(F^s_{p_1,q} \cap L^{r_2}) \times (F^s_{p_2,q} \cap L^{r_1})$ *defined by* $(f, g) \mapsto fg$ *is continuous.*

2. *Let* $0 < s < +\infty$, $1 < p < +\infty$, *and* $1 < q < +\infty$. *If* $f^n \to f$ *and* $g^n \to g$ *in* $F^s_{p,q}$, *and if* $(f^n)_{n\in\mathbb{N}}$ *and* $(g^n)_{n\in\mathbb{N}}$ *remain bounded in* L^∞, *then* $f^n g^n \to fg$ *in* $F^s_{p,q}$.

3. *Let* $0 < s < +\infty$, $1 < q < +\infty$, $1 < p, p_1 < +\infty$, $1 < r < +\infty$ *be such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{r}.$$

If $f^n \to f$ *in* $F^s_{p_1,r}$ *and* $(f^n)_{n\in\mathbb{N}}$ *remains bounded in* L^∞, *and if* $g^n \to g$ *in* $F^s_{p,q} \cap L^r$, *then* $f^n g^n \to fg$ *in* $F^s_{p,q}$.

Proof. We prove only the third item — which also happens to be the one we will need in the sequel. The proof of the second is similar, and the proof of the first is simpler, as we have full convergence in both lower order spaces, not only boundedness.

We start by writing

$$f^n g^n - fg = f^n(g^n - g) + (f^n - f)g.$$

From this decomposition, we see that we have reduced the problem to proving, first that if $f^n \to 0$ in $F^s_{p_1,q}$ and remains bounded in L^∞, then $f^n g \to 0$ for any fixed $g \in F^s_{p,q} \cap L^r$; and second that if $(f^n)_{n\in\mathbb{N}}$ remains bounded in $F^s_{p_1,q}$ and in L^∞ and if $g^n \to 0$ in $F^s_{p,q} \cap L^r$, then $f^n g^n \to 0$.

The second assertion is a clear consequence of the second estimate from Lemma 3.3.3. The first one require some more care. From the first estimate in Lemma 3.3.3 — and using Hölder and the maximal inequality on the first term — we have

$$\|f^n g\|_{F^s_{p,q}} \lesssim \|f^n\|_{F^s_{p_1,q}} \|g\|_{L^r} + \left\| Mf^n \|2^{sj} g_j\|_{\ell^q} \right\|_{L^p(\mathbb{R}^m)}.$$

The first term converges to 0 by assumption. Let us deal with the second. We define $F^n = Mf^n \|2^{sj} g_j\|_{\ell^q}$, which satisfies

$$|F^n| \lesssim \|2^{sj} g_j\|_{\ell^q} \in L^p(\mathbb{R}^m)$$

as a consequence of the L^∞ bound on the f^n. We now resort to the basic embedding $F^s_{p_1,q} \hookrightarrow L^{p_1}$. Therefore, the maximal inequality implies that $Mf^n \to 0$ in L^{p_1}, and hence, up to extraction of a subsequence, $Mf^n \to 0$ almost everywhere. We conclude that $\left\| Mf^n \|2^{sj} g_j\|_{\ell^q} \right\|_{L^p(\mathbb{R}^m)} \to 0$ by the means of the dominated convergence theorem, which finishes the proof. □

Having Oru's and the Runst–Sickel lemmas at our disposal, we are now fully equipped to prove our main objective, namely, the composition theorem featured as Theorem 3.1.2. Let us just mention that the approach we feature for $s > 1$ noninteger does not cover the case $p = 1$. For this, we refer to the approach in [MS02, BPVS13]. See also the proof of Theorem 4.1.1, which contains many similar ideas.

Proof of Theorem 3.1.2. We present various cases in increasing order of difficulty, as successive warm-ups, concluding with the most involved case, namely, $s > 1$ noninteger. But first, we observe that it suffices to prove the result when the domain is $\mathbb{R}^m$, the case of a bounded open set follows then by extension.

Case 1. — $s = 1$.

We assume that $u_n \to u$ in $W^{1,p}(\mathbb{R}^m)$. The function F being in particular Lipschitz, we have $|F \circ u_n - F \circ u| \lesssim |u_n - u|$, from which the L^p convergence follows directly. We now turn to the derivative. To justify very rigorously all steps, we somehow need to repeat twice the same argument, first with smooth maps, then in the general case.

Namely, we first assume that the u_n belong to $C^\infty_c(\mathbb{R}^m)$. In this case, the chain rule implies that

$$\mathrm{D}(F \circ u_n) = (\mathrm{D}F \circ u_n)\,\mathrm{D}u_n. \tag{3.3.3}$$

By the partial converse of the dominated convergence theorem, up to extracting a subsequence, we may assume that $\mathrm{D}u_n \to \mathrm{D}u$ and $u_n \to u$ almost everywhere, and that $|\mathrm{D}u_n| \le g$ for some $g \in L^p(\mathbb{R}^m)$. As $|\mathrm{D}(F \circ u_n)| \lesssim |\mathrm{D}u_n|$ as a consequence of (3.3.3) and the fact that $\mathrm{D}F$ is bounded, and as $(\mathrm{D}F \circ u_n)\,\mathrm{D}u_n \to (\mathrm{D}F \circ u)\,\mathrm{D}u$ almost everywhere, we deduce that $(\mathrm{D}F \circ u_n)\,\mathrm{D}u_n \to (\mathrm{D}F \circ u)\,\mathrm{D}u$ in L^p. The closure lemma implies that $F \circ u \in W^{1,p}$ and that $\mathrm{D}(F \circ u) = (\mathrm{D}F \circ u)\,\mathrm{D}u$.

Now that we know that the composition is well-defined on the whole $W^{1,p}$, we can repeat the same argument for general $u_n \in W^{1,p}$, and this yields the desired convergence up to a subsequence. However, since we have identified the limit as u, the convergence of the whole sequence follows via the subsequence principle.

Case 2. — $0 < s < 1$.

In this regime, the proof is somewhat different than in the other ranges, but the proof contains interesting ideas, so we present it in detail. For bibliographic notice, let us mention that the argument is due to Marcus and Mizel, and that we follow the presentation by Bousquet, Ponce, and Van Schaftingen [BPVS14, Lemma 2.5].

The fact that $F \circ u \in W^{s,p}$ for $u \in W^{s,p}$ is trivial, using the Lipschitz estimate on F in the numerator of the quotient used in the definition of the Gagliardo seminorm. This also shows that the composition operator is bounded, with constant at most the Lipschitz constant of F. But care is needed here, as the composition operator is not linear, so that boundedness does not imply continuity.

To prove the continuity, we start by taking u and $v \in W^{s,p}(\mathbb{R}^m)$, and we define

$$I(x,y) = \frac{|F(u(x)) - F(v(x)) - F(u(y)) + F(v(y))|^p}{|x-y|^{m+sp}},$$

the quotient that appears in the computation of the Gagliardo seminorm. There are two possible ways of bounding I. Either grouping together the u and the v, which yields

$$\begin{aligned} I(x,y) &\lesssim \frac{|u(x)-u(y)|^p}{|x-y|^{m+sp}} + \frac{|v(x)-v(y)|^p}{|x-y|^{m+sp}} \\ &\lesssim \frac{|u(x)-u(y)|^p}{|x-y|^{m+sp}} + \frac{|u(x)-v(x)-u(y)+v(y)|^p}{|x-y|^{m+sp}}. \end{aligned}$$

Or, grouping together the x and the y, which leads to

$$I(x,y) \lesssim \frac{|u(x)-v(x)|^p + |u(y)-v(y)|^p}{|x-y|^{m+sp}}.$$

Given $\varepsilon > 0$, let us define the set

$$A_{\varepsilon,v} = \{(x,y) \in \mathbb{R}^m \times \mathbb{R}^m \colon |u(x)-v(x)|^p + |u(y)-v(y)|^p \geq \varepsilon f(x,y)|x-y|^{m+sp}\},$$

where $f \in L^1(\mathbb{R}^m \times \mathbb{R}^m)$ is fixed once for all. We apply our first estimate on $A_{\varepsilon,v}$, and our second one on its complement, deriving

$$|F\circ u - F\circ v|^p_{W^{s,p}(\mathbb{R}^m)} \lesssim \varepsilon \iint_{\mathbb{R}^m\times\mathbb{R}^m} f(x,y)\,\mathrm{d}x\,\mathrm{d}y + \iint_{A_{\varepsilon,v}} \frac{|u(x)-u(y)|^p}{|x-y|^{m+sp}}\,\mathrm{d}x\,\mathrm{d}y + |u-v|^p_{W^{s,p}(\mathbb{R}^m)}.$$

For fixed $\varepsilon > 0$, we have $|A_{\varepsilon,v}| \to 0$ as $v \to u$ in L^p. Hence, letting $v \to u$ in $W^{s,p}$, we conclude from Lebesgue's lemma that

$$\limsup_{v\to u} |F\circ u - F\circ v|^p_{W^{s,p}(\mathbb{R}^m)} \lesssim \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, the conclusion follows.

Case 3. — $s = k \in \mathbb{N}_{\geq 2}$.

This case bears a lot of similarities with $s = 1$, using the Gagliardo–Nirenberg interpolation inequality as an extra ingredient to deal with higher order derivatives. First, we mention that the Faà di Bruno formula for smooth maps extends to $W^{k,p}$ maps, exactly as in the case $k = 1$. Hence, we shall directly consider the case of a sequence $(u_n)_{n\in\mathbb{N}}$ strongly converging towards u in $W^{k,p}$. By the Faà di Bruno formula and the bounds on the derivatives of F, we have

$$|\mathrm{D}^k(F\circ u_n)| \lesssim \sum_{j=1}^{k} \sum_{\substack{1\leq i_1,\ldots,i_j\leq k\\ i_1+\cdots+i_j=k}} |\mathrm{D}^{i_1}u_n|\cdots|\mathrm{D}^{i_j}u_n|.$$

By the Gagliardo–Nirenberg inequality, interpolating between $W^{k,p}$ and $W^{1,kp}$, we have $\mathrm{D}^t u_n \to \mathrm{D}^t u$ in $L^{\frac{kp}{t}}$ for every $1 \leq t \leq k$. Hence, up to extraction of a subsequence, we can assume that all convergences also occur almost everywhere, and that $|\mathrm{D}^t u_n| \leq g_t$ with $g_t \in L^{\frac{kp}{t}}$. By the generalised Hölder inequality, we have $g_{i_1}\cdots g_{i_j} \in L^p$ for $i_1+\cdots+i_j = k$. By the dominated convergence theorem, we conclude that $\mathrm{D}^k F\circ u_n \to \mathrm{D}^k(F\circ u)$ in L^p along the extracted subsequence. We conclude with the help of the subsequence principle.

Case 4. — $s > 1$ *noninteger, a naive unsuccessful approach.*

Before we present the actual proof of the most difficult case $s > 1$ noninteger, relying on the machinery we introduced above, let us first present a naive approach using only Sobolev spaces, and see at which point this fails. For simplicity, we assume that $1 < s < 2$, and write $s = 1 + \sigma$. Hence, $\mathrm{D}(F \circ u) = (\mathrm{D}F \circ u)\,\mathrm{D}u$. For the second factor, by the Gagliardo–Nirenberg inequality, we have $\mathrm{D}u \in W^{\sigma,p} \subset L^{(1+\sigma)p}$. For the first term, we have $\mathrm{D}F \circ u \in L^\infty$ by the uniform bound on $\mathrm{D}F$, and $\mathrm{D}F \circ u \in W^{1,(1+\sigma)p}$ by the Faà di Bruno formula. Applying Gagliardo–Nirenberg once more, we get $\mathrm{D}F \circ u \in W^{\sigma,\frac{1+\sigma}{\sigma}p}$. Let $U = \mathrm{D}u \in W^{\sigma,p} \cap L^{(1+\sigma)p}$ and $V \in W^{1,\frac{1+\sigma}{\sigma}p} \cap L^\infty$, and let us try to prove that $UV \in W^{\sigma,p}$. It is natural to write the decomposition

$$|U(x+h)V(x+h)-U(x)V(x)|^p \lesssim |V(x+h)|^p|U(x+h)-U(x)|^p+|U(x)|^p|V(x+h)-V(x)|^p.$$

Therefore, we have

$$|UV|^p_{W^{\sigma,p}} \lesssim \|V\|^p_{L^\infty}|U|^p_{W^{\sigma,p}} + \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \frac{|U(x)|^p|V(x+h)-V(x)|^p}{|h|^{m+\sigma p}}\,\mathrm{d}h\,\mathrm{d}x.$$

At this stage, it is again natural to invoke Hölder's inequality with exponents $\frac{1+\sigma}{\sigma}$ and $1+\sigma$ for the second term. This yields

$$\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \frac{|U(x)|^p|V(x+h)-V(x)|^p}{|h|^{m+\sigma p}}\,\mathrm{d}h\,\mathrm{d}x \leq |V|^p_{W^{\sigma,\frac{1+\sigma}{\sigma}p}}\biggl(\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\frac{|U(x)|^{(1+\sigma)p}}{|h|^m}\,\mathrm{d}h\,\mathrm{d}x\biggr)^{\frac{1}{1+\sigma}}.$$

The issue is that the latter integral diverges — even though barely. This gives some feeling that indeed only a microscopic improvement of regularity at a well-chosen point of the proof is required to save the argument, and this is precisely what the refined inequalities at the Triebel–Lizorkin scale we have presented so far do for us.

Case 5. — $s > 1$, *a successful proof.*

We already know, as shown in the very first part of the proof, that the composition map is well-defined and continuous into L^p. It therefore suffices to consider the first derivative, $\mathrm{D}(F \circ u) = (\mathrm{D}F \circ u)\,\mathrm{D}u$. By the standard Gagliardo–Nirenberg inequality, we have the continuous inclusion

$$W^{s,p} \cap W^{1,sp} \hookrightarrow W^{k,\frac{sp}{k}} \cap W^{1,sp}.$$

Applying the integer order case, we have, if $u_n \to u$ in $W^{s,p} \cap W^{1,sp}$,

$$\mathrm{D}F(u_n) \to \mathrm{D}F(u) \quad \text{in } W^{k,\frac{sp}{k}} = F^k_{\frac{sp}{k},2},$$

and this sequence is also bounded in L^∞ by the assumption on $\mathrm{D}F$. Using Oru's lemma $W^{s,p} \cap L^\infty \hookrightarrow F^{\theta s}_{p/\theta,q}$ with $\theta = \frac{s-1}{k}$, we get

$$\mathrm{D}F(u_n) \to \mathrm{D}F(u) \quad \text{in } F^{s-1}_{\frac{sp}{s-1},p}.$$

Here, we have used our freedom to change the microscopic regularity parameter when using the embedding. Meanwhile, we have

$$\mathrm{D}u_n \to \mathrm{D}u \quad \text{in } W^{s-1,p} \cap L^{sp} = F^{s-1}_{p,p} \cap L^{sp}.$$

The Runst–Sickel lemma now implies that $\mathrm{D}F(u_n)\,\mathrm{D}u_n \to \mathrm{D}F(u)\,\mathrm{D}u$ in $F^{s-1}_{p,p} = W^{s-1,p}$, which implies the conclusion.

Let us observe that the exact same proof implies the continuity of the composition operator when $W^{s,p}$ is replaced with $F^s_{p,q}$ for any q. □

3.4 Comments and perspectives

3.4.1 Another application for composition operators: The lifting problem

In this chapter, we did not use the full power of the composition theorem that we presented. Indeed, we had at our disposal a stronger assumption, namely that the maps we are composing with the nearest point projection belong to $W^{s,p} \cap L^\infty$ — the L^∞ bound being provided by the manifold constraint. In this framework, the result was already know before the work of H. Brezis and P. Mironescu, see e.g. T. Runst and W. Sickel [RS96, 5.3.7].

The original motivation from Brezis and Mironescu to establish a composition theorem for maps in $W^{s,p} \cap W^{1,sp}$ rather than for maps in $W^{s,p} \cap L^\infty$ came from the lifting problem. More specifically, let $u \colon \Omega \to \mathbb{S}^1$. If $u \in W^{s,p}$ and u has a lifting $\varphi \colon \Omega \to \mathbb{R}$ such that $\mathrm{e}^{\imath\varphi} = u$, then φ has no reason to be L^∞ — and will not, in general. However, u is in L^∞, and hence, by the Gagliardo–Nirenberg inequality, it belongs to $W^{1,sp}$. Therefore, as $\mathrm{D}u = \imath\,\mathrm{D}\varphi\mathrm{e}^{\imath\varphi}$, we have $|\mathrm{D}u| = |\mathrm{D}\varphi| \in L^{sp}$, so that the lifting *necessarily* belongs to $W^{1,sp}$.

Having a composition theorem at hand in this context allows notably to show that, if $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence of liftings converging to a lifting φ, then the projected maps converge, that is, $\mathrm{e}^{\imath\varphi_n} \to \mathrm{e}^{\imath\varphi}$. This is a simpler approach — specific to $\mathbb{S}^1$ — to prove a strong approximability property for circle-valued Sobolev maps; see [BM01b].

Other composition or product theorems, of various difficulties, were used in the problem of existence of a lifting, notably by J. Bourgain, H. Brezis, and P. Mironescu [BBM00], and by F. Bethuel and D. Chiron [BC07].

4 How to still apply the real-valued theory: The method of singular projection

In this chapter, we move towards the more difficult case of density when $sp < m$. But, before we address the case of a general target in Chapter 6, we first explain a very natural — and simpler — approach, that works when the target has a sufficiently simple topology. Our model example in Section 4.1 will be the case where $\mathscr{N} = \mathbb{S}^\ell \subset \mathbb{R}^{\ell+1}$, but we will explain in Section 4.2 what are actually the properties of the sphere that make this approach applicable, and how to extend it with minimal assumptions. Although the method of projection cannot cover the most general case for the strong approximation problem, it is nevertheless a very useful tool, also in other contexts; see [Det25a] for examples of applications.

4.1 A case study: Projecting a mapping back into the sphere

We start by mentioning that the approach presented in this section follows the idea first introduced by F. Bethuel and Zheng X. [BZ88]. Our objective is to study the strong approximation problem in $W^{1,p}(\mathbb{B}^m; \mathbb{S}^\ell)$, where $p < m$. As we already said, if there would exist a Lipschitz map $P\colon \mathbb{R}^{\ell+1} \to \mathbb{S}^\ell$ such that $P = \mathrm{id}$ on $\mathbb{S}^\ell$, we could simply apply the linear theory and project everything back into $\mathbb{S}^\ell$. However, the existence of such a map is forbidden by Brouwer's theorem. Nevertheless, there exists a map P that satisfies almost this property, defined as

$$P\colon \mathbb{R}^{\ell+1} \setminus \{0\}, \quad P(x) = \frac{x}{|x|}.$$

However, the presence of the singularity at 0 raises at least two[1] challenges. The first one is that the composition of a mapping with P will fail to be well-defined at any point where this mapping takes the value 0. The second is that, even if we can somehow ensure that the set of such point is small — for instance, a null set, so that the composition is almost everywhere defined — we cannot get estimates or convergence by an application of a composition theorem, such as Theorem 3.1.2, because the map P is singular at the

[1] It is always dangerous to start such an enumeration by announcing the number of items, one never knows if one will not think of extra items while enumerating them, and therefore be forced to correct the number and start again: https://www.youtube.com/watch?v=psMMKgvpGfg&pp=ygUgbW9udHkgcHl0aG9uIHNwYW5pc2ggaW5xdWlzaXRpb24%3D. (I promise, this is no rickroll!)

origin and therefore does not meet the assumption of such theorems. Overcoming these challenges is the point of the method of singular projection, which is due to R. Hardt and Lin F. [HL87] in the context of Sobolev mappings, with roots in the work of H. Federer and W. Fleming [FF60] on currents.

More specifically, our goal in this section is to prove (some cases of) the following theorem.

Theorem 4.1.1. *Assume that $s \geq 1$ and $\ell \leq sp < \ell + 1$. Then, $\mathscr{R}_{m-\ell-1}(\mathbb{B}^m; \mathbb{S}^\ell)$ is dense in $W^{s,p}(\mathbb{B}^m; \mathbb{S}^\ell)$.*

To be very precise, the class $\mathscr{R}$ that we will obtain from the proof is slightly different from the one we defined in the introduction: The singular set of its members is allowed to be any finite union of $(m - \lfloor sp \rfloor - 1)$-submanifolds, not necessarily affine spaces.

Before we move to proofs, let us do a quick sanity check, comparing with Theorem 1.2.6. Here, we have $\lfloor sp \rfloor = \ell$, so that the dimension of the class $\mathscr{R}$ is the correct one. Moreover, $\pi_\ell(\mathbb{S}^\ell) = \mathbb{Z}$ is nontrivial, so the strong approximation property fails. For $sp < \ell$, we would have $\pi_{\lfloor sp \rfloor}(\mathbb{S}^\ell) = \{0\}$, and hence smooth maps would be dense. (This case was actually also already known from the work of Bethuel and Zheng, with a much easier proof than in the case of a general target. The idea is to remove a small cap from the sphere to turn it into a ball and to "fold back" the mapping to be approximated into the rest of the sphere, bringing back to the linear case. Choosing suitably the cap being removed, one can ensure that the folded map converges to the original one.) For $sp \geq \ell + 1$, we may still have $\pi_{\lfloor sp \rfloor}(\mathbb{S}^\ell) \neq \{0\}$ — or not, homotopy groups of spheres are quite a mess — thwarting the strong approximation property. However, in this case, the method of projection cannot work for topological reasons, that will be explained in Section 4.2.

Proof of Theorem 4.1.1 when $s = 1$. As already explained, we rely on the projection $P\colon \mathbb{R}^{\ell+1} \setminus \{0\} \to \mathbb{S}^\ell$ defined by

$$P(x) = \frac{x}{|x|}.$$

We collect below the important properties of the map P:

(a) $P(x) \in \mathbb{S}^\ell$ for every $x \in \mathbb{R}^{\ell+1} \setminus \{0\}$;

(b) $P(x) = x$ for every $x \in \mathbb{S}^\ell$;

(c) $P \in C^\infty(\mathbb{R}^{\ell+1} \setminus \{0\})$ and

$$|DP(x)| \leq C \frac{1}{|x|} \quad \text{for every } x \in \mathbb{R}^{\ell+1} \setminus \{0\}.$$

Let $u \in W^{1,p}(\mathbb{B}^m; \mathbb{S}^\ell)$. By the linear approximation theorem, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $C^\infty(\overline{\mathbb{B}^m}; \mathbb{R}^{\ell+1})$ such that $u_n \to u$ in $W^{1,p}$. For every $a \in \mathbb{R}^{\ell+1}$, we define the map $u_{n,a} = P \circ (u_n - a)$. This map is well-defined and smooth on $\mathbb{B}^m \setminus u_n^{-1}(\{a\})$, and is $\mathbb{S}^\ell$-valued due to (a).

The first important step in the proof is to study the regularity of this set $u_n^{-1}(\{a\})$. We recall that a *regular value* for the map u_n is a value $a \in \mathbb{B}^m$ such that $\mathrm{D}u(x)$ is a surjective linear mapping for every $x \in u_n^{-1}(\{a\})$. The key property of regular values is that, according to the *submersion theorem* (see e.g. [Mil97, Lemma 1]), their preimage is a smooth submanifold of codimension $\ell + 1$. We combine this fact with the *Morse–Sard theorem* – also known as the *Sard–Brown lemma* – which ensures that, for a smooth map, almost every value is regular (see e.g. [Mil97, § 2]). Therefore (also using that a countable union of negligible sets is negligible), for almost every $a \in \mathbb{R}^{\ell+1}$, the set $u_n^{-1}(\{a\})$ is a smooth $(m - \ell - 1)$-submanifold of $\mathbb{B}^m$ for every $n \in \mathbb{N}$.

We now establish estimates for the maps $u_{n,a}$ to obtain their convergence. This relies mainly on the elegant *Federer-Fleming averaging argument*. Using the chain rule and (c), we estimate

$$|\mathrm{D}u_{n,a}| \lesssim \frac{|\mathrm{D}u_n|}{|u_n - a|}.$$

By the partial converse of the dominated convergence theorem, up to extraction of a subsequence, there exists a map $g \in L^p(\mathbb{B}^m)$ such that $|\mathrm{D}u_n| \leq g$ for every $n \in \mathbb{N}$. Since $p < \ell + 1$, for every $x \in \mathbb{B}^m$, we have

$$\int_{\mathbb{B}^{\ell+1}} |\mathrm{D}u_{n,a}(x)|^p \,\mathrm{d}a \lesssim \int_{\mathbb{B}^{\ell+1}} \frac{g^p}{|u_n(x) - a|^p} \,\mathrm{d}a \lesssim 1.$$

Actually, more is true: As the same argument repeated on a general subset of $\mathbb{B}^{\ell+1}$ shows, the sequence $(|\mathrm{D}u_{n,a}(x)|^p)_{n\in\mathbb{N}}$ is *equi-integrable* with respect to the a variable. Let $u_a = P \circ (u - a)$. Since (up to a further extraction) the sequence $(\mathrm{D}u_{n,a}(x))_{n\in\mathbb{N}}$ converges to $\mathrm{D}u_a(x)$ for almost every $x \in \mathbb{B}^m$, we deduce from the *Vitali convergence theorem* (see e.g. [Wil22, Theorem 3.1.9]) that

$$\lim_{n\to+\infty} \int_{\mathbb{B}^{\ell+1}} |\mathrm{D}u_{n,a}(x) - \mathrm{D}u_a(x)|^p \,\mathrm{d}a = 0 \quad \text{for almost every } x \in \mathbb{B}^m.$$

Moreover, by the same argument, we estimate

$$\int_{\mathbb{B}^{\ell+1}} |\mathrm{D}u_{n,a}(x) - \mathrm{D}u_a(x)|^p \,\mathrm{d}a \lesssim |g(x)|^p \quad \text{for almost every } x \in \mathbb{B}^m.$$

Hence, by the Lebesgue dominated convergence theorem, we conclude that

$$\lim_{n\to+\infty}\int_{\mathbb{B}^{\ell+1}}\int_{\mathbb{B}^m}|\mathrm{D}u_{n,a}(x)-\mathrm{D}u_a(x)|^p\,\mathrm{d}x\,\mathrm{d}a = \lim_{n\to+\infty}\int_{\mathbb{B}^m}\int_{\mathbb{B}^{\ell+1}}|\mathrm{D}u_{n,a}(x)-\mathrm{D}u_a(x)|^p\,\mathrm{d}a\,\mathrm{d}x = 0.$$

We conclude that, up to yet another extraction of a subsequence, there exists a sequence $(a_n)_{n\in\mathbb{N}}$ such that $a_n \to 0$ and such that

$$\lim_{n\to+\infty}\int_{\mathbb{B}^m}|\mathrm{D}u_{n,a_n}(x)-\mathrm{D}u_{a_n}(x)|^p\,\mathrm{d}x = 0.$$

Since $u_{a_n} \to u$ in $W^{1,p}$ thanks to (b), we conclude that $u_{n,a_n} \to u$, which finishes the proof. □

The proof for higher-order spaces is similar, relying on the Gagliardo–Nirenberg inequality to estimate the terms coming out when using the Faà di Bruno formula. The argument bears a lot of similarities with the corresponding proof of Theorem 3.1.2, and we omit it.

Proof when $1 < s < 2$. We use the same notation and constructions as in the case $s = 1$. Additionally, reprojecting onto $\overline{\mathbb{B}^{\ell+1}}$ if needed, we can assume our approximating sequence $(u_n)_{n\in\mathbb{N}}$ to be bounded, which will be useful for us to apply Gagliardo–Nirenberg in our argument.

By virtue of what we already proved, it suffices to establish the $W^{\sigma,p}$ convergence of the derivatives, where $s = 1+\sigma$. We focus on the new ideas, leaving aside some technicalities which are essentially as in the case $s = 1$. By the chain rule, we find $\mathrm{D}u_{n,a} = \mathrm{D}P(u_n - a)\,\mathrm{D}u_n$. By symmetry of the integrand in the definition of the Gagliardo seminorm, it suffices to work on the region where $|u_n(x)-a| \leq |u_n(y)-a|$. We denote this set as E_a, and denote its x-slices (that is, the set of y such that $(x,y)\in E$) as $E_{x,a,n}$.

We first write

$$|\mathrm{D}u_{n,a}(x)-\mathrm{D}u_{n,a}(y)|^p \lesssim A_n(x,y,a)+B_n(x,y,t),$$

with

$$A_n(x,y,a) = |\mathrm{D}P(u_n(x)-a)-\mathrm{D}P(u_n(y)-a)|^p|\mathrm{D}u_n(x)|^p$$

and

$$B_n(x,y,a) = |\mathrm{D}P(u_n(y)-a)|^p|\mathrm{D}u_n(x)-\mathrm{D}u_n(y)|^p.$$

We start by estimating the term containing A_n. For that, we observe that the difference between the derivatives of P can be estimated in two ways. A first option is to use directly the bound on $\mathrm{D}P$ and get

$$A_n(x,y,a) \lesssim \frac{|\mathrm{D}u_n(x)|^p}{|u_n(x)-a|^p}. \tag{4.1.1}$$

A second option is to use the fundamental theorem of calcul around a path that never approaches the origin closer than $u_n(x)-a$ combined with the bound on the second derivative of P, getting

$$A_n(x,y,a) \lesssim |\mathrm{D}u_n(x)|^p \frac{|u_n(x)-u_n(y)|^p}{|u_n(x)-a|^{2p}}. \tag{4.1.2}$$

We now consider the integrand in the Gagliardo seminorm, that is, $\frac{A(x,y,a)+B(x,y,a)}{|x-y|^{m+\sigma p}}$, that we integrate over $\mathbb{B}^m$ with respect to y. Let $\rho > 0$. Over $B_\rho(x)$, we want to rely on (4.1.2). By the fundamental theorem of calculus, we have

$$|u_n(x)-u_n(y)|^p \leq |x-y|^p \int_0^1 |\mathrm{D}u_n(x+t(y-x))|^p \,\mathrm{d}t.$$

Injecting this into the integral with respect to y and changing the variable as $h = y-x$, we find

$$\begin{aligned}\int_{E_{x,a,n}\cap B_\rho(x)} \frac{A_n(x,y,a)}{|x-y|^{m+\sigma p}}\,\mathrm{d}y &\leq \frac{|\mathrm{D}u_n(x)|^p}{|u_n(x)-a|^{2p}} \int_0^1 \int_{B_\rho} \frac{|\mathrm{D}u_n(x+th)|^p}{|h|^{m+(\sigma-1)p}}\,\mathrm{d}h\,\mathrm{d}t \\ &\leq \int_0^1 t^{(\sigma-1)p} \int_{B_{t\rho}(x)} \frac{|\mathrm{D}u_n(y)|^p}{|x-y|^{m+(\sigma-1)p}}\,\mathrm{d}y\,\mathrm{d}t.\end{aligned}$$

Here, we are in position to apply *Hedberg's lemma* [Hed72], which exactly asserts that

$$\int_{B_{t\rho}(x)} \frac{|\mathrm{D}u_n(y)|^p}{|x-y|^{m+(\sigma-1)p}}\,\mathrm{d}y \lesssim (t\rho)^{(1-\sigma)p} M(|\mathrm{D}u_n|^p)(x),$$

where again M denotes the Hardy–Littlewood maximal function. Therefore,

$$\int_{B_\rho(x)} \frac{A_n(x,y,a)}{|x-y|^{m+\sigma p}}\,\mathrm{d}y \lesssim \frac{|\mathrm{D}u_n(x)|^p}{|u_n(x)-a|^{2p}} \rho^{(1-\sigma)p} M(|\mathrm{D}u_n|^p)(x).$$

On the contrary, over $\mathbb{B}^m \setminus B_\rho(x)$, we want to use (4.1.1). For this, we rely on the straightforward estimate

$$\int_{\mathbb{B}^m \setminus B_\rho(x)} \frac{1}{|x-y|^{m+\sigma p}} \, \mathrm{d}y \lesssim \rho^{-\sigma p},$$

and we arrive at

$$\int_{E_{x,a,n} \setminus B_\rho(x)} \frac{A_n(x,y,a)}{|x-y|^{m+\sigma p}} \, \mathrm{d}y \lesssim \rho^{-\sigma p} \frac{|\mathrm{D}u_n(x)|^p}{|u_n(x)-a|^p}.$$

We are now at a situation where we have two different regimes, that can be tuned by the choice of ρ. Also, since our aim is to integrate with respect to a, we want the exponent falling on the $|u_n(x)-a|$ to be less than $\ell+1$ — ideally, we would like it to be sp, which satisfies this assumption. This is done by choosing ρ optimising the sum of both estimates. It can be checked that the optimum is realised by (a constant multiple of)

$$\rho = |u_n(x)-a|(M(|\mathrm{D}u_n|^p)(x))^{-\frac{1}{p}}.$$

Inserting this choice leads to

$$\int_{E_{x,a,n}} \frac{A_n(x,y,a)}{|x-y|^{m+\sigma p}} \, \mathrm{d}y \lesssim \frac{1}{|u_n(x)-a|^{sp}} |\mathrm{D}u_n(x)|^p (M(|\mathrm{D}u_n|^p)(x))^\sigma.$$

Now, we are done with this term. Indeed, arguing as for $s=1$, up to extracting a subsequence, we can find an integrable Lebesgue hat $g \in L^p$ for $|\mathrm{D}u_n|$, and $M(g^p)$ will then be a Lebesgue hat for $M(|\mathrm{D}u_n|^p)$. By Hölder's inequality with conjugate exponents $s > 1$ and $\frac{s}{\sigma}$, and by the Hardy–Littlewood maximal inequality in L^s, the function $g^p(M(g^p))^\sigma$ is then integrable in x. We conclude as for the case $s=1$, using this Lebesgue hat to justify exchanging the limits and the integral in x and y, and deducing equi-integrability from the assumption $sp < \ell+1$ to justify the exchange of the limit and the integral with respect to a. We omit the details.

The term containing $B_n(x,y,a)$ is easier to handle. Indeed, by construction of P, we have

$$|\mathrm{D}P(u_n(y)-a)|^p \lesssim \frac{1}{|u_n-a|^p},$$

which is integrable, and even uniformly equi-integrable, with respect to a, while the term $|\mathrm{D}u_n(x) - \mathrm{D}u_n(y)|^p$, once we divide by $|x-y|^{m+\sigma p}$, will give exactly the integrand in the Gagliardo seminorm. We can therefore once again apply the partial converse of the dominated convergence theorem to obtain a Lebesgue hat for this term, and conclude with the help of the dominated convergence theorem and Vitali's convergence theorem. □

We have restricted the proof to $1 < s < 2$ to avoid excessive technicality, as this case already contains essentially all the ideas of the fractional case. If $s > 2$, the same argument works, one just has to deal with all the product of derivatives coming from the use of the Faà di Bruno to compute the k-th order derivative. Maybe surprisingly, the most significant new difficult comes from the B_n term, which will involve the difference between products of derivatives at points x and y. This term is handled via the following lemma.

Lemma 4.1.2. *Let $0 < \sigma < 1$ and $1 \leq p < +\infty$. For every $\alpha \in \{1, \dots, i\}$, let $v_\alpha \in L^{q_\alpha}$ be such that $\mathrm{D}v_\alpha \in L^{r_\alpha}$, with $1 < r_\alpha < q_\alpha$ such that*

$$\frac{1-\sigma}{q_\alpha} + \frac{\sigma}{r_\alpha} + \sum_{\substack{1 \leq \beta \leq i \\ \beta \neq \alpha}} \frac{1}{q_\beta} = \frac{1}{p}.$$

Then,

$$\prod_{\alpha=1}^{i} v_\alpha \in W^{\sigma,p}.$$

This lemma was proved by V. Maz'ya and T. Shaposhnikova [MS02], using only elementary Sobolev spaces techniques. As observed notably in [BPVS13], there is also a quick proof using the Triebel–Lizorkin machinery we introduced in Chapter 3. It is a good exercise to try doing this proof.

4.2 General singular projection

In the previous section, we have been working with the specific projection map $P\colon \mathbb{R}^{\ell+1} \setminus \{0\} \to \mathbb{S}^\ell$ defined as $P(x) = \frac{x}{|x|}$. However, as can be seen from the proof, the only features of P that we really used are that $P\colon \mathbb{R}^\nu \setminus \Sigma \to \mathscr{N}$ is a smooth map satisfying $P = \mathrm{id}$ on $\mathscr{N}$ and

$$|\mathrm{D}^j P(x)| \leq \frac{C_j}{\operatorname{dist}(x, \Sigma)^j} \quad \text{for every } j \in \mathbb{N}_*,$$

for some constant $C_j > 0$ depending on j, where $\Sigma \subset \mathbb{R}^\nu \setminus \mathscr{N}$ is a finite union of closedly embedded $(\nu - \ell - 1)$-submanifolds. A map satisfying such properties is called a *singular projection*.

We shall not repeat the proof of the counterpart of Theorem 4.1.1 in the case where $\mathbb{S}^\ell$ is replaced with a more general target admitting a singular projection, however, it is a good exercise to check how to implement the required modifications.

In this section, we rather focus on which manifolds do admit a singular projection. As we will see, this condition implies rather strong constraints on the topology of $\mathscr{N}$. We start with the following, rather natural, necessary condition.

Lemma 4.2.1. *If there exists a continuous map $P\colon \mathbb{R}^\nu \setminus \Sigma \to \mathscr{N}$ such that $P_{|\mathscr{N}} = \mathrm{id}_{\mathscr{N}}$, where Σ is a finite union of closedly embedded $(\nu-\ell)$-submanifolds of $\mathbb{R}^\nu$, then $\mathscr{N}$ is $(\ell-2)$-connected.*

Proof. The key observation is the fact that $\mathbb{R}^\nu \setminus \Sigma$ is $(\ell-2)$-connected. Taking this for granted, the conclusion follows directly from the fact that a retract of an $(\ell-2)$-connected space is itself $(\ell-2)$-connected. Namely, for every $i \in \{0,\dots,\ell-2\}$, we have the commutative diagram

$$\begin{tikzcd}
 & \{0\} = \pi_i(\mathbb{R}^\nu \setminus \Sigma) \arrow[dr, "P_*"] & \\
\pi_i(\mathscr{N}) \arrow[ur, hook] \arrow[rr, "\mathrm{id}_{\mathscr{N}}"] & & \pi_i(\mathscr{N})
\end{tikzcd}$$

for the maps induced between the homotopy groups. This implies that the identity map on the group $\pi_i(\mathscr{N})$ is the zero map, whence $\pi_i(\mathscr{N}) = \{0\}$ for every $i \in \{0,\dots,\ell-2\}$.

We now show that $\mathbb{R}^\nu \setminus \Sigma$ is $(\ell-2)$-connected. We show that, if Ω is an $(\ell-2)$-connected open subset of $\mathbb{R}^\nu$ and $\mathscr{M}$ a closedly embedded $(\nu-\ell)$-submanifold of Ω, then $\Omega \setminus \mathscr{M}$ is $(\ell-2)$-connected. The conclusion then follows by removing inductively each manifold constituting Σ, and using this claim at each step to show that the resulting set remains $(\ell-2)$-connected.

To prove the claim, let $i \in \{0,\dots,\ell-2\}$ and $f\colon \mathbb{S}^i \to \Omega \setminus \mathscr{M}$ be a continuous map. Since Ω is $(\ell-2)$-connected, there exists a continuous map $g\colon \overline{\mathbb{B}^{i+1}} \to \Omega$ such that $g = f$ on $\mathbb{S}^i$. Moreover, by a standard regularization process, we may assume that g is smooth on $\mathbb{B}^{i+1}$. Since $f(\mathbb{S}^i)$ is closed and $\Omega \setminus \mathscr{M}$ is open, there exists $\delta > 0$ sufficiently small such that, for any $a \in B_\delta$, $(f(\mathbb{S}^i) - a) \cap (\Omega \setminus \mathscr{M}) = \varnothing$. Now, by Sard's lemma, since $i+1 \leq \ell-1 < \ell$, for almost every $a \in B_\delta$, we have $g^{-1}(\mathscr{M}+a) = \varnothing$. This implies that, for any such $a \in B_\delta$, $g-a$ is a continuous extension to $\overline{\mathbb{B}^{i+1}}$ of $f-a$, whence $f-a$ is nullhomotopic. But by our choice of δ, $f-a$ and f are homotopic, which implies that f itself is nullhomotopic. This concludes the proof of the lemma. □

We now show that this condition is actually sufficient.

Lemma 4.2.2. *If $\mathscr{N}$ is $(\ell-2)$-connected, then it admits an ℓ-singular projection, whose singular set is the underlying set of a finite $(\nu-\ell)$-subskeleton in $\mathbb{R}^\nu$.*

For the sake of simplicity, we only explain how to construct a Lipschitz projection, also omitting technical details, only sketching the core of the construction. To make it smooth, more care is required, using the *thickening construction*, which will be presented in Chapter 6.

In the proof, we adopt the following convention for families of cubes: A straight letter indicates sets of cubes, and the corresponding cursive letter indicates the underlying set, made of the union of those cubes. Our terminology for skeletons will be introduced more carefully in Chapter 6, when we will make extensive use of grid decompositions.

Proof for a Lipschitz projection. We follow the approach in [VS19, Proposition 4.4]. Let $\iota > 0$ be sufficiently small so that the nearest point projection Π onto $\mathscr{N}$ is well-defined on $\mathscr{N}_\iota$. Let K^ν be a cubication of $\mathbb{R}^\nu$ of radius $r > 0$. Choosing $r > 0$ sufficiently small, we may find a subskeleton U^ν of K^ν such that $\mathscr{N} \subset \mathscr{U}^\nu \subset \mathscr{N} + B_{\iota/2}$. We let V^ν be the subskeleton of K^ν consisting in all cubes in K^ν that do not intersect some cube in U^ν, and $W^\nu = K^\nu \setminus (U^\nu \cup V^\nu)$.

We define P on $\mathscr{U}^\nu$ by $P = \Pi$, and on $\mathscr{V}^\nu$, we set $P = b$ for some arbitrary $b \in \mathscr{N}$. Hence, it remains to define P on $\mathscr{W}^\nu$. We proceed by induction. For any $\sigma^0 \in W^0$ such that P is not yet defined on σ^0, we let $P = b$ at σ^0. Then, for any $\sigma^1 \in W^1$ on which P is not yet defined, we use the assumption $\pi_0(\mathscr{N}) = \{0\}$ to extend P on σ^1 from its values on W^0. Repeating this process up to dimension $\ell - 1$, we define P on the whole $\mathscr{W}^{\ell-1}$. Finally, we use successive homogeneous extensions to extend P on $\mathscr{W}^\nu \setminus \mathscr{T}^{(\ell-1)^*}$, where $T^{(\ell-1)^*}$ is the dual skeleton to $W^{\ell-1}$. Here the homogeneous extension to $\overline{Q^i} \setminus \{0\}$ of a map f defined on ∂Q^i is given by $x \mapsto f(x/|x|_\infty)$. Hence, a first step of homogeneous extension allows us to define P on $\mathscr{W}^\ell$, with one singularity at the center of each ℓ-cell. A second step extends P on $\mathscr{W}^{\ell+1}$, with a singular set given by a finite union of segments, whose endpoints are located at the centers of the $(\ell+1)$-cells and at the centers of the ℓ-cells from the previous step. We pursue this construction up to dimension ν, each step increasing the dimension of the singular set by 1. By the properties of homogeneous extension, the map P that we constructed is indeed a singular projection, with singular set given by $\Sigma = \mathscr{T}^{(\ell-1)^*}$, which concludes the proof. □

4.3 The case $0 < s < 1$: A hint at new difficulties

The attentive reader will have noticed that Theorem 4.1.1 was only stated for $s \geq 1$. It would therefore be legitimate to ask what happens when $0 < s < 1$. Of course, the *statement* itself remains true, as it is a special case of the general strong approximation theorem stated as Theorem 1.2.6. However, this chapter is more about a proof via the method of singular projection than about the statement itself, so the question remains.

If $p < \ell + 1$, then it is easy to convince oneself — and not much more difficult to write a precise argument, see for instance [Vin25] — that the method still works. Indeed, one can readily bound the numerator of the Gagliardo seminorm of the projection by a quantity of the form $|u(x) - u(y)|^p$ times a power p of a term involving the derivative of the projection. This requires therefore $p < \ell + 1$ to be integrated with respect to the translation variable a. However, when $s \geq 1$, we have seen that the appropriate condition to ask is $sp \leq \ell + 1$. For $0 < s < 1$, this would be a less strict condition than $p < \ell + 1$.

But, in a somehow surprising way, the condition appearing in the range $0 < s < 1$ is indeed $p < \ell + 1$, not $sp < \ell + 1$. Even though constructing a counterexample will bring us a bit further apart from our main concern, which is about the strong approximation property, it is a very good excuse to introduce another very useful tool from geometric analysis, which plays a crucial role in the construction of a counterexample — and many others — namely, the *nonlinear uniform boundedness principle*. This will be the topic of our next chapter, before we come back to the strong approximation problem and give the idea of the proof in the general case in Chapter 6.

4.4 Comments and perspectives

4.4.1 At the origins of the method: Integral and normal currents

As we already mentioned, the averaging argument used in the implementation of the method of singular projection has roots in the work of H. Federer and W. Fleming [FF60] on currents. Let us be slightly more specific here. As this is just a comments section, we use some notions from geometric measure theory without introducing the required definitions.

One of the important results contained in the work of Federer and Fleming is the *deformation theorem*, which states that a normal current T can be written as $T = P + Q + \partial S$, where P is a *polyhedral* current, and where Q and S have small mass. In other words, any normal current can be written as a polyhedral current, plus a rest, which is made of a small current plus the boundary of a small current. This, in turn, is a key ingredient in the proof of the *polyhedral approximation theorem*, which then leads to the important closure and compactness theorems for normal currents.

The proof of the deformation relies on a similar projection and averaging method. Indeed, in short, the strategy is to divide the domain where T leaves into a grid of small mesh, and then to project T onto the k-skeleton of the mesh, where k is the dimension of T. This way, the resulting current is a k-current supported on a k-dimensional polyhedral set, and can therefore be shown to be a polyhedral current. But, doing this for an arbitrary grid will not give a controlled reminder. To obtain this conclusion, the idea introduced by Federer and Fleming is to average over all possible grids — by translation — and to prove an averaged estimate. This, in turn, implies that the projection centre can be chosen to obtain a controlled reminder.

4.4.2 Other uses of the method of singular projection

Extension of traces Just as we did for the strong approximation problem, the method of projection can be used in the study of the extension of traces problem. Namely, if $\mathscr{N}$ admits a singular projection $P\colon \mathbb{R}^\nu \setminus \Sigma \to \mathscr{N}$, to construct the trace extension of a mapping $u \in W^{1-1/p,p}(\partial\Omega;\mathscr{N})$, one may consider a linear extension $V \in W^{1,p}(\Omega;\mathscr{N})$, obtained by applying Theorem 2.1.6, and then consider the projection $P(V-a)+a$, where a is a suitable translation parameter obtained by averaging. After a small technical adjustment to correct the trace, one obtains an $\mathscr{N}$-valued trace extension of u. This is the original context in which the method of singular projection was introduced for Sobolev mappings, in the work of R. Hardt and Lin F. [HL87], see also [BD95]. Before the work of J. Van Schaftingen [VS24], which is to the extension of traces problem what Bethuel's 1991 paper [Bet91] is to the strong approximation problem, this was essentially the only known technique to prove extension of traces with constraint values.

More recently, this technique has been used by R. Caniato and T. Rivière [CR24] to study an approximation problem for weak Yang–Mills connections. The method is used to extend gauge changes, which are specific examples of mappings into a manifold, the target being a Lie group G. Importantly, in addition to the energy estimates inside the domain, they also required slicing-type estimates for the extension.

The singular set of Sobolev mappings In [PR03], M. R. Pakzad and T. Rivière defined the singular set of Sobolev mappings as a flat chain with values into the group $\pi_\ell(\mathscr{N})$ — see Section 2.5. Apart from the $(\ell-1)$-connectedness assumption in their work, which is inherent to their approach, they also needed an extra restriction on the possible dimensions they could consider, because of their use of a deep geometric measure theory result that is not available in arbitrary dimension. This extra assumption was removed by G. Canevari and G. Orlandi [CO19]. Crucial to their strategy of proof is to be able to define the singular set not only of $\mathscr{N}$-valued maps, but also of general $\mathbb{R}^\nu$-valued maps, as the singular set of their projection onto $\mathscr{N}$ via a singular projection. Also here, the method of singular projection turned out to be a crucial tool.

The weak approximation problem In [PR03], M. R. Pakzad and T. Rivière devised a new method to prove weak approximation of Sobolev mappings with smooth maps. Their approach is via minimal connections and dipole insertions, as we will explain in

Chapter 7. To implement this for more general targets than the sphere, they relied on the method of singular projection in the following way. First, they wrote the target $\mathscr{N}$ as a cellular complex — this can always be done for smooth manifolds. Then, viewing a j-cell of this complex as the ball $\mathbb{B}^j$, one may consider the projection $\mathbb{B}^j \to \mathbb{S}^{j-1}$, $x \mapsto \frac{x}{|x|}$. Composing a map with this projection yields an $\mathbb{S}^{j-1}$-valued map. Doing this for every j-cell allows to transform a map valued into the j -skeleton to a map valued into the $(j-1)$-skeleton. Iterating this construction from j being the dimension of $\mathscr{N}$ to $j = \ell + 1$, one ends up with a $W^{1,\ell}$ map valued into the ℓ-skeleton of $\mathscr{N}$. But now, under suitable assumptions, this ℓ-skeleton is a bouquet of spheres, and one can the apply the theory available for sphere-valued maps to construct minimal connections for the map. To ensure suitable estimates at each step of this dimension reduction, the method of projection was used once more, choosing the centres of projection by an averaging argument.

Competitors for minimising harmonic maps Also in the study of minimising harmonic maps, the method of projection has proved its usefulness. Typically, in the study of such variational problems, once may be willing to construct competitors from a prescribed map, by modifying it on a given region. This can involve filling in a disk with a harmonic extension, or gluing two different competitors via a cut-off procedure. In both case, this is a linear construction, which will therefore not be compatible with the constraint that the map has to be $\mathscr{N}$-valued. But, if the target admits a singular projection, this can be corrected by reprojecting the linear construction back onto the target.

This strategy was notably used by R. Hardt and Lin F. [HL87] to obtain compactness of minimisers, or by R. Hardt, D. Kinderlehrer, and Lin F. [HKL86] to obtain energy estimates for minimisers on balls. We also refer the reader to [MMS18] for a survey of such results — and others — and to [Gas16] for higher-order results and [Vin25] for fractional results.

4.4.3 The method of almost projection

The method of almost projection is an analogous technique, which was first introduced by P. Hajłasz [Haj94] — with roots in the work of F. Bethuel and Zheng X. [BZ88] in the case of sphere-valued maps — to give a simpler proof of Theorem 1.2.5 in the case where the target is $\lfloor p \rfloor$-connected. Compared with the general case of Theorem 1.2.5 for $s = 1$, one requires not only $\pi_{\lfloor p \rfloor}$ to be trivial, but also all the lower order homotopy groups. We also refer to [BPVS14] for higher-order results.

Unlike singular projections, which is the identity on *the whole* $\mathscr{N}$ but displays a singular set, almost projections are well-defined and smooth on the whole ambient space $\mathbb{R}^\nu$, but in exchange, they fail to coincide with the identity on a small subset of $\mathcal{N}$. Then, if P is an almost projection and if, for instance $(u_n)_{n\in\mathbb{N}}$ is a sequence converging towards u, it is straightforward to obtain the convergence of the $P \circ u_n$ towards $P \circ u$, but one needs to estimate the distance between $P \circ u$ and u itself. This is also done by the means of an averaging argument, similar as for the method of singular projection.

5 The Swiss-knife tool to produce analytical obstructions: The nonlinear uniform boundedness principle

In this chapter, we present the general version of the *nonlinear uniform boundedness principle* to produce *analytical obstructions* to several types of problems. This principle was already present in the work of Hang F. and Lin F. [HL03b] on the weak approximation problem. It subsequently showed up in the study of other problems, such as the extension of traces [Bet14] and the lifting [MVS21a]. A general version of this principle was formalised by A. Monteil and J. Van Schaftingen [MVS19]. A loose way to state the principle is as follows: Under some suitable assumptions, to rule out a qualitative property, it suffices to rule out the associated quantitative property. In this chapter, we explain the general version of the principle, following closely Monteil and Van Schaftingen, and use as an excuse the applicability of the method of singular projection to give an illustration of it, following [Det25b].

5.1 A nonlinear version of the Banach–Steinhaus theorem

Let us come back to the method of singular projection. The question that we left at the end of Chapter 4 is the following: Assuming that $sp < \ell + 1$ and $0 < s < 1$, can we apply the method of singular projection in $W^{s,p}(\mathbb{B}^m; \mathbb{S}^\ell)$ with $P(x) = \frac{x}{|x|}$. Before even talking about convergence and technical details, a first question that arises is the well-definiteness of the projection. That is, if $u \in W^{s,p}(\mathbb{B}^m; \mathbb{R}^{\ell+1})$, can we find $a \in \mathbb{B}^{\ell+1}$ such that $P \circ (u - a) \in W^{s,p}(\mathbb{B}^m; \mathbb{S}^\ell)$?

There is a quantitative version of this problem. Namely, one can define the best energy of projection as

$$\mathcal{G}(u) = \inf_{a \in \mathbb{B}^{\ell+1}} |P \circ (u - a)|^p_{W^{s,p}(\mathbb{B}^m)},$$

and ask whether or not it is possible to get an estimate of $\mathcal{G}(u)$ in terms of $|u|^p_{W^{s,p}(\mathbb{B}^m)}$. What our proof of Theorem 4.1.1 actually show is that, when $s = 1$, a *linear* estimate does hold:

$$\mathcal{G}(u) \leq C|u|^p_{W^{s,p}(\mathbb{B}^m)}.$$

The nonlinear uniform boundedness principle asserts that, for a large class of problems with an associated energy, a similar relationship can be established. First, let us define precisely the notion of energy we are using. We call an *energy* with state space $\mathcal{N}$ a map $\mathcal{G}$ which sends an open set $A \subset \mathbb{R}^m$ and a measurable map $u\colon A \to \mathcal{N}$ to a number $\mathcal{G}(u, A) \in [0, +\infty]$ in such a way that the following monotonicity property holds: if $A \subset B$ are open and if $u\colon B \to \mathcal{N}$ is measurable, then

$$\mathcal{G}(u_{|A}, A) \leq \mathcal{G}(u, B).$$

We shall omit to write the restriction in the sequel when it is clear from the context.

We also consider the two following important properties for the energy:

1. (superadditivity) if A and B are open with disjoint closures, then
$$\mathcal{G}(u, A \cup B) \geq \mathcal{G}(u, A) + \mathcal{G}(u, B);$$
2. (scaling and translation with parameters s and p) for every $\lambda > 0$, $h \in \mathbb{R}^m$, and $A \subset \mathbb{R}^m$ open,
$$\mathcal{G}(u, h + \lambda A) = \lambda^{m-sp}\mathcal{G}(u, A).$$

Let us observe that the scaling and translation property is exactly the one satisfied by the Sobolev energy. Moreover, the least energy of projection that we introduced above indeed satisfies all these assumptions, as is readily checked.

Also, to have somehow uniform notation, we denote by $\mathcal{E}^{s,p}(u, A)$ the Sobolev energy of the map u over the set A, that is, its Gagliardo seminorm if $0 < s < 1$, and the L^p norm of its gradient if $s = 1$.

With these preliminaries, we are in position to state the nonlinear uniform boundedness principle.

Theorem 5.1.1. *Let $0 < s \leq 1$ and $1 \leq p < +\infty$ be such that $sp < m$. Let $\mathcal{G}$ be an energy over state space $\mathcal{N}$, where $\mathcal{N}$ is a connected Riemannian manifold (here, we do not assume compactness!), satisfying superadditivity and scaling and translation with parameters s and p. If for every measurable function $u\colon \mathbb{B}^m \to \mathcal{N}$, $\mathcal{E}^{s,p}(u, \mathbb{B}^m) < +\infty$ implies $\mathcal{G}(u, \mathbb{B}^m) < +\infty$ and $\mathcal{E}^{s,p}(u, \mathbb{B}^m) = 0$ implies $\mathcal{G}(u, \mathbb{B}^m) = 0$, then there exists a constant $C > 0$ such that*

$$\mathcal{G}(u, \mathbb{B}^m) \leq C\mathcal{E}^{s,p}(u, \mathbb{B}^m) \quad \textit{for every measurable map } u\colon \mathbb{B}^m \to \mathcal{N}.$$

This theorem is often used in its contrapositive version: The failure of a linear estimate $\mathcal{G}(u, \mathbb{B}^m) \leq C\mathcal{E}^{s,p}(u, \mathbb{B}^m)$ for some constant $C > 0$ implies the existence of a map such that $\mathcal{E}^{s,p}(u, \mathbb{B}^m) < +\infty$ but $\mathcal{G}(u, \mathbb{B}^m) = +\infty$. This is also how the result is proved.

The basic idea behind the proof is strikingly simple. Assuming that the linear estimate fails, one can find a sequence $(u_n)_{n\in\mathbb{N}}$ whose $\mathcal{G}$ energy blows up superlinearly with respect to their Sobolev energy. Using the scaling property, we can scale those maps to very small balls in order to make the Sobolev energies summable, and the $\mathcal{G}$ energies diverging to $+\infty$. Patching all those scaled versions of the u_n together, using the monotonicity property, should yield the desired counterexample, a map with finite Sobolev energy but infinite $\mathcal{G}$ energy.

However, the devil lies in the details. In particular, there is no reason for the u_n to have compatible values at the boundary in any way, so that gluing them is compromised. This issue is bypassed in two steps. First, we modify those maps to make them constant on the boundary, by the means of the *opening* procedure — another very useful tool, that this chapter gives us the opportunity to present. But this is still not enough, as these boundary values need not be the same. We would like to correct this by adding a kind of collar to the maps to bring them to the same constant. However, this is going to cost some energy, which might ruin the efforts we have made so far to adjust the Sobolev and $\mathcal{G}$ energies with respect to each other. We overcome this difficulty by clustering together several copies of the opened versions of each fixed u_n to increase their energy until the one of the collar become negligible in comparison. This is where the superadditivity assumption is used.

Finally, a last difficulty arises when $0 < s < 1$, namely, the nonlocal character of the Gagliardo seminorm. That is, the Gagliardo seminorm on the union of two domains is not the sum of the Gagliardo seminorms on each of these domains, unlike what happens for the norm of the gradient. This makes every patching step, where an estimate has to be provided for the energy of several maps glued together, much difficult to perform. This is a recurrent difficulty when working with nonlocal quantities such as the Gagliardo seminorm, and we take the opportunity of this chapter to illustrate some classical techniques to deal with it.

The next sections are devoted to the presentation of the aforementioned tools, before we move to the proof of Theorem 5.1.1.

5.2 Opening

The goal of this section is to introduce the opening procedure. Let us start with a naive idea. We are given a map $u \in W^{s,p}(B_2;\mathcal{N})$, and we would like to make it constant near ∂B_2, without modifying it on $\mathbb{B}^m$. Because of the constraint that our map should be valued into $\mathcal{N}$, we *cannot* rely on a standard cut-off procedure. An idea could be to make modifications "on the domain". That is, take $\varphi\colon B_2 \to B_2$ a Lipschitz map such that $\varphi(x) = x$ on $\mathbb{B}^m$ and $\varphi = 0$ on a neighbourhood of ∂B_2. We would like to consider the map $u \circ \varphi$, which would indeed coincide with u on $\mathbb{B}^m$ and be constantly equal to $u(0)$ near ∂B_2. The problem is that we are not composing with a diffeomorphism — and we cannot, as we want to make the composition constant on an open set — so that there is no guarantee that this composition preserves Sobolev regularity. Actually, the value $u(0)$ could not be representative at all of the values nearby, hence blowing up the energy of the composition.

This is somehow a very similar problem to the one we encountered with the method of singular projection — even though here the order of composition is reversed — and we are going to solve it with a similar idea, namely, translating the map we compose with, and using an averaging argument. This is the content of the opening construction, that we now present with a precise statement below.

Proposition 5.2.1. *Let $0 < s \leq 1$ and $1 \leq p < +\infty$. Let $\lambda > 1$ and $\eta \in (0, \lambda)$. For every $\rho > 0$, every measurable map $u \colon B_{\lambda\rho} \to \mathcal{N}$, and every Lipschitz map $\varphi \colon B_{(1+\eta)\rho} \to B_{(\lambda-\eta)\rho}$, there exists a point $a \in B_{\eta\rho}$ such that*

$$\mathcal{E}^{s,p}(u \circ (\varphi(\cdot - a) + a), B_\rho) \leq C \operatorname{Lip}(\varphi)^{sp} \mathcal{E}^{s,p}(u, B_{\lambda\rho}),$$

for some constant $C > 0$ depending on s, p, m, λ, and η.

We insist on the fact that the point a depends on the map u itself. Even more, modifying u only on a null set could change the point a. In particular, in this setting, we are working with a representative of the map u, not with the full equivalence class.

Compared to our informal introduction, the parameters η and λ serve to arrange some margin of security for our constructions. In particular, the choice of the sizes of all balls implies that all compositions are always well-defined, for every possible choice of a.

Proof. The proof of the case $s = 1$ already contains all the ideas of the averaging argument at work. However, the proof of the fractional case $0 < s < 1$ contains important technical ideas to deal with the Gagliardo seminorm, so that we present it in details as well.

It suffices to prove an averaged estimate of the form

$$\fint_{B_{\eta\rho/2}} \mathcal{E}^{s,p}(u \circ \varphi_a, B_\rho) \, \mathrm{d}a \lesssim \operatorname{Lip}(\varphi)^{sp} \mathcal{E}^{s,p}(u, B_{\lambda\rho}),$$

where we have written $\varphi_a = \varphi(\cdot - a) + a$ for the conciseness of notation. Indeed, from this averaged estimate, it is readily deduced that we can choose $a \in B_{\eta\rho/2}$ out of a subset of $B_{\eta\rho/2}$ of positive measure in order to reach the conclusion. We can also take the measure of this set to be an arbitrarily high fraction of the total measure of $B_{\eta\rho/2}$, the price being that the hidden constant in the estimate on $\mathcal{E}^{s,p}(u \circ \varphi_a, B_\rho)$ will increase accordingly.

We start with $s = 1$. The chain rule implies that $|\mathrm{D}(u \circ \varphi_a)| \le \mathrm{Lip}(\varphi)|\mathrm{D}u| \circ \varphi_a$. (Actually, one should justify the appropriate Sobolev regularity allowing to apply the chain rule. This can be done via a genericity argument, the reasoning that follows also allowing that almost every choice of a will give an admissible map.) By definition of the map φ_a, by the change of variable $y = x - a$, and by Tonelli's theorem, we find

$$\begin{aligned}\int_{B_{\eta\rho}} \mathcal{E}^{1,p}(u \circ \varphi_a, B_\rho)\,\mathrm{d}a &\le \mathrm{Lip}(\varphi)^p \int_{B_{\eta\rho}} \int_{B_\rho} |\mathrm{D}u(\varphi(x-a)+a)|^p\,\mathrm{d}x\,\mathrm{d}a \\ &\le \mathrm{Lip}(\varphi)^p \int_{B_{\eta\rho}} \int_{B_{(1+\eta)\rho}} |\mathrm{D}u(\varphi(y)+a)|^p\,\mathrm{d}y\,\mathrm{d}a \\ &\le \mathrm{Lip}(\varphi)^p \int_{B_{(1+\eta)\rho}} \int_{B_{\eta\rho}(\varphi(y))} |\mathrm{D}u(a)|^p\,\mathrm{d}a\,\mathrm{d}y.\end{aligned}$$

Therefore, we arrive at

$$\int_{B_{\eta\rho}} \mathcal{E}^{1,p}(u \circ \varphi_a, B_\rho)\,\mathrm{d}a \le \mathrm{Lip}(\varphi)^p |B_{(1+\eta)\rho}| \mathcal{E}^{1,p}(u, B_{\lambda\rho}).$$

The conclusion follows — with a hidden constant depending on η — by the comparison between the volume of $B_{(1+\eta)\rho}$ and $B_{\eta\rho}$.

We now turn to the case $0 < s < 1$, which is technically more demanding because of the presence of the Gagliardo seminorm. The trick here — which is reminiscent of the proof of Besov's lemma [Ada75, Proof of 44. Lemma] — is to insert an extra z variable via the triangle inequality, and to average over this extra variable. Namely, we introduce the domain

$$B_{x,y,a} = B_{\frac{|\varphi(x)-\varphi(y)|}{\beta}}\Big(\frac{\varphi(x)+\varphi(y)}{2}\Big), \quad \text{where } \beta = \frac{4\lambda}{\eta} - 2,$$

and we write

$$|u \circ \varphi_a(x) - u \circ \varphi_a(y)|^p \lesssim |u \circ \varphi_a(x) - u \circ \varphi_a(z)|^p + |u \circ \varphi_a(z) - u \circ \varphi_a(y)|^p.$$

Let us first observe that, for x, $y \in B_\rho$ and for $a \in B_{\eta\rho/2}$, we have $|\varphi_a(x) \pm \varphi_a(y)| \le \rho(2\lambda - \eta)$, whence

$$\frac{|\varphi(x)-\varphi(y)|}{\beta} + \frac{|\varphi(x)+\varphi(y)|}{2} \le \rho(2\lambda-\eta)\Big(\frac{1}{2} + \frac{1}{\beta}\Big) = \lambda\rho,$$

implying that $B_{x,y,a} \subset B_{\lambda\rho}$. Averaging over $z \in B_{x,y,a}$ — and using the symmetry of the integrand — we obtain

$$\mathcal{E}^{s,p}(u, B_\rho) \lesssim \int_{B_\rho} \int_{B_\rho} \fint_{B_{x,y,a}} \frac{|u \circ \varphi_a(x) - u(z)|^p}{|x-y|^{m+sp}} \,\mathrm{d}z \,\mathrm{d}x \,\mathrm{d}y.$$

To get rid of the volume of the set $B_{x,y,z}$, we observe that, for $z \in B_{x,y,a}$, it holds that

$$\begin{aligned} |\varphi_a(x) - z| &\leq \left|\frac{\varphi_a(x) - \varphi_a(y)}{2}\right| + \left|\frac{\varphi_a(x) + \varphi_a(y)}{2} - z\right| \\ &\leq \left(\frac{1}{2} + \frac{1}{\beta}\right)|\varphi_a(x) - \varphi_a(y)| = \frac{\lambda}{2\eta - \lambda}|\varphi_a(x) - \varphi_a(y)|. \end{aligned}$$

Hence,

$$\mathcal{E}^{s,p}(u, B_\rho) \lesssim \int_{B_\rho} \int_{B_\rho} \int_{B_{x,y,a}} \frac{|u \circ \varphi_a(x) - u(z)|^p}{|\varphi_a(x) - z|^m |x-y|^{m+sp}} \,\mathrm{d}z \,\mathrm{d}x \,\mathrm{d}y.$$

Letting $Y_{a,x,z}$ be the set of all y such that $z \in B_{x,y,a}$, we can use Tonelli's theorem to rewrite the above as

$$\mathcal{E}^{s,p}(u, B_\rho) \lesssim \int_{B_{\lambda\rho}} \int_{B_\rho} \int_{Y_{a,x,z}} \frac{|u \circ \varphi_a(x) - u(z)|^p}{|\varphi_a(x) - z|^m |x-y|^{m+sp}} \,\mathrm{d}y \,\mathrm{d}x \,\mathrm{d}z.$$

Observing that $|\varphi_a(x) + \varphi_a(y) - 2z| \geq 2|\varphi_a(x) - z| - |\varphi_a(x) - \varphi_a(y)|$, we can write

$$Y_{a,x,z} \subset \{y \in \mathbb{R}^m : |\varphi_a(x) - z| \leq C\mathrm{Lip}(\varphi)|x-y|\},$$

so that the inner integral with respect to y can be estimated as

$$\mathcal{E}^{s,p}(u, B_\rho) \lesssim \mathrm{Lip}(\varphi)^{sp} \int_{B_{\lambda\rho}} \int_{B_\rho} \frac{|u \circ \varphi_a(x) - u(z)|^p}{|\varphi_a(x) - z|^{m+sp}} \,\mathrm{d}x \,\mathrm{d}z.$$

We now integrate over $a \in B_{\eta\rho/2}$ and use the changes of variable $y = x - a \in B_{(1+\eta/2)\rho}$ and $w = \varphi(y) + a \in B_{(\lambda-\eta/2)\rho}$ to find

$$\begin{aligned} \int_{B_{\eta\rho/2}} \mathcal{E}^{s,p}(u, B_\rho) \,\mathrm{d}a &\lesssim \mathrm{Lip}(\varphi)^{sp} \int_{B_{\eta\rho/2}} \int_{B_{\lambda\rho}} \int_{B_{(1+\eta/2)\rho}} \frac{|u(\varphi(y) + a) - u(z)|^p}{|\varphi(y) + a - z|^{m+sp}} \,\mathrm{d}y \,\mathrm{d}z \,\mathrm{d}a \\ &\leq \mathrm{Lip}(\varphi)^{sp} \int_{B_{(1+\eta/2)\rho}} \int_{B_{(\lambda-\eta/2)\rho}} \int_{B_{\lambda\rho}} \frac{|u(w) - u(z)|^p}{|w - z|^{m+sp}} \,\mathrm{d}z \,\mathrm{d}w \,\mathrm{d}y \\ &\leq \mathrm{Lip}(\varphi)^{sp} |B_{(1+\eta/2)\rho}| \mathcal{E}^{s,p}(u, B_{\lambda\rho}). \end{aligned}$$

This yields the conclusion as in the case $s = 1$. □

For historical notes, let us mention that the original idea of the opening was introduced by H. Brezis and Li Y. [BL01]. Here, the idea was to open a map around a point. The construction was later on generalised by P. Bousquet, A. Ponce, and J. Van Schaftingen [BPVS15, BPVS25] to deal with openings defined as the composition with a more general map. Here, we followed the presentation by A. Monteil and J. Van Schaftingen [MVS19], who were also the first to consider the opening for fractional Sobolev maps.

5.3 Patching Sobolev mappings

In this section, we present some lemma that serve to estimate the Gagliardo seminorm of several maps glued together. As we already said, such difficulties are typical when working with nonlocal quantities, and are not encountered with, for instance, the norm of the gradient.

The first lemma explains how to glue to maps, without any kind of compatibility assumption. The price to pay is that we need to prescribe some margin of security, and that the estimate that we obtain comes with a constant that deteriorates as this margin of security shrinks.

Lemma 5.3.1. *Let $0 < s \leq 1$ and $1 \leq p < +\infty$. For every $0 < \eta < 1$, every open set $A \subset \mathbb{R}^m$, every measurable map $u \colon A \to \mathscr{N}$, and every $\rho > 0$ such that $B_\rho \setminus B_{\eta\rho}$, it holds*

$$\mathcal{E}^{s,p}(u, A) \leq \Big(1 + \frac{C}{(1-\eta)^{1+sp}}\Big)\mathcal{E}^{s,p}(u, A \cap B_\rho) + \Big(1 + \frac{C\eta^m}{1-\eta}\Big)\mathcal{E}^{s,p}(u, A \setminus B_{\eta\rho}),$$

for some constant $C > 0$ depending only on m, s, and p.

Proof. For $s = 1$, the conclusion follows directly from the additivity of the integral — and one could even take $C = 0$, so that the constants would not blow up as $\eta \to 0$.

Let us now consider $0 < s < 1$. Applying again the additivity of the integral, we get

$$\mathcal{E}^{s,p}(u, A) \leq \mathcal{E}^{s,p}(u, A \cap B_\rho) + \mathcal{E}^{s,p}(u, A \setminus B_{\eta\rho}) + 2\int_{A \setminus B_\rho}\int_{A \cap B_{\eta\rho}} \frac{|u(x) - u(y)|^p}{|x - y|^{m+sp}}\,\mathrm{d}x\,\mathrm{d}y.$$

Hence, we just have to estimate the last double integral on the right. We again introduce an extra variable $z \in B_\rho \setminus B_{\eta\rho}$ over which we average, getting

$$\begin{aligned}\int_{A\setminus B_\rho}\int_{A\cap B_{\eta\rho}} &\frac{|u(x)-u(y)|^p}{|x-y|^{m+sp}}\,\mathrm{d}x\,\mathrm{d}y\\ &\lesssim \fint_{B_\rho\setminus B_{\eta\rho}}\int_{A\setminus B_\rho}\int_{A\cap B_{\eta\rho}} \frac{|u(x)-u(z)|^p}{|x-y|^{m+sp}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\ &\quad+\fint_{B_\rho\setminus B_{\eta\rho}}\int_{A\setminus B_\rho}\int_{A\cap B_{\eta\rho}} \frac{|u(z)-u(y)|^p}{|x-y|^{m+sp}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.\end{aligned}$$

For the first integral, we observe that $|x-y| \geq (1-\eta)\rho$ and first integrate over y, which yields

$$\begin{aligned}\fint_{B_\rho\setminus B_{\eta\rho}}\int_{A\setminus B_\rho}\int_{A\cap B_{\eta\rho}} &\frac{|u(x)-u(z)|^p}{|x-y|^{m+sp}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\ &\lesssim \frac{1}{(1-\eta)^{sp}\rho^{sp}}\fint_{B_\rho\setminus B_{\eta\rho}}\int_{A\cap B_{\eta\rho}}|u(x)-u(z)|^p\,\mathrm{d}x\,\mathrm{d}z.\end{aligned}$$

For the observation that, for $x \in A\cap B_{\eta\rho}$ and $z \in B_\rho\setminus B_{\eta\rho}$ it holds $|x-z| \leq 2\rho$, we get

$$\fint_{B_\rho\setminus B_{\eta\rho}}\int_{A\setminus B_\rho}\int_{A\cap B_{\eta\rho}} \frac{|u(x)-u(z)|^p}{|x-y|^{m+sp}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \lesssim \frac{1}{(1-\eta)^{sp+1}\rho^{sp}}\mathcal{E}^{s,p}(u, A\cap B_\rho).$$

For the second integral, we use the inequality $|x-y| \geq |y| - \eta\rho$ to find

$$\begin{aligned}\fint_{B_\rho\setminus B_{\eta\rho}}\int_{A\setminus B_\rho}\int_{A\cap B_{\eta\rho}} &\frac{|u(z)-u(y)|^p}{|x-y|^{m+sp}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\ &\lesssim \eta^m\rho^m\fint_{B_\rho\setminus B_{\eta\rho}}\int_{A\setminus B_\rho}\frac{|u(z)-u(y)|^p}{(|y|-\eta\rho)^{m+sp}}\,\mathrm{d}y\,\mathrm{d}z.\end{aligned}$$

But now, we also have

$$|y-z| \leq \operatorname{dist}(y, B_{\eta\rho}) + \operatorname{dist}(z, B_{\eta\rho}) \leq 2(|y|-\eta\rho).$$

This yields

$$\fint_{B_\rho\setminus B_{\eta\rho}}\int_{A\setminus B_\rho}\int_{A\cap B_{\eta\rho}} \frac{|u(z)-u(y)|^p}{|x-y|^{m+sp}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \lesssim \frac{\eta^m}{1-\eta}\mathcal{E}^{s,p}(u, A\setminus B_{\eta\rho}),$$

and we have reached the conclusion. □

Apart from the issue coming from the margin of security, Lemma 5.3.1 has another drawback: If one is willing to iterate it to get a gluing estimate for several maps, the constants will blow up with the number of maps under consideration. We therefore present another gluing lemma, which allows to patch countably many Sobolev maps, but requires a compatibility condition.

Lemma 5.3.2. *Let $0 < s \leq 1$, $1 \leq p < +\infty$, and $\Omega \subset \mathbb{R}^m$ be an open set. Let I be a finite or countable set, and for every $i \in I$, let $u_i : \Omega \to \mathcal{N}$ be a measurable map. Assume there exists a family $\{A_i\}_{i \in I}$ of open subsets of Ω with disjoint closures and a point $b \in \mathcal{N}$ such that $u_i = b$ outside of A_i for every $i \in I$. Define*

$$u = \begin{cases} u_i & \text{on } A_i, \\ b & \text{otherwise.} \end{cases}$$

Then,

$$\mathcal{E}^{s,p}(u, \Omega) \leq C \sum_{i \in I} \mathcal{E}^{s,p}(u_i, \Omega),$$

with $C = 1$ if $s = 1$, and $C = 2^p$ otherwise.

We emphasise that it is crucial to consider the $\mathcal{E}^{s,p}$ of the maps u_i on the *whole* Ω, not only on A_i. One can then reduce this domain by Lemma 5.3.1, but a margin of security around A_i will still be required.

Proof. The case $s = 1$ is essentially a straightforward consequence of the additivity of the integral (though, to be very precise, one needs to justify the weak differentiability).

Let us turn to $0 < s < 1$. By the additivity of the integral, we get

$$\mathcal{E}^{s,p}(u, \Omega) \leq \sum_{i \in I} \mathcal{E}^{s,p}(u, A_i) + \sum_{\substack{i,j \in I \\ i \neq j}} \int_{A_i} \int_{A_j} \frac{|u(x) - u(y)|^p}{|x - y|^{m+sp}} \, \mathrm{d}y \, \mathrm{d}x \\ + 2 \sum_{i \in I} \int_{A_i} \int_{\Omega \setminus \bigcup_{j \in I} A_j} \frac{|u(x) - u(y)|^p}{|x - y|^{m+sp}} \, \mathrm{d}y \, \mathrm{d}x.$$

On the first and last term, u may be replaced by u_i by construction of u. It remains to deal with the second term. For this, we use the triangle inequality to introduce an extra

b, and get

$$\int_{A_i}\int_{A_j}\frac{|u(x)-u(y)|^p}{|x-y|^{m+sp}}\,\mathrm{d}x\,\mathrm{d}y \leq 2^{p-1}\left(\int_{A_i}\int_{A_j}\frac{|u(x)-b|^p}{|x-y|^{m+sp}}\,\mathrm{d}y\,\mathrm{d}x+\int_{A_i}\int_{A_j}\frac{|b-u(y)|^p}{|x-y|^{m+sp}}\,\mathrm{d}y\,\mathrm{d}x\right).$$

In the first integral, we have $u(x) = u_i(x)$ and $b = u_i(y)$, and in the second, same with j. Gathering all inequalities then yields the conclusion. □

5.4 Proof of Theorem 5.1.1

We now exploit the tools we have introduced so far to present the proof of the nonlinear uniform boundedness principle stated as Theorem 5.1.1.

Proof of Theorem 5.1.1. We divide the proof into several steps.

Step 1. — Construction of a sequence with superlinearly growing energy.

As announced, we argue by contraposition. Therefore, the assumption that the linear bound fails implies that, for each $n \in \mathbb{N}$, we can find a measurable map $u_n\colon \mathbb{B}^m \to \mathscr{N}$ such that

$$0 < \mathcal{E}^{s,p}(u,\mathbb{B}^m) \leq \mu^{-n}\mathcal{G}(u_n,\mathbb{B}^m) < +\infty,$$

where $\mu > 1$ is to be suitably chosen later on.

Step 2. — Opening the maps.

We now open the maps u_n in order to make them constant on the boundary of a ball, to be able to glue them together later on. But first, by a standard extension procedure — for instance, by reflection — we can construct maps $u_n^{\mathrm{ext}}\colon B_3 \to \mathscr{N}$ satisfying $u_n^{\mathrm{ext}} = u_n$ on $\mathbb{B}^m$ and

$$\mathcal{E}^{s,p}(u_n^{\mathrm{ext}}, B_3) \lesssim \mathcal{E}^{s,p}(u_n,\mathbb{B}^m).$$

Now, we pick a Lipschitz map $\varphi\colon B_6 \to B_2$ such that $\varphi(x) = x$ if $|x| \leq 2$ and $\varphi(x) = 0$ if $|x| \geq 3$, and we apply Proposition 5.2.1 with parameters $\rho = 5$, $\eta = \frac{1}{5}$, and $\lambda = \frac{3}{5}$, and we obtain the existence of a point $a_n \in \mathbb{B}^m$ such that, if we define

$$u_n^{\mathrm{op}}(x) = \begin{cases} u_n^{\mathrm{ext}}(\varphi(x-a_n)+a_n) & \text{if } |x| \leq 4,\\ b_n & \text{otherwise,}\end{cases}$$

then

$$\mathcal{E}^{s,p}(u_n^{\mathrm{op}}, B_5) \lesssim \mathcal{E}^{s,p}(u_n^{\mathrm{ext}}, B_3).$$

Moreover, by our choice of φ, we have $u_n^{\mathrm{op}} = u_n$ on $\mathbb{B}^m$ and $u_n^{\mathrm{op}} = b_n$ outside of B_4. In particular, applying Lemma 5.3.1, we get

$$\mathcal{E}^{s,p}(u_n^{\mathrm{op}}, \mathbb{R}^m) \lesssim \mathcal{E}^{s,p}(u_n^{\mathrm{op}}, B_5).$$

We conclude that

$$\mathcal{E}^{s,p}(u_n^{\mathrm{op}}, \mathbb{R}^m) \lesssim \mu^{-n} \mathcal{G}(u_n^{\mathrm{op}}, \mathbb{B}^m). \tag{5.4.1}$$

Step 3. — Clustering the maps.

We now with to correct the fact that the constant value b_n on the boundary depends on n. For that, we will add a collar around our maps to change this values. To ensure that the energy of this collar does not affect significantly the energy of our maps, we will cluster several of them to increase their energy.

We fix a point $b \in \mathcal{N}$. By connectedness assumption, there exists a smooth map $v_n \colon \mathbb{R}^m \to \mathcal{N}$ such that $v_n = b_n$ on $B_{\frac{1}{2}}$ and $v_n = b$ outside of B_1. Moreover,

$$\mathcal{E}^{s,p}(v_n, \mathbb{R}^m) < +\infty$$

— this can be proved e.g. again using Lemma 5.3.1. For $k \in \mathbb{N}_*$, we can fit inside $B_{\frac{1}{2}}$ a set P_k of k^m points — for instance, along a grid — in such a way that the balls $B_{1/(2k\sqrt{m})}(c)$ for $c \in P_k$ are disjoint subsets of $B_{\frac{1}{2}}$. For each $c \in P_k$, we define $v_{n,k,c}(x) = u_n^{\mathrm{op}}(16k\sqrt{m}(x-c))$. Finally, we define

$$v_{n,k}(x) = \begin{cases} v_{n,k,c}(x) & \text{if } x \in B_{1/(2k\sqrt{m})}(c), \\ b_n & \text{otherwise.} \end{cases}$$

By virtue of Lemma 5.3.2, we have

$$\begin{aligned} \mathcal{E}^{s,p}(v_{n,k}, \mathbb{R}^m) &\lesssim \mathcal{E}^{s,p}(v_n, \mathbb{R}^m) + \sum_{c \in P_k} \mathcal{E}^{s,p}(v_{n,k,c}, \mathbb{R}^m) \\ &= \mathcal{E}^{s,p}(v_n, \mathbb{R}^m) + \frac{k^{sp}}{(16\sqrt{m})^{m-sp}} \mathcal{E}^{s,p}(u_n^{\mathrm{op}}, \mathbb{R}^m). \end{aligned}$$

On the other hand, we can write

$$\mathcal{E}^{s,p}(v_{n,k},\mathbb{R}^m) \geq \sum_{c\in P_k} \mathcal{E}^{s,p}(v_{n,k,c}, B_{1/(16k\sqrt{m})}(c)) = \frac{k^{sp}}{(16\sqrt{m})^{m-sp}}\mathcal{E}^{s,p}(u_n^{\mathrm{op}},\mathbb{B}^m).$$

As $\mathcal{E}^{s,p}(u_n^{\mathrm{op}},\mathbb{B}^m) > 0$, we can pick $k \in \mathbb{N}_*$ — depending on n — in such a way that

$$\nu \leq \mathcal{E}^{s,p}(v_{n,k},\mathbb{R}^m) \lesssim \frac{k^{sp}}{(16\sqrt{m})^{m-sp}}\mathcal{E}^{s,p}(u_n^{\mathrm{op}},\mathbb{R}^m),$$

where $\nu > 0$ is to be chosen suitably later on. Meanwhile, by the subadditivity of the energy, we have

$$\mathcal{G}(v_{k,n},\mathbb{B}^m) \geq \sum_{c\in P_k} \mathcal{G}(v_{n,k,c}, B_{1/(16k\sqrt{m})}(c)) = \frac{k^{sp}}{(16\sqrt{m})^{m-sp}}\mathcal{G}(u_n^{\mathrm{op}},\mathbb{B}^m).$$

Using (5.4.1), we arrive at

$$\nu \leq \mathcal{E}^{s,p}(v_{n,k},\mathbb{R}^m) \lesssim \mu^{-n}\mathcal{G}(v_{k,n},\mathbb{B}^m).$$

Finally, we define the clustered map $u_n^{\mathrm{clstr}}(x) = v_{n,k}(x/\lambda)$, where the scaling parameter $0 < \lambda \leq 1$ is chosen in such a way that

$$\nu = \mathcal{E}^{s,p}(u_n^{\mathrm{clstr}},\mathbb{R}^m) \lesssim \mu^{-n}\mathcal{G}(u_n^{\mathrm{clstr}},\mathbb{B}^m).$$

Here we use the fact that $sp < m$, so that scaling a map to a smaller ball decreases its energy.

Step 4. — Gluing the maps.

We now glue together the clustered maps for every n, in order to produce our final counterexample. Namely, setting $\rho_n = 2^{-n-2}/\sqrt{m}$, we can find points $a_n \to 0$ in such a way that the balls $\overline{B_{\rho_n}}(a_n)$ are disjoint and contained in $B_{1/2}$. We define

$$u(x) = \begin{cases} u_n^{\mathrm{clstr}}\left(\frac{x-a_n}{\rho_n}\right) & \text{if } x \in B_{\rho_n}(a_n), \\ b & \text{otherwise.} \end{cases}$$

We choose $\mu = 2^{m-sp}$. By superadditivity, translation invariance, and scaling of the energy, we get

$$\mathcal{G}(u,\mathbb{B}^m) \geq \sum_{n\in\mathbb{N}} \mathcal{G}(u, B_{\rho_n}(a_n)) = \sum_{n\in\mathbb{N}} \rho_n^{m-sp}\mathcal{G}(u_n^{\mathrm{clstr}},\mathbb{B}^m) \gtrsim \sum_{n\in\mathbb{N}} \frac{\nu\mu^n}{2^{n(m-sp)}} = +\infty.$$

On the other hand, by Lemma 5.3.2, we obtain

$$\mathcal{E}^{s,p}(u,\mathbb{R}^m) \lesssim \sum_{n\in\mathbb{N}} \rho_n^{m-sp}\mathcal{E}^{s,p}(u_n^{\mathrm{clstr}},\mathbb{R}^m) \lesssim \sum_{n\in\mathbb{N}} \frac{\nu}{2^{n(m-sp)}} \lesssim \nu,$$

where we have used $sp < m$. This concludes the proof. □

Let us observe that the proof gives actually slightly more than the statement. Indeed, it shows that the counterexamples can be constructed so that they are constant near the boundary of $\mathbb{B}^m$ — which allows to glue them — and their Sobolev energy can be made arbitrarily small.

5.5 An application: A more stringent threshold for the method of singular projection when $0 < s < 1$

In this last section, we come back to what was our initial motivation, namely, the method of singular projection. We use the nonlinear uniform boundedness principle to show that the threshold of applicability of this method when $0 < s < 1$ is indeed the more stringent one given by $p < \ell+1$, and not $sp < \ell+1$ as would be naturally expected. More precisely, we prove the following result.

Theorem 5.5.1. *Assume that $0 < s < 1$ and $1 \leq p < +\infty$ are such that $sp < \ell+1$ but $p \geq \ell+1$, and let $P\colon \mathbb{R}^{\ell+1}\setminus\{0\} \to \mathbb{S}^\ell$ be given by $P(x) = \frac{x}{|x|}$. There exists a map $u \in W^{s,p}(\mathbb{B}^{\ell+1};\mathbb{R}^{\ell+1}) \cap L^\infty(\mathbb{B}^{\ell+1};\mathbb{R}^{\ell+1})$ such that $P\circ(u-a) \notin W^{s,p}$ for every $a \in \mathbb{B}^{\ell+1}$.*

We start by introducing some notation. We let e_i be the i-th vector of the canonical basis of $\mathbb{R}^{\ell+1}$, and given $c \in \mathbb{B}^{\ell+1}$ and $n \in \mathbb{N}$, we let $c^+ = c_n^+ = c + 2^{1-n}e_1$ and $c^- = c_n^- = c - 2^{1-n}e_1$.

The building block of our construction is a function taking both values c^+ and c^- on two close cubes, and whose instrumental properties are collected in the following lemma.

Lemma 5.5.2. *For every $n \in \mathbb{N}$, every $c \in \mathbb{B}^{\ell+1}$, and every $0 < s < 1$ and $1 \leq p < +\infty$, there exists a map $v_c \in C^\infty_{\mathrm{c}}(\mathbb{R}^{\ell+1};\mathbb{R}^{\ell+1})$ with support contained in $Q^{\ell+1}$ and such that*

$$|v_c|^p_{W^{s,p}(\mathbb{R}^{\ell+1})} \leq C \tag{5.5.1}$$

while, for every $a \in \mathbb{B}^{\ell+1}$,

$$|P\circ(v_c-a)|^p_{W^{s,p}(\mathbb{B}^{\ell+1})} \geq C' \frac{|P(c^+-a)-P(c^--a)|^p}{2^{(1-n)p}}, \tag{5.5.2}$$

for some constants $C, C' > 0$ depending on s, p, and ℓ.

Proof. We fix a cutoff function $\psi \in C^\infty_c(\mathbb{R}^{\ell+1})$ whose support is contained in $\mathbb{B}^{\ell+1}$ and such that $0 \leq \psi \leq 1$ and $\psi = 1$ on $B_{1/2}$. We define the map w_c by $w_c(x) = c + 2^{1-n}(\psi(x - e_1) - \psi(x + e_1))e_1$, so that $w_c = c^-$ on $B_{1/2}(-e_1)$ and $w_c = c^+$ on $B_{1/2}(e_1)$. It is straightforward to observe that $|w_c|_{W^{s,p}(\mathbb{R}^{\ell+1})} \lesssim 2^{(1-n)p}$ while $|P \circ (w_c - a)|^p_{W^{s,p}(\mathbb{B}^{\ell+1})} \gtrsim |P(c^+ - a) - P(c^- - a)|^p$.

We would like to transform w_c into a compactly supported map while preserving the order of the ratio between the projected map and the original one. However, we need to cope with the fact that the transition towards 0 will bring a fixed cost to the Sobolev energy. We bypass this issue by clustering several scaled copies of the map w_c, using an analogue argument to the proof of Theorem 5.1.1. Namely, we construct a map $\tilde{w}_c$ by arranging scaled copies of the map w_c on $k^{\ell+1}$ balls of radius of order $\frac{1}{k}$ arranged in a regular grid inside $B_{1/2}$, with $k \in \mathbb{N}_*$ to be chosen later on. In particular, the map $\tilde{w}_c$ takes the value c outside of $B_{1/2}$. By scaling and the countable patching property of Sobolev maps, it holds that

$$|\tilde{w}_c|^p_{W^{s,p}(\mathbb{R}^{\ell+1})} \lesssim k^{\ell+1} k^{sp-\ell-1} 2^{(1-n)p} = k^{sp} 2^{(1-n)p},$$

while, by superadditivity,

$$|P \circ (\tilde{w}_c - a)|^p_{W^{s,p}(B_{1/2})} \gtrsim k^{sp} |P(c^+ - a) - P(c^- - a)|^p.$$

We choose k sufficiently large so that $k^{sp} \simeq 2^{(n-1)p}$.

Finally, we choose a function $\varphi_c \in C^\infty_c(\mathbb{R}^{\ell+1})$ whose support is contained in B_2 and such that $\varphi_c = c$ on $\mathbb{B}^{\ell+1}$ and $|\varphi_c|^p_{W^{s,p}(\mathbb{R}^{\ell+1})} \lesssim 1$, and we define v_c by $v_c = \tilde{w}_c$ on $\mathbb{B}^{\ell+1}$ and $v_c = \varphi_c$ outside of $\mathbb{B}^{\ell+1}$. It follows from Lemma 5.3.1 that

$$|v_c|^p_{W^{s,p}(\mathbb{R}^{\ell+1})} \lesssim |\tilde{w}_c|^p_{W^{s,p}(\mathbb{R}^{\ell+1})} + |\varphi_c|^p_{W^{s,p}(\mathbb{R}^{\ell+1})} \lesssim 1,$$

while

$$|P \circ (v_c - a)|^p_{W^{s,p}(\mathbb{B}^{\ell+1})} \gtrsim k^{sp} |P(c^+ - a) - P(c^- - a)|^p \gtrsim \frac{|P(c^+ - a) - P(c^- - a)|^p}{2^{(1-n)p}},$$

which concludes the proof. □

To make use of Lemma 5.5.2, we need to suitably estimate the quantity $|P(c^+ - a) - P(c^- - a)|$. We present two different arguments: The first one is simpler, but it only covers the range $p > \ell + 1$; the second one is slightly more involved, but allows to handle also the limiting case $p = \ell + 1$. These arguments each rely on a lemma involving planar geometry, respectively Lemma 5.5.3 and Lemma 5.5.4 below.

Lemma 5.5.3. *For every* $n \in \mathbb{N}$*, every* $c \in \mathbb{B}^{\ell+1}$*, and every* $a \in \overline{Q}_{2^{-n}}(c)$*, we have*

$$|P(c^+ - a) - P(c^- - a)| \geq C,$$

for some constant $C > 0$ *depending only on* ℓ*.*

Lemma 5.5.4. *For every* $n \in \mathbb{N}$*, every* $c \in \mathbb{B}^{\ell+1}$*, and every* $a \in \mathbb{B}^{\ell+1}$ *such that (i) the angle* φ *formed by the vectors* $a - c$ *and* e_1 *satisfies* $|\cos\varphi| \leq \frac{1}{8}$*, (ii)* $|a - c| \geq 2^{-n}$*, we have*

$$|P(c^+ - a) - P(c^- - a)| \geq C\frac{2^{1-n}}{|a-c|},$$

for some constant $C > 0$ *depending only on* ℓ*.*

Taking Lemmas 5.5.3 and 5.5.4 for granted, we proceed with the proof of Theorem 5.5.1.

Proof of Theorem 5.5.1. As we already announced, our goal is to apply the nonlinear uniform boundedness principle with the energy $\mathscr{G}$ over $\mathbb{R}^{\ell+1}$ with state space $\mathbb{R}^{\ell+1}$ given by

$$\mathscr{G}(u, A) = \inf_{a \in \mathbb{B}^{\ell+1}} \mathcal{E}^{s,p}(P \circ (u - a), A).$$

It is readily seen that such a quantity indeed satisfies the monotonicity, scaling, and superadditivity assumptions. We construct a sequence $(u_n)_{n\in\mathbb{N}}$ in $W^{s,p}(\mathbb{B}^{\ell+1}; \mathbb{R}^{\ell+1})$ such that

$$\lim_{n\to+\infty} \frac{\mathscr{G}(u_n, \mathbb{B}^{\ell+1})}{\mathcal{E}^{s,p}(u_n, \mathbb{B}^{\ell+1})} = +\infty,$$

and the conclusion then follows at once from the nonlinear uniform boundedness principle.

The key idea of the proof is to suitably glue together different "patches", each of them bringing an important contribution for some values of a. For every $n \in \mathbb{N}$, we consider the standard decomposition of the unit cube $Q^{\ell+1}$ into $2^{n(\ell+1)}$ cubes of sidelength 2^{1-n}. We let $(c_{n,k})_{1\leq k\leq 2^{n(\ell+1)}}$ be an enumeration of the centers of those cubes. For every k, we define

$$c^+_{n,k} = c_{n,k} + 2^{1-n}e_1 \quad \text{and} \quad c^-_{n,k} = c_{n,k} - 2^{1-n}e_1,$$

following the notation introduced before the statement of Lemma 5.5.2. We define

$$v_{n,k} = v_{c_{n,k}},$$

where $v_{c_{n,k}}$ is the map provided by Lemma 5.5.2. Combining (5.5.2) with Lemma 5.5.3, we find that

$$\mathcal{E}^{s,p}(P \circ (v_{n,k} - a), \mathbb{B}^{\ell+1}) \gtrsim 2^{np} \quad \text{for every } a \in \overline{Q}_{2^{-n}}(c_{n,k}). \tag{5.5.3}$$

We now exploit this lower bound to obtain the required estimate for our counterexample. We construct a map v_n by gluing together translates of the $v_{n,k}$, one for each $1 \leq k \leq 2^{n(\ell+1)}$, so that their supports do not overlap. A straightforward lower bound yields

$$\mathcal{E}^{s,p}(P \circ (v_n - a), \operatorname{supp} v_n) \gtrsim 2^{np} \quad \text{for every } a \in \mathbb{B}^{\ell+1},$$

by using the estimate (5.5.3) for a k such that $a \in \overline{Q}_{2^{-n}}(c_{n,k})$ — this k is unique unless a is on the boundary of some cube of the grid. On the other hand, the countable patching property of Sobolev mappings implies that

$$\mathcal{E}^{s,p}(v_n, \mathbb{R}^{\ell+1}) \lesssim \sum_{1 \leq k \leq 2^{n(\ell+1)}} \mathcal{E}^{s,p}(v_{n,k}, \mathbb{R}^{\ell+1}) \lesssim 2^{n(\ell+1)}. \tag{5.5.4}$$

In the case where $p > \ell + 1$, we find

$$\frac{\inf\limits_{a \in \mathbb{B}^{\ell+1}} \mathcal{E}^{s,p}(P \circ (v_n - a), \operatorname{supp} v_n)}{\mathcal{E}^{s,p}(v_n, \mathbb{R}^{\ell+1})} \to +\infty \quad \text{as } n \to +\infty. \tag{5.5.5}$$

We conclude by letting u_n be a scaled copy of v_n so that its support is contained in $\mathbb{B}^{\ell+1}$ and invoking the nonlinear uniform boundedness principle.

If $p = \ell + 1$, one needs to rely on a more careful estimate, and to take into account the contribution of several centers, relying on Lemma 5.5.4 instead to deduce the required lower bound on the $\mathscr{G}$ energy of v_n. More specifically, for a fixed point $a \in \mathbb{B}^{\ell+1}$ and a fixed $d \in 2^{1-n}\mathbb{N}_*$, we invoke Lemma 5.5.4 with $c = c_{n,k}$ for every k such that the ℓ^∞ distance between the center of the cube containing a and $c_{n,k}$ is d and such that $|\varphi - \frac{\pi}{2}|$ is sufficiently small, and we find

$$|P(c_{n,k}^{+} - a) - P(c_{n,k}^{-} - a)| \gtrsim \frac{2^{1-n}}{d},$$

using also the fact that $d \simeq |c_{n,k} - a|$. The number of corresponding such indices k behaves like $(2^{n-1}d)^\ell$ — this can be seen as a discrete isoperimetric inequality. We take into account all these contributions, for $d = j2^{1-n}$ with $j \in \mathbb{N}_*$ ranging from 1 to 2^{n-1},

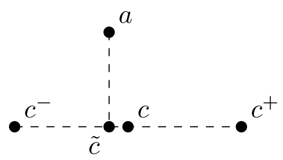

Figure 5.1: Estimate of $|P(c^+ - a) - P(c^- - a)|$ in Lemma 5.5.3

and we get from (5.5.2)

$$\mathcal{E}^{s,p}(P \circ (v_n - a), \operatorname{supp} v_n) \gtrsim 2^{np} \sum_{j=1}^{2^{n-1}} \frac{1}{j^p} j^\ell \gtrsim 2^{np} \ln 2^{n-1}.$$

Combining this with (5.5.4) and the fact that $p = \ell + 1$, we deduce that we are again in position to apply the uniform nonlinear boundedness principle to the maps u_n defined as scaled copies of v_n so that their supports are contained in $\mathbb{B}^{\ell+1}$, and this concludes the proof. □

Let us make a comment about the strategy for estimating the energy of $P \circ (v_n - a)$ in the case where $p = \ell + 1$ in the above proof. The idea was to take into account the contribution to the energy of not only one "patch" but of many of them. More specifically, we took into account the contribution of every patch whose associated center $c_{n,k}$ lies in a cone with vertex a and whose aperture is small, but fixed independently of the scale n. Alternatively, we could have relied on a similar construction, but instead of using as a building block the maps $v_{n,k}$ with values concentrated around the two points $c^+_{n,k}$ and $c^-_{n,k}$ which are aligned along the direction e_1, using rather maps with values concentrated around *several* points, pairwise aligned along different directions. Using a number of directions possibly large, but fixed independently of the scale n, it is possible to ensure that *every* center then leads a significant contribution to the energy of the projected map, eliminating the need of selecting only points in a fixed cone.

From a geometric point of view, this discussion amounts to say that one may either consider all centers in a region small but of fixed size determined by one axis, or use a number of axis large but independent of the scale to be in position to consider all centers.

We now prove Lemmata 5.5.3 and 5.5.4.

Proof of Lemma 5.5.3. We refer the reader to Figure 5.1 for an illustration of our constructions and notation. We let $\tilde{c}$ be the orthogonal projection of a onto the line passing

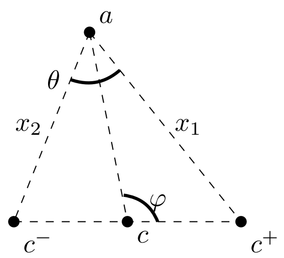

Figure 5.2: Estimate of $|P(c^+ - a) - P(c^- - a)|$ in Lemma 5.5.4

through c^+ and c^-, and we observe that $\tilde{c} \in \overline{Q}_{2^{-n}}(c)$. We compute

$$P(c^+ - a) - P(c^- - a) = \frac{(c^+ - a)|c^- - a| - (c^- - a)|c^+ - a|}{|c^+ - a||c^- - a|}. \tag{5.5.6}$$

Let N denote the numerator of the right-hand side in (5.5.6). We compute

$$|N|^2 = 2|c^+ - a|^2|c^- - a|^2 - 2|c^+ - a||c^- - a|(c^+ - a)\cdot(c^- - a). \tag{5.5.7}$$

Writing the orthogonal decomposition $c^\pm - a = c^\pm - \tilde{c} + \tilde{c} - a$, we find

$$(c^+ - a)\cdot(c^- - a) = |\tilde{c} - a|^2 - |c^+ - \tilde{c}||c^- - \tilde{c}|,$$

where we have used the fact that $c^+ - \tilde{c}$ and $c^- - \tilde{c}$ are aligned and have opposite directions. Since $\tilde{c}$ is an orthogonal projection, we have $|\tilde{c} - a| \leq |c^\pm - a|$. Hence, we find

$$|P(c^+ - a) - P(c^- - a)| \geq 2\frac{|c^+ - \tilde{c}||c^- - \tilde{c}|}{|c^+ - a||c^- - a|}.$$

Finally, since $a \in \overline{Q}_{2^{-n}}(c)$, we find that $|c^\pm - \tilde{c}| \gtrsim 2^{-n}$ while $|c^\pm - a| \lesssim 2^{1-n}$, whence we conclude that

$$|P(c^+ - a) - P(c^- - a)| \gtrsim 1,$$

finishing the proof of the lemma. □

Proof of Lemma 5.5.4. Since everything takes place inside one fixed plane — depending on c and a — we may assume that $\ell + 1 = 2$. We refer the reader to Figure 5.2 for an illustration of our constructions and notation inside the plane. Let θ denote the angle between the vectors $c^+ - a$ and $c^- - a$, and let $x_1 = |c^+ - a|$ and $x_2 = |c^- - a|$. By Al

Kashi's formula, we know that

$$|P(c^+ - a) - P(c^- - a)|^2 = 2 - 2\cos\theta,$$

while

$$2^{2(1-n)} = x_1^2 + x_2^2 - 2x_1x_2\cos\theta = (x_1 - x_2)^2 + x_1x_2(2 - 2\cos\theta).$$

Hence,

$$|P(c^+ - a) - P(c^- - a)|^2 = \frac{2^{2(1-n)} - (x_1 - x_2)^2}{x_1x_2}. \tag{5.5.8}$$

By another application of Al Kashi's formula, we find

$$x_1^2 = 2^{2(1-n)} + |a - c|^2 - 2 \cdot 2^{1-n}|a - c|\cos\varphi$$

and

$$x_2^2 = 2^{2(1-n)} + |a - c|^2 + 2 \cdot 2^{1-n}|a - c|\cos\varphi.$$

Therefore, as either $x_1 \geq |a - c|$ or $x_2 \geq |a - c|$, we find

$$|x_1 - x_2||a - c| \leq |x_1^2 - x_2^2| = 4 \cdot 2^{1-n}|a - c||\cos\varphi|,$$

so that, using (i), $|x_1 - x_2| \leq \frac{1}{2}2^{1-n}$. Combining (i) and (ii), we also find that $x_1 \simeq |a - c| \simeq x_2$. We conclude from (5.5.8) that

$$|P(c^+ - a) - P(c^- - a)| \gtrsim \frac{2^{1-n}}{|a - c|},$$

hence finishing the proof of the lemma. □

In view of the counterexample provided by Theorem 5.5.1, the reader might wonder whether it would not be possible to work with a different singular projection, which would satisfy improved estimates near its singular set, to overcome the above issue. We conclude this section with a short discussion that strongly suggests that there is no such hope.

We again work in the model case where $\mathscr{N} = \mathbb{S}^\ell \subset \mathbb{R}^{\ell+1}$. In order to perform the approach by projection, we are looking for a retraction P into $\mathbb{S}^\ell$, which is defined on the whole $\mathbb{R}^{\ell+1}$ except on a finite set of point singularities. When restricting to $\overline{\mathbb{B}^{\ell+1}}$, by a well-known fact from topology, there must exist at least one singular point $a \in \mathbb{B}^{\ell+1}$

of P such that, on any sufficiently small sphere centered at a, P covers the whole $\mathbb{S}^\ell$. It is then easy to show that

$$\lim_{r\to 0} \sup_{x\in\partial B_a(r)} |x-a||\mathrm{D}P(x)| \gtrsim 1.$$

In some sense, this means that the rate of blow up

$$|\mathrm{D}P(x)| \simeq \frac{1}{|x-a|}$$

near the singularity a that was required for P to be an $(\ell+1)$-singular projection cannot be improved. As our proof of Theorem 5.5.1 essentially relies on the estimate satisfied by P, this suggests that there is no hope to find a singular projection of $\mathbb{R}^{\ell+1}$ onto $\mathbb{S}^\ell$ that would avoid the obstruction observed above, although we shall not attempt to present a complete argument here to avoid excessive technicality.

5.6 Comments and perspectives

5.6.1 The genesis of the nonlinear uniform boundedness principle: The Banach–Steinhaus theorem

As we already explained, the nonlinear uniform boundedness principle can be thought of as a nonlinear version of the Banach–Steinhaus theorem from Functional Analysis I. But more is true: The proof of the nonlinear uniform boundedness principle, which consists of packing together maps with higher and higher $\mathcal{G}$ energy in such a way that their Sobolev energies remain summable, is reminiscent of the original proof of the Banach–Steinhaus theorem[1]. The main difference is that the proof of the Banach–Steinhaus theorem relies on *summing* the maps being stacked together, which obviously is no option for mappings into manifolds. This explains the need of all the tools we have presented above, to modify the maps to make them compatible so that they can be glued without changing their values. Otherwise stated, modifications had to be performed by composition *on the domain*, and not on the target.

5.6.2 Other versions

In this exposition, we limited ourselves to the version of the nonlinear uniform boundedness principle for $sp < m$. However, there are already similar statements for $sp \geq m$.

[1] It turns out that the Banach–Steinhaus theorem was not proved first by S. Banach and H. Steinhaus, but 4 years ealier by T. H. Hildebrandt, in 1923. Banach and Steinhaus reproved it in 1927, apparently unaware that the result was already known. Hildebrandt's proof [Hil23] is by stacking worse and worse maps, while Banach and Steinhaus' one [BS27] relies on the Baire category theory.

The only important difference is that, as scaling maps to smaller balls does not decrease their energy in this regime, a smallness assumption has to be included in the statement of the theorem. Moreover, to achieve the step where the maps are transformed so that they have a common boundary value, the target needs to be assumed to be compact when $sp > m$, otherwise there is no way to neglect the energy of this modification in this regime.

Actually, this is not a problem from the proof, but there is really a qualitatively different behaviour of such problems when $sp \geq m$. For lifting or extension of traces for instance, there are situations where constructions for $sp \geq m$ *cannot* come with a linear estimate, but come with a superlinear estimate. We refer the reader to [VS25b, VS24] for such examples.

In addition, we mention that, as a byproduct of the proof, one obtains that counterexamples provided by the nonlinear uniform boundedness principle are dense in $W^{s,p}(\Omega;\mathcal{N})$.

Finally, we mention that there are also variants of this theorem for $s = 0$ and $s > 1$, see [MVS19].

5.6.3 Other uses of the nonlinear uniform boundedness principle

To illustrate the usefulness of the nonlinear uniform boundedness principle, let us present other problems to which it applies. The moral of the story is that, each time an energy can be associated with a qualitative question to turn it into a quantitative problem, there is a chance that the nonlinear uniform boundedness principle might be applied.

Extension of traces To the existence problem of whether every $u \in W^{1-1/p,p}(\partial\Omega;\mathcal{N})$ admits a trace extension $U \in W^{1,p}(\Omega;\mathcal{N})$, one may associate the minimal extension energy as

$$\mathcal{E}^{1,p}_{\mathrm{ext}}(u) = \inf\left\{\int_\Omega |\mathrm{D}u|^p \colon U \in W^{1,p}(\Omega;\mathcal{N}),\, \operatorname{tr} U = u \text{ on } \partial\Omega\right\}.$$

This energy satisfies the monotonicity, superadditivity, and translation and scaling assumptions, and therefore the nonlinear uniform boundedness principle can be applied to it.

This was notably used by F. Bethuel [Bet14], and by P. Mironescu and J. Van Schaftingen [MVS21b], to prove the existence of *analytical obstructions* to the extension of traces problem in presence of an infinite homotopy group.

Lifting of Sobolev mappings In Section 1.3, we mentioned the lifting problem, which, in its general formulation, is concerned about the existence of a lifting $\tilde{u} \in W^{s,p}(\Omega; \tilde{\mathcal{N}})$ for a Sobolev map $u \in W^{s,p}(\Omega; \mathcal{N})$, where $\pi\colon \tilde{\mathcal{N}} \to \mathcal{N}$ is a Riemannian covering.

Here as well, this problem may be quantified, defining the lifting energy

$$\mathcal{E}^{s,p}_{\mathrm{lift}}(u) = \inf\{\mathcal{E}^{s,p}(\tilde{u}, \Omega) \colon \tilde{u}\colon \Omega \to \tilde{\mathcal{N}} \text{ measurable}, \pi \circ \tilde{u} = u\},$$

also satisfying the monotonicity, superadditivity, and translation and scaling properties.

The nonlinear uniform boundedness principle was applied in this context by P. Mironescu and J. Van Schaftingen [MVS21a] to prove the existence of *analytical obstructions* to the lifting property in $W^{s,p}$ for $0 < s < 1$ when $sp = 1$.

The weak approximation problem In Chapter 7, we will discuss the weak approximation problem, which asks whether any $W^{s,p}(\Omega; \mathcal{N})$ map can be *weakly* approximated with a sequence of smooth $\mathcal{N}$-valued maps. To this problem is associated the *relaxed energy*, defined as

$$\mathcal{E}^{s,p}_{\mathrm{rel}}(u) = \inf\Big\{\liminf_{n\to+\infty} \mathcal{E}^{s,p}(u_n, \Omega) \colon u_n \in C^\infty(\Omega; \mathcal{N}), u_n \to u \text{ in measure}\Big\}.$$

This measures the minimal energy to approximate weakly the target map u. In turn, this energy has been intensively studied and used, but we shall come back to this in Section 5.6.

From a historical perspective, the first version of a nonlinear uniform boundedness principle was formulated for this very specific energy, by Hang F. and Lin F. [HL03b]. It was then a key ingredient in Bethuel's groundbreaking counterexample to the weak approximation property [Bet20], and to subsequent families of counterexamples [DVS24].

6 The general strong approximation theorem: The method of good and bad cubes

In this section, we explain the main ingredients of the proof of the general case of the strong density theorems, namely Theorem 1.2.5 and Theorem 1.2.6. We shall however not present the proof of the most general case in full details[1]. Instead, we will restrict to the technically simplest case, namely, $s = 1$ and $m - 1 < p < m$. This case already contains all the main ideas of the method of good and bad cubes, which is the standard tool to prove those density theorems. Of course, going from that special case to the general one of $0 < s < +\infty$ and $sp < m$ still requires nontrivial work: Going from C^0 to VMO to include the case where $sp \in \mathbb{N}_*$, dealing with the extra rigidity of higher order spaces to treat $s \in \mathbb{N}_*$, and dealing with the nonlocal character of the Gagliardo seminorm to cover s noninteger. But, these difficulties and some classical techniques to overcome them were already explained in other contexts, in other chapters. In this chapter, we wish to focus on the method of good and bad cubes and its implementation, whence the choice of a special case to work with.

Also, for technical reasons, it will be easier to work with the domain being a cube rather than a ball. That being said, the special version of the strong density theorem we prove in this section is the following.

Theorem 6.0.1. *Assume that $m - 1 < p < m$. The class $C^\infty(\overline{Q^m}; \mathscr{N})$ is dense in $W^{1,p}(Q^m; \mathscr{N})$ if and only if $\pi_{m-1}(\mathscr{N}) \simeq \{0\}$. Moreover, the class of $W^{1,p}(Q^m; \mathscr{N})$ maps that are smooth outside of a finite set of points is always dense in $W^{1,p}(Q^m; \mathscr{N})$.*

Before delving into the proof, let us give a few words about it. As already explained, we cannot rely on a basic convolution procedure to regularize a Sobolev mapping on the whole Q^m while preserving the geometric constraint. On the other hand, since $p > m - 1$, we know that a $W^{1,p}$ map is continuous on $(m-1)$-dimensional sets. The

[1] Bethuel's proof [Bet91] for $s = 1$ is already around 30 pages, with some parts being only sketched. Bousquet, Ponce, and Van Schaftingen's proof [BPVS15] for $s \in \mathbb{N}_*$ reaches 65 pages with the additional tools to deal with higher order Sobolev spaces. The general proof [Det26], also including the techniques to handle the nonlocal Gagliardo seminorm, is almost 100 pages. Meanwhile, Brezis and Mironescu's alternative argument for $0 < s < 1$, despite being conceptually simpler, also reaches around 50 pages, because of the technical computations needed to deal with the nonlocal fractional Sobolev spaces. For this reason, it should be clear why an attempt of presenting the most general case in full details would not be reasonable in this course... But, I put nice drawings in the complements section to illustrate the general case, as a consolation!

idea is therefore to find a suitable $(m-1)$-dimensional grid, and to take advantage of the nice behaviour of the mapping we wish to approximate on this grid.

Another important idea is the following. Since convolution is a kind of averaging, its enemy is oscillation. If a map does not oscillate too much, then a convolution procedure at small scale should not disrupt the geometric constraint too much (and we know we can deal with small perturbations by projecting back onto the target). A key observation is that, being allowed a fixed amount of energy, a Sobolev map cannot wildly oscillate everywhere. We shall therefore distinguish between the regions where our map does not oscillate too much, and where we can actually rely on convolution, and the regions where it oscillates wildly and we have to proceed otherwise. This is enforced by the *method of good and bad cubes*, devised by F. Bethuel in his seminal 1991 contribution, that we explain in the next section.

6.1 Good and bad cubes

In the rest of the chapter, we take $u \in W^{1,p}(Q^m;\mathscr{N})$, where $m-1 < p < m$. By a classical reflection or dilation argument, we may always assume that u is defined on a slightly larger set than Q^m, so that we can perform constructions that rely on values of u slightly outside of Q^m. For convenience, here we shall assume that $u \in W^{1,p}(2Q^m;\mathscr{N})$, where $2Q^m$ is the cube with same centre as Q^m and double sidelength.

As we explained, we wish to find a suitable grid on which u is well-behaved. For this purpose, we introduce some notation. From now on, we fix $\eta > 0$. We denote $K^m_\eta = Q_\eta + \eta\mathbb{Z}^m$ the standard decomposition of $\mathbb{R}^m$ by cubes of sidelength η. (We note that K^m_η is a set of cubes.) For $a \in \mathbb{R}^m$, we write $K^m_{\eta,a} = K^m_\eta + a$ the set of all translates of the cubes in K^m_η by a. We also write $K^{m-1}_{\eta,a}$ to denote the set of all faces of cubes in $K^m_{\eta,a}$. Finally, given a family of cubes, denoted by a Roman letter, the corresponding calligraphic letter will denote the set obtained by taking the union of all cubes in the family. For instance, $\mathscr{K}^{m-1}_{\eta,a}$ is an $(m-1)$-dimensional set in $\mathbb{R}^m$.

We now prove the following result concerning the existence of a suitable grid on which u is well-behaved.

Proposition 6.1.1. *There exists $a \in \mathbb{R}^m$ such that $u \in W^{1,p}(\mathscr{K}^{m-1}_{\eta,a} \cap 2Q^m;\mathscr{N})$ with*

$$\int_{\mathscr{K}^{m-1}_{\eta,a}\cap 2Q^m} |\mathrm{D}u|^p \lesssim \frac{1}{\eta}\int_{2Q^m} |\mathrm{D}u|^p.$$

Proof. The claims follows readily from a genericity and averaging argument. Indeed, by Tonelli's theorem, we estimate

$$\int_{Q_\eta}\int_{\mathscr{K}^{m-1}_{\eta,a}\cap 2Q^m}|\mathrm{D}u|^p \lesssim \eta^{m-1}\int_{2Q^m}|\mathrm{D}u|^p.$$

This implies that

$$\fint_{Q_\eta}\int_{\mathscr{K}^{m-1}_{\eta,a}\cap 2Q^m}|\mathrm{D}u|^p = \frac{1}{\eta}\int_{2Q^m}|\mathrm{D}u|^p,$$

which ensures the existence of a subset of Q_η of positive measure on which

$$\int_{\mathscr{K}^{m-1}_{\eta,a}\cap 2Q^m}|\mathrm{D}u|^p \lesssim \frac{1}{\eta}\int_{2Q^m}|\mathrm{D}u|^p.$$

This concludes the proof. □

Let us note that, in the above, there are a few technicalities to be checked to make the argument completely rigorous. First of all, it is not completely clear what it means to be Sobolev on the faces of a grid, as this is not a regular set. Here, we adopt the definition that being Sobolev on faces of a grid means to be Sobolev on each face and having compatible traces on the boundary. One can check that this implies, for instance, that if one identifies the boundary of a cube of the grid with the sphere, the corresponding map will be Sobolev on this sphere. However, the argument we presented only establishes the estimate on the norm of the gradient, not this regularity. This could require to exclude a further set of zero measure of Q_η. It is a good exercise to write the complete genericity argument justifying this.

We shall now define the *good and bad cubes*. We let $\kappa > 0$ to be chosen later on, and we say that $\sigma \in K^m_{\eta,a}$ that is entirely contained in $2Q^m$ is a *good cube* whenever

$$\frac{1}{\eta^{m-p}}\int_\sigma|\mathrm{D}u|^p \leq \kappa^p \quad \text{and} \quad \frac{1}{\eta^{m-p-1}}\int_{\partial\sigma}|\mathrm{D}u|^p \leq \kappa^p. \tag{6.1.1}$$

Otherwise, we call σ a *bad cube*. We let $B_{\eta,a}$ be the set of bad cubes, and $G_{\eta,a}$ be the set of good cubes. (Although a is fixed from now, we keep it in the notation to minimize possible confusion between the set of bad cubes and the ball.) We note that the above quantities are rescaled energies, respectively on the cubes and their boundaries.

An important feature of bad cubes is that there are not too many of them.

Proposition 6.1.2. *The number of bad cubes satisfies the estimate*

$$\operatorname{card} B_{\eta,a} \lesssim \eta^{p-m} \kappa^{-p} \int_{2Q^m} |\mathrm{D}u|^p.$$

Proof. By definition of bad cubes, we have

$$\operatorname{card} B_{\eta,a} = \sum_{\sigma \in B_{\eta,a}} 1 \leq \sum_{\sigma \in B_{\eta,a}} \eta^{p-m} \kappa^{-p} \int_{\sigma} |\mathrm{D}u|^p + \eta^{p+1-m} \kappa^{-p} \int_{\partial\sigma} |\mathrm{D}u|^p.$$

The second term above is estimated using Proposition 6.1.1:

$$\sum_{\sigma \in B_{\eta,a}} \int_{\partial\sigma} |\mathrm{D}u|^p \lesssim \int_{\mathscr{K}^{m-1}_{\eta,a}} |\mathrm{D}u|^p \lesssim \frac{1}{\eta} \int_{2Q^m} |\mathrm{D}u|^p.$$

This concludes the proof. □

6.2 Filling in the bad cubes: the thickening procedure

Since we do not have much control on the behaviour of u on bad cubes, it is unclear whether we can perform an accurate approximation on this region. Good news is that, since there are not too many bad cubes, it is actually sufficient to approximate our map very roughly in this region. The main result in this section is the following proposition, which features the so-called *thickening procedure*. The idea is to fill in cubes with values of u only on their boundary, where it is well-behaved.

Proposition 6.2.1. *Let $v \in W^{1,p}(Q^m; \mathscr{N})$ be such that $v \in W^{1,p}(\partial Q^m; \mathscr{N})$. There exists a map $w \in W^{1,p}(Q^m; \mathscr{N})$ that is continuous outside of a finite number of points and such that $\operatorname{tr}_{\partial Q^m} w = \operatorname{tr}_{\partial Q^m} v$ and*

$$\int_{Q^m} |\mathrm{D}w|^p \lesssim \int_{Q^m} |\mathrm{D}v|^p.$$

The key ingredient to prove Proposition 6.2.1 is the following thickening procedure on *one* cube.

Lemma 6.2.2. *Let $Q(r)$ denote the cube centered at 0 with inradius equal to r. Given $v \in W^{1,p}(\partial Q(r))$, we define $v^{\mathrm{th}}(x) = v\bigl(\frac{rx}{|x|_\infty}\bigr)$. Then, $v^{\mathrm{th}} \in W^{1,p}(\partial Q(r))$ and it satisfies the estimate*

$$\int_{Q(r)} |\mathrm{D}v^{\mathrm{th}}|^p \lesssim r \int_{\partial Q(r)} |\mathrm{D}v|^p.$$

Proof. The proof is a simple polar integration computation. Indeed, we estimate

$$\mathrm{D}v^{\mathrm{th}}(x) \lesssim \frac{r}{|x|_\infty} \mathrm{D}v\Big(\frac{rx}{|x|_\infty}\Big),$$

and the polar integration formula yields

$$\begin{aligned} \int_{Q(r)} |\mathrm{D}v^{\mathrm{th}}|^p &= \int_0^r \int_{\partial Q(\rho)} \frac{r^p}{|x|_\infty^p} \Big|\mathrm{D}v\Big(\frac{rx}{|x|_\infty}\Big)\Big| \,\mathrm{d}x \,\mathrm{d}\rho \\ &= r^{p-m+1} \int_0^r \rho^{m-1-p} \Big(\int_{\partial Q(r)} |\mathrm{D}v|^p\Big) \,\mathrm{d}\rho \lesssim r \int_{\partial Q(r)} |\mathrm{D}v|^p. \end{aligned}$$

We note that only the condition $p < m$ is used, to ensure the convergence of the above integral. This lemma does not rely on $p > m - 1$.

We importantly mention that we have only proved the required estimate on the derivative of v. However, it remains to prove the actual Sobolev regularity. In particular, for a complete argument, one should prove the weak differentiability of the map that we have constructed. Filling in the missing details is a good exercise. □

With this basic tool at hand, we may now prove Proposition 6.2.1. The idea is to perform the previous homogeneous extension construction on each cube of a suitable grid.

Proof of Proposition 6.2.1. Given $\eta > 0$ as in Proposition 6.1.1, we obtain a number $a \in \mathbb{R}^m$ such that $v \in W^{1,p}(\mathscr{K}^{m-1}_{\eta,a} \cap Q^m; \mathscr{N})$ and

$$\int_{\mathscr{K}^{m-1}_{\eta,a} \cap Q^m} |\mathrm{D}v|^p \lesssim \frac{1}{\eta} \int_{Q^m} |\mathrm{D}v|^p.$$

We apply the construction from Lemma 6.2.2 to each cube $\sigma \in K^m_{\eta,a}$ — if σ does not lie completely in Q^m, we intersect it with Q^m, and still identify it with a cube via a bi-Lipschitz diffeomorphism — and this yields a map $w \in W^{1,p}$ on Q^m and with values into $\mathscr{N}$ that satisfies

$$\begin{aligned} \int_{Q^m} |\mathrm{D}w|^p &\lesssim \eta \int_{\mathscr{K}^{m-1}_{\eta,a} \cap Q^m} |\mathrm{D}v|^p + \eta \int_{\partial Q^m} |\mathrm{D}v|^p \\ &\lesssim \int_{Q^m} |\mathrm{D}v|^p + \eta \int_{\partial Q^m} |\mathrm{D}v|^p. \end{aligned}$$

Choosing η sufficiently small so that the second term is absorbed by the first one yields the required estimate.

It is clear from its construction and the Morrey embedding that the map w is continuous outside of a finite union of points, and the coincidence of traces also follows directly from the construction of our map. □

We now explain how to conclude the approximation on bad cubes. We apply Proposition 6.2.1 to the map u on each bad cube σ (after scaling), and we replace u by the corresponding map w on σ. The resulting map is Sobolev and continuous on the whole cube σ except on a finite number of points. The convergence follows from the energy bound provided by Proposition 6.2.1 and Lebesgue's lemma, since the volume of the bad cubes converges to 0 as $\eta \to 0$.

6.3 Approximation on good cubes: a projection technique and convolution

We now turn to approximation on good cubes. Since this region accounts for most of the cube on which we are working, we need to construct a more precise approximation there. But luckily, we have the energy estimates available on good cubes at our disposal to help us in this task.

The idea is in two steps. First, we prove that, on each good cube, u takes its value on a given small ball except on a set of small measure. Therefore, projecting radially onto this ball, we do not mess up too much with the values of u. Then, once this projection has been performed, we regularize by convolution. Since this procedure preserves convex constraints, we know that we remain in this ball, which allows to conclude by the classical reprojection onto the manifold trick.

The first result is the following. We let $\delta > 0$ be sufficiently small so that $\delta < \iota$, where ι is the radius of a neighborhood of $\mathscr{N}$ on which there is a smooth retraction onto $\mathscr{N}$.

Proposition 6.3.1. *For every sufficiently small $\eta > 0$, if $\kappa > 0$ is chosen sufficiently small in the definition of good cubes, there exists a map $w_\eta \in W^{1,p}(\mathscr{G}^m_{\eta,a};\mathscr{N})$ defined on the set of all good cubes, such that*

1. *for every $\sigma \in G_{\eta,a}$, there exists a point $y_\sigma \in \mathscr{N}$ such that $w_\eta \in B_\delta(y_\sigma)$;*
2. *$w_\eta = u$ on $\partial\sigma$ for every $\sigma \in G_{\eta,a}$;*
3. *there exists a measurable set $A_\eta \subset \mathscr{G}^m_{\eta,a}$ such that $|A_\eta| \lesssim \kappa^p\delta^{-p}$ and*

$$\int_{\mathscr{G}^m_{\eta,a}} |u - w_\eta|^p + |\mathrm{D}u - \mathrm{D}w_\eta|^p \lesssim \int_{A_\eta} |\mathrm{D}u|^p.$$

Proof. By the Morrey inequality and the second condition in the definition of good cubes (6.1.1), we may choose κ sufficiently small, depending on δ, so that the oscillation of u on every $\partial\sigma$ with $\sigma \in G_{\eta,a}$ is less than $\frac{\delta}{2}$. Choose y_σ to be any point in the image of $\partial\sigma$ by u.

We let P_σ be the projection onto the ball $B_\delta(y_\sigma)$, defined by

$$P_\sigma(y) = \begin{cases} y & \text{if } y \in B_\delta(y_\sigma), \\ y_\sigma + \delta \frac{y - y_\sigma}{|y - y_\sigma|} & \text{else.} \end{cases}$$

We define $w_\eta = P_\sigma \circ u$ on every $\sigma \in G_{\eta,a}$, and we show that it has the required properties.

Property 1 is obvious from the definition. Similarly, property 2 follows from the choices of κ and y_σ. It only remains to prove 3.

Let us define $\mathcal{U}_{\eta,\sigma} = \{x \in \sigma \colon u(x) \notin B_\delta(y_\sigma)\}$. By the construction of w_η, we have $u = w_\eta$ and $\mathrm{D}u = \mathrm{D}w_\eta$ almost everywhere on $\sigma \setminus \mathcal{U}_{\eta,\sigma}$. We therefore estimate

$$\int_\sigma |\mathrm{D}u - \mathrm{D}w_\eta|^p \lesssim \int_{\mathcal{U}_{\eta,\sigma}} |\mathrm{D}u|^p.$$

Moreover, since u and w_η coincide on $\partial\sigma$, the Poincaré inequality ensures that

$$\int_\sigma |u - w_\eta|^p \lesssim \eta^p \int_\sigma |\mathrm{D}u - \mathrm{D}w_\eta|^p \lesssim \eta^p \int_{\mathcal{U}_{\eta,\sigma}} |\mathrm{D}u|^p.$$

Summing over good cubes,

$$\int_{\mathcal{G}^m_{\eta,a}} |u - w_\eta|^p + |\mathrm{D}u - \mathrm{D}w_\eta|^p \lesssim \int_{A_\eta} |\mathrm{D}u|^p,$$

with

$$A_\eta = \bigcup_{\sigma \in G_{\eta,a}} \mathcal{U}_{\eta,\sigma}.$$

We conclude by estimating the measure of A_η.

This relies on a nice truncation argument. We consider a truncation function $\varphi \colon \mathbb{R} \to [0,1]$ such that $\varphi \in C^\infty(\mathbb{R})$, $\varphi(x) = 0$ if $x \leq \frac{\delta}{2}$, $\varphi(x) = 1$ if $x \geq \delta$, and $|\varphi'| \lesssim \delta^{-1}$. We observe that $\varphi(|u - y_\sigma|)^p = 1$ on $\mathcal{U}_{\eta,\sigma}$. Hence,

$$|\mathcal{U}_{\eta,\sigma}| \leq \int_\sigma \varphi(|u - y_\sigma|)^p.$$

Since $\varphi(|u - y_\sigma|) = 0$ on $\partial\sigma$, the Poincaré inequality implies that

$$\int_\sigma \varphi(|u - y_\sigma|)^p \lesssim \eta^p \int_\sigma |\mathrm{D}(\varphi(|u - y_\sigma|))|^p \lesssim \eta^p \delta^{-p} \int_\sigma |\mathrm{D}u|^p.$$

Now we use the first condition in the definition of good cubes (6.1.1) and a change of variable to find that

$$|\mathscr{U}_{\eta,\sigma}| \lesssim \kappa^p \delta^{-p} \eta^m = \kappa^p \delta^{-p} |\sigma|.$$

It suffices to sum over all good cubes to finish the proof. □

We now explain how to finish the approximation procedure on good cubes. For this, as we already announced, we rely on regularization by convolution, which is possible in this setting since we know that it will not leave the ball $B_\delta(y_\sigma)$. More precisely, on each σ, the map w_η can be approximated with continuous $B_\delta(y_\sigma)$-valued maps that coincide with w_η — and hence with u — on $\partial\sigma$. (It is a good exercise to write carefully this approximation step.) Meanwhile, choosing κ sufficiently small, by Lebesgue's lemma, we can make $\int_{A_\eta} |\mathrm{D}u|^p$ arbitrarily small, ensuring that u and w_η are closed.

Gathering the constructions on good and bad cubes, we conclude that u can be approximated in $W^{1,p}$ with maps taking their values into a δ-neighbourhood of $\mathscr{N}$, that are continuous except at point singularities. A further regularisation step allows to turn this into an approximation with smooth maps (writing the details is a good exercise), and we conclude by reprojecting onto $\mathscr{N}$.

This proves the density of the class $\mathscr{R}_0(Q^m)$ in $W^{1,p}(Q^m; \mathscr{N})$. Let us note importantly that, as the proof suggests, the number of singularities along an approximating sequence might diverge to infinity.

6.4 Removing the singularities: shrinking

To finish the proof of Theorem 6.0.1, we only need to explain how to exploit the assumption $\pi_{m-1}(\mathscr{N})$ to remove the pointwise singularities created by the above procedure. Since the construction is local, it suffices to deal with the case of a Sobolev mapping v on Q^m that is continuous except at the origin. Therefore, the following proposition will suffice to conclude the argument.

Proposition 6.4.1. *Let $v \in W^{1,p}(Q^m; \mathscr{N})$ be continuous in $Q^m \setminus \{0\}$. If $\pi_{m-1}(\mathscr{N}) \simeq \{0\}$, then for every $\eta > 0$, there exists a map $w_\eta \in C^\infty(Q^m; \mathscr{N})$ such that $\|v - w_\eta\|_{W^{1,p}(Q^m)} \leq \eta$ and that coincides with v outside of an arbitrarily small neighborhood of* 0.

Proof. Fix $\delta > 0$, and let w_δ^{ext} be a continuous Sobolev extension of $v_{|\partial Q_\delta}$ to Q_δ. (The existence of a continuous extension follows directly from the topological assumption $\pi_{m-1}(\mathscr{N}) \simeq \{0\}$, as explained in Section 2.3. The fact that it can be taken to be Sobolev follows from a straightforward regularisation argument. Here, convolution can be used since the map we start with is continuous. Filling in the details is a good exercise.) The problem here is that we have no control on the Sobolev energy of w_δ^{ext}: Even though it is defined on a very small set, its derivative could have an arbitrarily high norm. We shall correct this by the shrinking procedure.

We define $w_{\delta,t}^{\mathrm{sh}}$ by

$$w_{\delta,t}^{\mathrm{sh}}(x) = \begin{cases} v(x) & \text{if } x \notin Q_\delta, \\ v(\frac{\delta x}{|x|_\infty}) & \text{if } x \in Q_\delta \setminus Q_{t\delta}, \\ w_\delta^{\mathrm{ext}}(\frac{x}{t}) & \text{if } x \in Q_{t\delta}. \end{cases}$$

We wish to estimate the Sobolev distance between v and $w_{\delta,t}^{\mathrm{sh}}$. We exemplify the computation by showing the estimate for the derivatives (the L^p estimate being much simpler, as the maps are uniformly bounded and differ only on a small set). We have

$$\int_{Q^m} |\mathrm{D}v - \mathrm{D}w_{\delta,t}^{\mathrm{sh}}|^p \lesssim \int_{Q_\delta} |\mathrm{D}v|^p + \int_{Q_\delta \setminus Q_{t\delta}} |\mathrm{D}w_{\delta,t}^{\mathrm{sh}}|^p + \int_{Q_{t\delta}} |\mathrm{D}w_{\delta,t}^{\mathrm{sh}}|^p.$$

From Lemma 6.2.2, we have

$$\int_{Q_\delta \setminus Q_{t\delta}} |\mathrm{D}w_{\delta,t}^{\mathrm{sh}}|^p \lesssim \delta \int_{\partial Q_\delta} |\mathrm{D}v|^p.$$

From a simple change of variable, we find

$$\int_{Q_{t\delta}} |\mathrm{D}w_{\delta,t}^{\mathrm{sh}}|^p \lesssim t^{m-p} \int_{Q_\delta} |\mathrm{D}w_\delta^{\mathrm{ext}}|^p.$$

Here it is important that $p < m$: Letting $t \to 0$, we may make the right-hand-side as small as we please. Choosing δ appropriately, and then t sufficiently small, we obtain the required map through $w_\eta = w_{\delta,t}^{\mathrm{sh}}$. □

As a final remark, let us mention that the above proof shows that, regardless of the topology of the target, a *given* map $u \in W^{1,p}(Q^m; \mathscr{N})$ may be approximated by a sequence of smooth mappings if and only if its restriction to a *generic* square is homotopic to a constant. By generic, we mean that the above property holds for $\partial Q_r(a)$ for almost every $a \in Q^m$ and almost every $r \in (0,1)$ such that $Q_r(a) \subset Q^m$.

6.5 Comments and perspectives

6.5.1 The general case of the strong density theorem

Let us explain, with the help of figures as an illustration, what are the main difficulties to prove the most general case of the strong density theorem, and what are the main tools to overcome them.

We identify at least four extra layers of difficulties. First, when going to $sp < m - 1$, we are invited to go down to a skeleton of dimension $\lfloor sp \rfloor$, to exploit the Morrey embedding. However, this makes gluing the constructions on neighbouring cubes much more challenging. For instance, filling in a cube with the values of the map on the $(m-2)$-skeleton will modify the values of the map not only inside the cube, but also inside the faces, making the resulting map possibly incompatible with the one constructed on the cube next to it — this arises typically for bad cubes adjacent to a good one.

Next, when considering the limiting case $sp \in \mathbb{N}_*$, an extra difficulty comes from the failure of the Morrey embedding. One therefore needs to somehow smoothen the map also on the $\lfloor sp \rfloor$-skeleton to make it continuous.

When considering higher order spaces $W^{k,p}$ with $k \in \mathbb{N}$, an extra challenge arises when willing to glue constructions on neighbouring cubes. Indeed, matching the traces does not suffice anymore, and one also has to ensure that the derivatives match up to order $k - 1$. This would typically not be ensured by the homogeneous extension procedure, which will break the first derivative along the interface.

Finally, the case s noninteger requires some extra technical work to obtain the convergence estimates not only for derivatives, but also for the nonlocal Gagliardo seminorm.

These difficulties were tackled in several steps, with along the way several technical modifications of the implementation of the argument. We explain the strategy devised by P. Bousquet, A. Ponce, and J. Van Schaftingen [BPVS15] for the case $s \in \mathbb{N}_*$, focusing on the core idea behind all the tools involved in the procedure.

The starting point is again the division of the domain into a grid, and the distinction between good and bad cubes. However, here only an interior energy estimate is required, not assumption on the energy on the boundary is made. Namely, the good cubes are the cubes such that

$$\frac{1}{\eta^{m-sp}} \int_{\sigma} |\mathrm{D}u|^{sp} \leq \kappa^{sp},$$

with κ to be chosen sufficiently small later on.

Opening The first tool used in the argument is the opening procedure, initially introduced by H. Brezis and Li Y. [BL01], and further developed by P. Bousquet, A. Ponce, and J. Van Schaftingen [BPVS15], that we already presented in Section 6.1. Here, this construction is used to lower the number of variables on which the map u depends near the ℓ-skeleton $\mathscr{K}_\eta^\ell$ of the grid, with $\ell = \lfloor sp \rfloor$. More specifically, using as an opening map φ such that φ is constant near the centre of a cube, and such that $\varphi = \mathrm{id}$ outside of a slightly larger region near the center, one can make u constant near the center of a cube, without affecting its value outside of a slightly larger region. Doing the same along an edge, using the variable parametrising the edge as a "dummy variable", one can analogously make u depend only on one variable — the one parametrising the edge — near this edge. Combining these constructions iterated up to dimension ℓ around bad cubes, one produces an opened map u_η^{op}, which depends only locally on ℓ variable near $\mathscr{B}_\eta^\ell$. The construction is illustrated, for $m = 2$ and $\ell = 1$, on Figure 6.5.1. The dark blue region is the one where the map effectively depends on less variables. The light blue region is the transition region, outside of which the map remains unaffected.

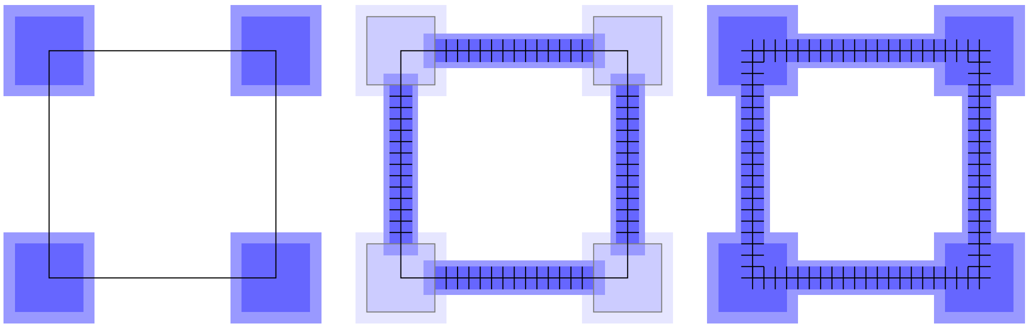

Figure 6.1: The opening procedure for $m = 2$ and $\ell = 1$

In particular, having $sp \geq \ell$, the map u_η^{op} is now VMO in the dark blue region, which allow to apply the VMO machinery, notably approximating by convolution with small parameter and remaining close to the target manifold.

Adaptive convolution By virtue of the estimate on good cubes along with the Poincaré–Wirtinger inequality, we know that if we convolve u_η^{op} with parameter of order η on good cubes, the convolved map will remain close to $\mathscr{N}$. On the other hand, near $\mathscr{B}_\eta^\ell$, as the map u_η^{op} is VMO, we are invited to choose our convolution parameter to be very small, also in order to make the regularised map close to the target manifold $\mathscr{N}$. To deal with these two regime, the idea is to rely on adaptive convolution. Already implicitly present in the proof of the $H = W$ theorem by N. Meyers and J. Serrin [MS64], this technique has been introduced under the form that we present and in the context of mappings to manifolds by R. Schoen and K. Uhlenbeck [SU82]. The idea is to define

$$u_\eta^{\mathrm{sm}}(x) = \int_{\mathbb{B}^m} u(x + \psi_\eta(x)z)\varphi(z)\,\mathrm{d}z,$$

where φ is a standard mollifying kernel and ψ_η a positive function that dictates the size of the (variable) convolution parameter. Classical convolution would correspond to $\psi_\eta = h$ constant. Here, we have the freedom to take ψ_η of different order depending on the region, and we will exploit this to take it very small on the opened region, and of order η on good cubes. Doing so, the map u_η^{sm} is smooth, and close to $\mathscr{N}$ except on the bad cubes, far from the ℓ-skeleton. (Here, we are deliberately omitting to explain how to deal with the transition region, where ψ_η moves from one regime to the other. This is an extra technical difficulty, which requires to introduce a third family of cubes, we refer the reader to [BPVS15] for details.)

Thickening At this stage, it remains to fill in the interior of the bad cubes with suitable values. In the case $s = 1$, this was done via homogeneous extension of the values on the skeleton. To make this procedure smoother, Bousquet, Ponce, and Van Schaftingen introduced the notion of thicking. For $\ell = m - 1$, this would correspond to define on a cube σ

$$v^{\mathrm{th}}(x) = v(\lambda(x)x),$$

where λ is a smooth function chosen so that $\frac{\lambda(x)}{|x|}$ is radially increasing and $\lambda = 1$ near $\partial\sigma$ — the homogeneous extension corresponds to $\lambda(x) = |x|$. This way, v^{th} coincides with v on an *open neighbourhood* of $\partial\sigma$, and not only on $\partial\sigma$. This allows to glue smoothly constructions on various neighbouring cubes. Moreover, the fact that we are using values on a neighbourhood of $\partial\sigma$ to fill in σ implies that the energy of the thickened map is controlled by the energy of the original map on this open set, hence avoiding the need of resorting to another slicing and averaging argument at this stage.

For a smaller value of ℓ, we iterate the thickening construction with respect to the dimension. We use the values near the ℓ-skeleton to fill in the $(\ell + 1)$-faces, then the $(\ell + 2)$-faces, and so on until the bad cubes are completely filled in. The procedure is illustrated for $m = 2$ and $\ell = 0$. The values in dark blue on the left are propagated into the dark blue region on the right. One can see the creation of a singular set, of dimension $m - \ell - 1$, in red.

Applying thickening to u_η^{sm} on all bad cubes, one obtain a map u_η^{th}, which is smooth outside of the singular set, and is close to the target manifold everywhere.

I then remains to project this map back onto $\mathscr{N}$. Moreover, establishing suitable estimates at each step, it can be shown that this map converges towards u as $\eta \to 0$, which concludes the proof of the density of the class $\mathscr{R}_{m-\lfloor sp \rfloor-1}$. More precisely, the opening only modifies u near the bad cubes, with a controlled energy. Then, the smoothened map converges towards u_η^{op} as $\eta \to 0$. Finally, the thickening map only differs from u_η^{sm} near the bad cubes, also with controlled energy. The conclusion follows from the fact that the measure of the bad cubes converges to 0, combined with Lebesgue's lemma.

Shrinking To conclude the proof of the strong approximation theorem, it remains to explain how to remove the singularities in the presence of the topological assumption. This relies on the shrinking procedure, which, as thickening is for the homogeneous extension, is a smoothened version of the procedure we used in the case $s = 1$. More specifically, we start with a topological extension — whose existence is guaranteed by the topological assumption — to fill in a small region near the dual $(m - \lfloor sp \rfloor - 1)$-skeleton, where the singular set lies. Then, we shrink this extension to an even much smaller region, to decrease its energy so that it does not affect the estimates we have established so far. This yields the required approximation with smooth maps. The procedure is illustrated on Figure 6.5.1. The topological extension on the left, in light blue, is shrinked to the light blue region on the right.

A graphical summary of the construction is available on Figure 6.5.1. The domain is first divided into good (in green) and bad (in red) cubes. Then, the opening procedure makes the map u of vanishing mean oscillation on the blue region, and the map is smoothened everywhere except on the red region. Next, the bad red region is filled in with good values from the blue region. Finally, the singularities in red are patched with values in light blue using the topological assumption.

Domains with topology An issue that we eluded so far regards the case where the domain has topology. The proof of the strong density of the class $\mathscr{R}$ works as well in this setting, using the smoothness of the domain to slightly enlarge it, covering it with a grid, an repeating the same argument. Extra difficulties arise when considering the approximation with smooth maps under suitable topological assumptions, and more specifically, at the topological extension step. This is not seen in the special case $m - 1 < p < m$ (with $s = 1$) that we presented in detail, because the procedure is local there: The map we want to approximate has point singularities, and we can patch all of them individually. However, when the singular set has higher dimension, the problem becomes global. Essentially, we have a continuous map defined on the skeleton of dimension $\ell = \lfloor sp \rfloor$ — or a neighbourhood of it — and we want to extend it to the whole domain, filling in the singular set of dimension $m - \ell - 1$ — or a neighbourhood of it.

This is precisely the topological assumption that has to be imposed: Every continuous map from the ℓ-skeleton of Ω into $\mathscr{N}$ has to admit a continuous $\mathscr{N}$-valued extension on the whole Ω. This was understood by Hang F. and Lin F. [HL03a] (in the case $s = 1$).

As special cases, we note that this condition in particular implies that $\pi_\ell(\mathscr{N}) = \{0\}$, and is equivalent to it when the domain is a cube, or more generally contractible. Both conditions also coincide when $\ell = m - 1$, as then the extension can be constructed cell by cell, from the boundary to the interior of the cell. However, examples exists where the extension condition required by Hang and Lin is strictly stronger than $\pi_\ell(\mathscr{N}) = \{0\}$, for instance when both the domain and the target are suitably chosen projective planes.

The case $0 < s < 1$: When homogeneous extensions suffice In the detailed proof we presented for $s = 1$ and $m - 1 < p < m$ as well as in the general version that we sketched in this complements section, we used homogeneous extension — or a variant of it — to fill in the bad cubes. Our comment was that this does not provide a good approximation, but this is fine as there are not too many bad cubes. When $0 < s < 1$ however, a miracle occurs: Homogeneous extension *can* indeed provide a good approximation. This was the key observation by H. Brezis and P. Mironescu in their proof of the strong density theorem for $0 < s < 1$ [BM15]. More precisely, they showed that, if the domain is covered by a grid, and if the grid is chosen appropriately via an averaging argument, then one obtains a good approximation just by filling in the whole domain with values taken from the ℓ-skeleton by homogeneous extension. This provides a conceptually simpler proof — although still quite demanding, because of the tedious computations inherent to the Gagliardo seminorm — as the only tool being used is essentially homogeneous extensions. Moreover, apart from providing a simpler argument, it also has the advantage of making easier working on problems regarding to the singular set of Sobolev mappings for instance, as it leaves *unchanged* the values of the map on the ℓ-skeleton. On the contrary, the approach devised by Bousquet, Ponce, and Van Schaftingen that we sketched above modifies the map on the ℓ-skeleton at the opening and adaptive smoothing steps, which makes for instance taking track of the topology of the restriction of the map to the ℓ-skeleton more delicate.

6.5.2 Other uses of the method of good and bad cubes in geometric analysis

Good and bad cubes-type constructions are not specific to the strong approximation problem for Sobolev mappings. They have also proved useful in other contexts, in the study of different approximation problems for instance. We present below two such instances. But before, on a historical note, let us mention that the prototypical example of this construction is nothing else but. . . the Calderón–Zygmund decomposition, that was notably presented in Functional Analysis II. There is however one major difference between Calderón–Zygmund decompositions and the method of good and bad cubes that we explained above. In Calderón–Zygmund decompositions, the good and bad regions are constructed from cubes of different sizes, taken from a dyadic scale. On the contrary, in the proof of the strong approximation theorem, one relies on a grid made of cubes *of constant size*.

Sobolev homeomorphisms As we already explained, an important question related to nonlinear elasticity is to know whether a Sobolev homeomorphism $f \in W^{1,p}(\Omega; \mathbb{R}^m)$ can be approximated, in the $W^{1,p}$ topology, with piecewise affine homeomorphisms. After various contributions in different settings — see [Hen26] for a survey — the case $W^{1,1}(\Omega; \mathbb{R}^2)$ (that is, $m = 2$ and $p = 1$) was solved by S. Hencl and A. Pratelli, using a good and bad cubes construction.

In their context, good cubes are cubes on which the map is close to an affine map. More specifically, they require the condition

$$\fint_\sigma |\mathrm{D}f - M| \leq \kappa,$$

for some 2×2 matrix M and some sufficiently small number $\kappa > 0$. On good cubes, it is therefore easier to approximate f with piecewise affine maps, at it is itself close to being affine. On bad cubes, they instead merely fill in the cube with a piecewise affine extension of the boundary values with controlled energy. As for the strong approximation problem, the conclusion then follows from the fact that there are not too many bad cubes. Here, this essentially follows from the Lebesgue differentiation theorem applied to $\mathrm{D}f \in L^1(\Omega)$.

An important part of the technical work, that we do not mention here, is about constructing piecewise affine extensions of a piecewise affine map on the boundary on the one hand, and suitably modifying the map f on the 1-skeleton of the grid without distorting it too much on the other hand.

Weak Yang–Mills connections As we briefly mentioned in Section 1.3, in the study of problems related to the Yang–Mills functional, an important step is to define an appropriate class of *weak Yang–Mills connections*, on which the corresponding variational problem is well-posed and enjoys nice properties. On the other hand, just as Sobolev functions, weak Yang–Mills connections are not always easy to manipulate, and there is a need for an approximation theorem for these connections with smooth — or "almost smooth" — connections. Such a result was obtained in dimension $m = 5$ — which is the first supercritical dimension, the critical dimension for the Yang–Mills functional being

$m = 4$, just as for $W^{1,4}$ or $W^{2,2}$ — by R. Caniato and T. Rivière [CR24], also relying on a good and bad cubes approach.

In [CR24], good cubes are defined by a combination of multiple conditions: Essentially, both the curvature and its connection form should be small, and close to their average. However, just as for Sobolev mappings, these conditions are such that the volume of the set formed by all the bad cubes goes to 0 as the size of the grid goes to 0. Another important difference between this setting and mappings to manifolds is that, in [CR24], there is nothing such as a regularisation of the connection inside the cubes. Instead, the approximating connections are entirely constructed from the values on the grid, using suitable extensions.

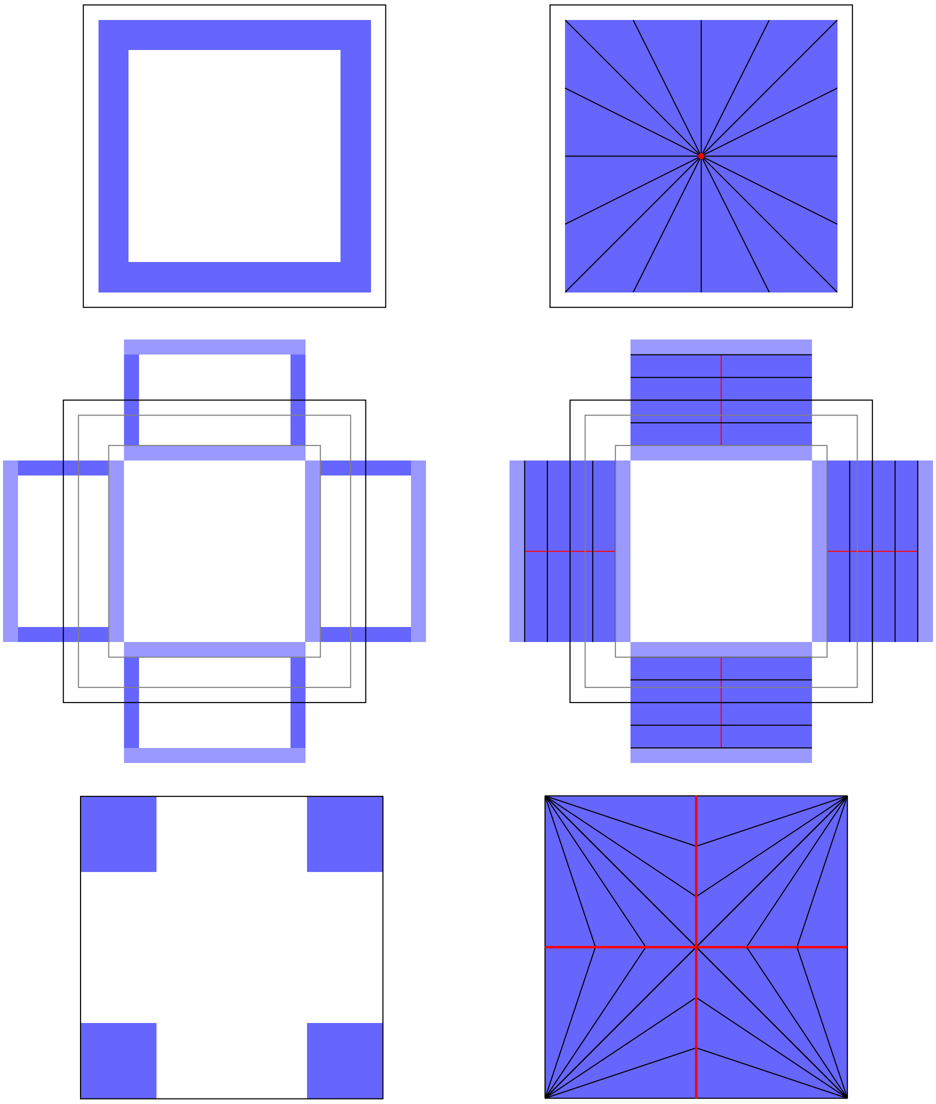

Figure 6.2: Thickening for $m = 2$ and $\ell = 0$

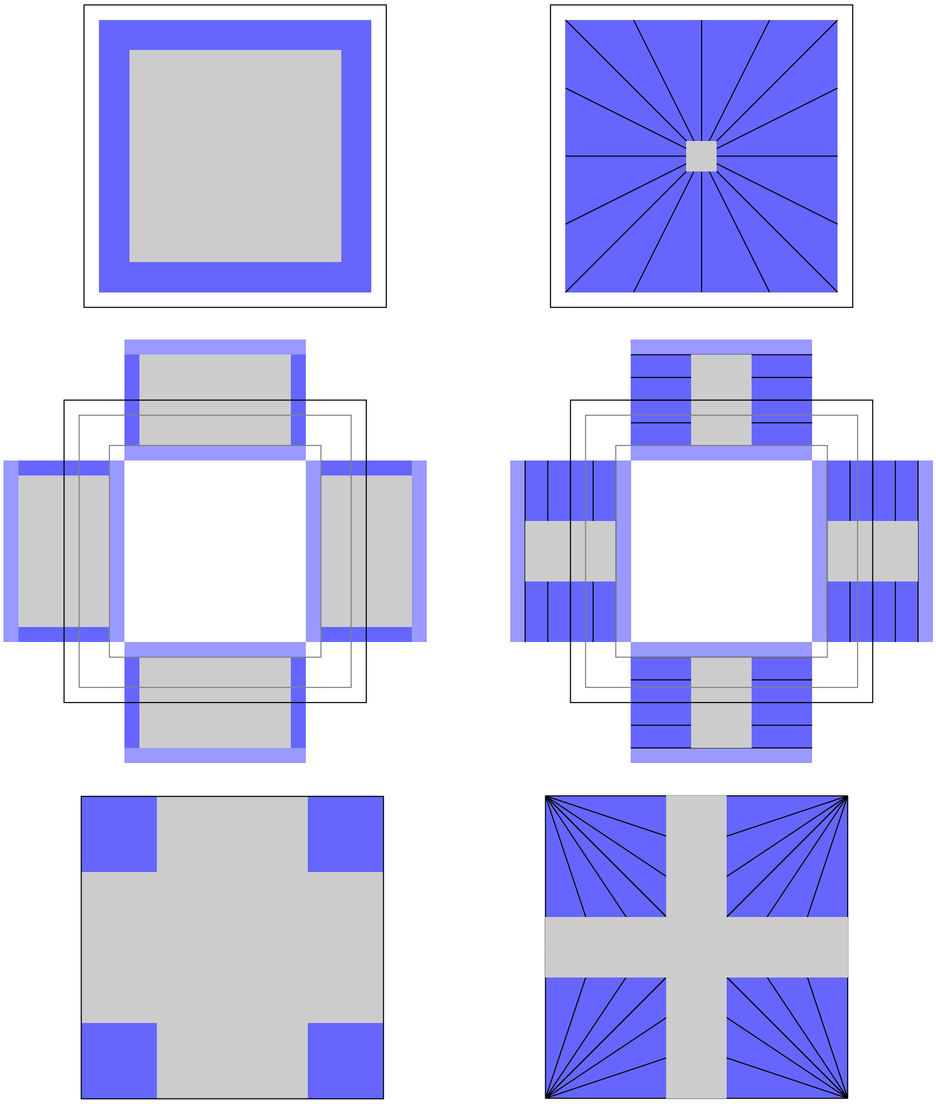

Figure 6.3: Shrinking for $m = 2$ and $\ell = 0$

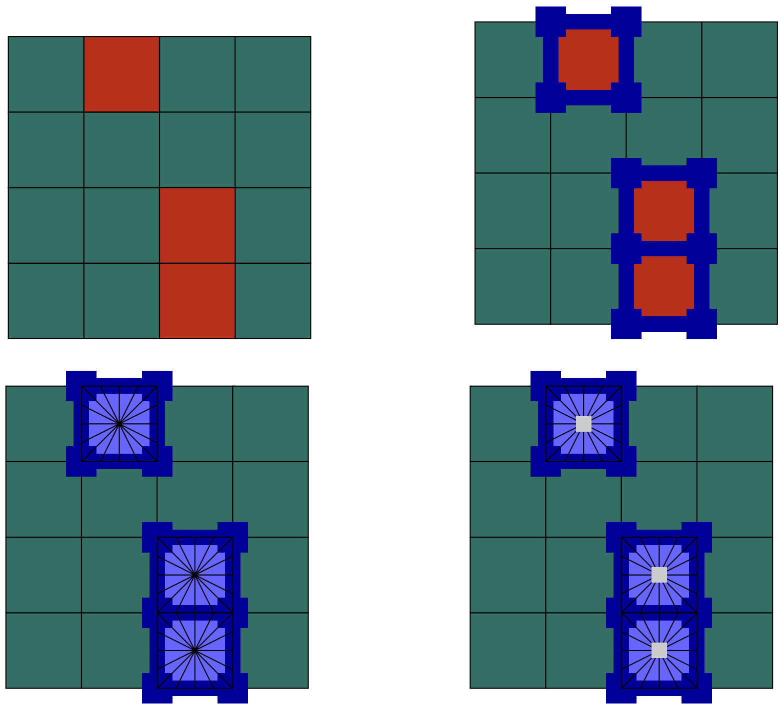

Figure 6.4: Strong approximation for $m = 2$ and $\ell = 1$

7 A glimpse at the next problem: Weak approximation, and the dipole construction

In this last chapter, we slightly broaden our perspective by considering the weak approximation problem. This problem arises naturally once being faced with the failure of the strong approximation property for Sobolev mappings. Indeed, a natural thought is then that we are simply demanding too much with strong convergence, and that we should relax the notion of convergence that we are working with.

Unlike the strong approximation problem, which is nowadays considered as completely solved and comes with a quite neat answer, with one clear necessary and sufficient condition and a nice class of functions that is always dense, the weak approximation problem is much more delicate. To start with, it is still largely open nowadays[1], but there is not even a conjectural quite neat necessary and sufficient condition that has emerged yet.

For this reason, we are only going to see a small portion of the existing results and techniques related to this problem.

7.1 The weak approximation problem

As we already mentioned, the main point of this chapter is to consider a weaker notion of convergence. For the sake of simplicity, here we restrict to $s = 1$.

We say that a sequence $(u_n)_{n\in\mathbb{N}}$ in $W^{1,p}(\mathbb{B}^m;\mathscr{N})$ *weakly converges* to u, and we write $u_n \rightharpoonup u$, whenever

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{B}^m} |\mathrm{D}u_n|^p < +\infty$$

and $u_n \to u$ in measure.

In other words, weak convergence amounts to boundedness in $W^{1,p}$ along with a very weak convergence for the functions themselves. We could also have required convergence in L^1 or almost everywhere and this would have given rise to the same notion, possibly up to extraction of a subsequence.

An important remark concerns the relation between this notion and the usual notion

[1] Which means that if you are looking for cool problems to work on for research, this one can be a fun option.

of linear weak convergence, inherited from the dual space of L^p, that is taught for instance in Functional Analysis I. When $1 < p < +\infty$, both these notions coincide thanks to the reflexivity of L^p, via the Banach–Alaoglu theorem. When $p = 1$ however, weak convergence is somehow ill-behaved due to the lack of reflexivity of L^1, and both notions cease to coincide. Actually, the linear notion is stronger than the one that we adopted, and is equivalent to it under an equi-integrability requirement. From now on, we shall work with weak convergence as defined above without any further comments. It is somehow more intrinsic in spirit, since it does not rely on the fact that $\mathscr{N}$ is embedded to make sense of the linear notion of weak convergence induced by the dual space.

It is straightforward from the definition that strong convergence implies weak convergence. Additionally, weak convergence is somehow a natural notion to work with in the context of problems of partial differential equations or calculus of variations, since it is in general much more easy to obtain weak convergence for sequences of approximate solutions or almost minimizers than strong convergence. It can even happen that weak convergence holds while strong convergence fails.

The natural question to ask is whether smooth mappings are always weakly dense in $W^{1,p}(\mathbb{B}^m; \mathscr{N})$, and if not, if it is possible to characterize the cases where weak density happens. Let us importantly mention that here, by *weak density*, we implicitly mean *weak sequential density*. Indeed, since weak convergence is not metrizable, density and sequential density do not coincide. It has been proved by F. Bethuel [Bet91] that the closure of the space of smooth mappings with respect to the topology induced by weak convergence is the whole Sobolev space of mappings. This comes from the fact that the mappings in the class $\mathscr{R}$ are always weakly approximable, as will become clear later on, combined with the strong density of the class $\mathscr{R}$. On the other hand, as we will see, if we look for weak approximability, things become more involved. For the sake of conciseness, we shall make the abuse of language of speaking about *weak density* when denoting weak sequential density. To use a notation similar as for strong density, we write $H^{1,p}_{\mathrm{W}}(\Omega; \mathscr{N})$ for the set of all $u \in W^{1,p}(\Omega; \mathscr{N})$ such that there exists a sequence of smooth $\mathscr{N}$-valued mappings on Ω weakly converging to u.

7.2 Here we go again: Topological obstructions strike back

We start with some bad news, that is, when $p \notin \mathbb{N}_*$, we gained *nothing* by relaxing the notion of convergence. To illustrate the phenomenon at work, we invoke again our old friend the hedgehog map.

Example 7.2.1. We define once again $u_0 \in W^{1,p}(\mathbb{B}^3; \mathbb{S}^2)$, with $1 \leq p < 3$, by

$$u_0(x) = \frac{x}{|x|}.$$

We claim that u_0 cannot be weakly approximated by smooth mappings in $W^{1,p}$ when $p > 2$. Indeed, assume by contradiction that there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $C^\infty(\mathbb{B}^3;\mathbb{S}^2)$ such that $u_n \rightharpoonup u$. By the polar integration formula and Fatou's lemma, we find

$$\int_0^1 \Big(\liminf_{n\to+\infty} \int_{\partial B_r} |\mathrm{D}u_n|^p\Big)\,\mathrm{d}r \leq \liminf_{n\to+\infty} \int_0^1 \Big(\int_{\partial B_r} |\mathrm{D}u_n|^p\Big)\,\mathrm{d}r = \liminf_{n\to+\infty} \int_{\mathbb{B}^3} |\mathrm{D}u_n|^p < +\infty.$$

Therefore, for every $r \in (0,1)$, up to extraction of a subsequence (here possibly depending on r), we have

$$\sup_{n\in\mathbb{N}} \int_{\partial B_r} |\mathrm{D}u_n|^p < +\infty.$$

Another easy Tonelli argument provides convergence in measure, whence we deduce weak convergence on ∂B_r. By the Rellich–Kondrashov theorem, this implies the uniform convergence of $u_{n|\partial B_r}$ to $u_{|\partial B_r}$, which allows to derive the required contradiction by a homotopy argument as for strong density. □

Exactly as for strong density, this can be extended to the general setting of an arbitrary target with nontrivial $\pi_{\lfloor p\rfloor}$, leading to the following theorem — here, we state the result for an arbitrary value of the regularity parameter.

Theorem 7.2.2. *Assume that $sp < m$ and $sp \notin \mathbb{N}_*$. Then, $H^{s,p}_{\mathrm{W}}(\Omega;\mathscr{N}) = W^{s,p}(\Omega;\mathscr{N})$ implies that $\pi_{\lfloor sp\rfloor}(\mathscr{N}) \simeq \{0\}$.*

When $s = 1$, Theorem 7.2.2 is due to F. Bethuel [Bet91][2].

In Theorem 2.4.1, the argument using the Morrey embedding fails when $p = m$, but this can be overcome by using instead the limiting embedding into VMO. As far as weak density is concerned, this failure is more dramatic, since the argument relies crucially on the *compactness* of the embedding, which is lost in the limiting case even when using VMO as a target space for the embedding. As we will see in the next section, this is not just a matter of the proof, but instead comes from a qualitative difference between $p \notin \mathbb{N}_*$ and $p \in \mathbb{N}_*$ when studying weak approximation.

[2]This theorem has somehow the unfortunate status of folklore unproven result. In the case $s = 1$, it is only sketch in Bethuel's paper. To the best of my knowledge, the general case is proven nowhere in the existing literature. It often happens that such results exist, that experts know for sure to be true but that were not proved anywhere in the literature. They then end up in this awkward situation where everyone would like a clean proof to be completely safe and have a reference to quote, but no one bothers to do it, because this would be a high writing effort for a quite low reward. And of course, I am making myself guilty as well, by reproducing this theorem in these notes without providing a proof of it...

7.3 Moving singularities: The dipole construction

When $p \in \mathbb{N}_*$, the weak approximation problem exhibits a new behaviour. We start with the following model example, featuring once again the hedgehog map.

Example 7.3.1. Let u_0 again denote the hedgehog map. We claim that u_0 can be weakly approximated by smooth maps (although we already know that strong approximation is not feasible).

For this purpose, given any $\eta > 0$, we define a map $\varphi_\eta \colon \mathbb{S}^2 \to \mathbb{S}^2$ such that $\varphi_\eta = \mathrm{id}$ outside of a ball of radius η around the north pole, φ_η has degree zero, and $\sup_{\eta>0} \|\varphi_\eta\|_{W^{1,2}} < +\infty$. For instance, we can rely on the map $\psi \colon \mathbb{B}^2 \to \mathbb{S}^2$ defined by

$$\psi(x) = \Big(\sin(\pi|x|) \cdot \frac{x}{|x|}, \cos(\pi(1-|x|))\Big),$$

which has degree -1 and maps $\overline{\mathbb{B}^2}$ on the north pole. It then suffices to scale ψ to a ball of radius η, and then to proceed to a suitable truncation and rescaling to match it with the identity.

We then define $u_\eta \in W^{1,2}(\mathbb{B}^3; \mathbb{S}^2)$ as the homogeneous extension of φ_η:

$$u_\eta(x) = \varphi_\eta\Big(\frac{x}{|x|}\Big).$$

By the properties of the homogeneous extension, these maps enjoy the following features: (i) $\sup_{\eta>0} \|u_\eta\|_{W^{1,2}} < +\infty$; (ii) $u_\eta \to u_0$ in measure; (iii) $u_{\eta|\partial B_r}$ has degree 0 for every η and every $r \in (0,1)$. By this third property, we may apply the shrinking procedure to approximate each u_η *strongly* in $W^{1,2}$ by smooth mappings into $\mathbb{S}^2$. A diagonal argument allows us to conclude. □

As this is the core of the argument used to prove the non approximability of Sobolev mappings, it is interesting to look at what happens in this example on spheres centred around the origin. Since the map we construct is homogeneous, the behaviour is the same on every such sphere. Over there, the approximating maps have degree 0, but they are made of a degree 1 and a degree -1 map glued together. As $\eta \to 0$, the degree 1 map covers more and more of the sphere, while the degree -1 map shrinks to a point. By the scaling invariance of the Sobolev energy in critical dimension, this can be done with constant energy. At the limit, the degree -1 contribution disappears at the North pole, and we are left with a degree 1 map.

This is a special instance of a general phenomenon that arises in many critical problems in geometric analysis, which is called the *bubbling* phenomenon. The name evokes the fact that something, here topology, concentrates in a smaller and smaller "bubble", which disappears at the limit. In many problems, proving statements involve ruling out bubbling by various considerations, or on the contrary showing that it happens and analysing carefully the outcome. More shall be said in the comments and perspectives section.

Let us come back to Example 7.3.1. One way to understand the construction is that we connected the topological singularity at the centre to another point, here on the boundary, and we modified the map on a small neighbourhood of the connecting line to "move" the topological singularity and evacuate it at the boundary. This is a special case of the *dipole construction*. A more general statement is given below.

For this purpose, we need to define a notion that we call a *pencil-shaped neighborhood*. Let A and $B \in \overline{\mathbb{B}^{\ell+1}}$. We assume that the coordinates $x = (x_1, x')$ in $\mathbb{R}^{\ell+1}$ have been chosen so that x_1 is the coordinate in the direction of the segment from A to B, and x' is the coordinate in a hyperplane orthogonal to this segment. Given $\delta > 0$ and $\gamma > 0$, we define $g_{\delta,\lambda}(x_1) = \min\{\delta, \gamma \operatorname{dist}(x_1, \{A, B\})\}$. The set $U_{\delta,\gamma}$ is defined as

$$U_{\delta,\gamma} = \{x \in \overline{\mathbb{B}^{\ell+1}} \colon x_1 \in (A, B), |x'| < g_{\delta,\gamma}(x_1)\}.$$

Here, $x_1 \in (A, B)$ is an abuse of notation for the fact that the orthogonal projection of x on the line passing through A and B actually lies between A and B. This construction is taken from the work of G. Alberti, S. Baldi, and G. Orlandi [ABO03], with roots in the pioneering work of H. Brezis, J.-M. Coron, and E. Lieb [BCL86] who devised the dipole construction. We refer to [ABO03, Figure 2] for a very nice depiction of the construction.

Proposition 7.3.2. *Let A, $B \in \overline{\mathbb{B}^{\ell+1}}$, let $a \in \mathcal{N}$, and let $\alpha \in \pi_\ell(\mathcal{N})$. For every $\delta > 0$ and $\gamma > 0$, there exists a map $u_{\delta,\gamma} \in W^{1,\ell}(\mathbb{B}^{\ell+1}; \mathcal{N})$ which satisfies the following properties*:

1. *$u_{\delta,\gamma} = a$ outside of $U_{\delta,\gamma}$;*
2. *$u_{\delta,\gamma} \in C(\overline{\mathbb{B}^{\ell+1}} \setminus \{A, B\}; \mathcal{N})$;*
3. *$[u_{\delta,\gamma}]_A = -\alpha$ and $[u_{\delta,\gamma}]_B = \alpha$;*

where $[u_{\delta,\gamma}]_P$ denotes the homotopy class of the restriction of $u_{\delta,\gamma}$ on a small sphere centered on P, which does not depend on the choice of the radius provided that it is taken to be sufficiently small. In addition, for any $\eta > 0$, there exists a choice of $\delta > 0$ and $\gamma > 0$ sufficiently small so that $|U_{\delta,\gamma}| \leq \eta$ and

$$\int_{\mathbb{B}^{\ell+1}} |Du_{\delta,\gamma}|^\ell \leq C_\alpha |B - A|,$$

where $C_\alpha > 0$ is a constant depending only on α.

The proof is intuitively clear, but technically a bit painful, so we omit the details. The idea is to fix a representative of the homotopy class α on $\mathbb{B}^\ell$ with constant boundary value a, and to fill in the pencil shaped neighbourhood with scaled copies of this representative. By the scaling invariance of the Sobolev energy in critical dimension, all scaled copies have the same energy, and the total energy is therefore this energy multiplied by the length of the dipole, by Tonelli's theorem.

The main technicality is how to deal with the region where the width is shrinking to 0, and where notably the derivative also has a component in the direction of the dipole. This is where one needs to take γ and δ, so that this contribution is negligible in front of the radial one.

In particular, this shows that the constant C_α can essentially be taken as the least Sobolev energy to realise a representative of α on $\mathbb{B}^\ell$ with boundary value a.

7.4 Weak approximation in $W^{1,2}(\mathbb{B}^3;\mathbb{S}^2)$: When optimal transport meets geometric analysis

We saw in Example 7.3.1 that the hedgehog map is weakly approximable in $W^{1,2}(\mathbb{B}^3;\mathbb{S}^2)$. This is actually not an isolated phenomenon, but is on the contrary a special instance of the fact that the weak approximation property holds in $W^{1,2}(\mathbb{B}^3;\mathbb{S}^2)$. This was proved by F. Bethuel [Bet90]. We state a version valid for any dimension of spheres.

Theorem 7.4.1. *For every $\ell \in \mathbb{N}_*$,*

$$H^{1,\ell}_{\mathrm{W}}(\mathbb{B}^{\ell+1};\mathbb{S}^\ell) = W^{1,\ell}(\mathbb{B}^{\ell+1};\mathbb{S}^\ell).$$

There are two main ingredients entering the proof of Theorem 7.4.1. The first one is the dipole construction from the previous section. The second one is the ability to connect singularities together with a controlled length for the connecting set, so that we can apply the dipole construction along this connecting set. The exact statement that we use is as follows.

Proposition 7.4.2. *Let $u \in \mathscr{R}_0(\mathbb{B}^{\ell+1};\mathbb{S}^\ell)$. There exists $y \in \mathbb{S}^\ell$ a regular image for u such that the set $u^{-1}(\{y\})$ is a smooth 1-dimensional manifold whose length satisfies*

$$\mathscr{H}^1(u^{-1}(\{y\})) \lesssim \int_{\mathbb{B}^{\ell+1}} |\mathrm{D}u|^\ell.$$

Before we prove this proposition, let us explain why it provides the desired connection. First of all, we observe that the curves constituting $u^{-1}(\{y\})$ may be given a canonical orientation induced by u. Indeed, at each point of these curves, the pullback by u of a basis of the tangent plane of the sphere at the point y orients a plane transversal to the curve at this point, and it suffices to take an orientation on the curve compatible with this plane and the orientation on the ambient space.

Then, by the formula for the degree given by the sum of the signs of the Jacobian at all points in the preimage of a regular image, we know that any sufficiently small sphere centered around a singularity a of u should cross $u^{-1}(\{y\})$ a number of times, counted with a sign accounting for the orientation, equal to the degree of u on this sphere. Otherwise stated, in the language of geometric measure theory,

$$\partial(u^{-1}(\{y\})) = S_u \quad \text{in } \mathbb{B}^{\ell+1}.$$

We shall not attempt to give the formal definitions required to explain rigorously this identity — this will be discussed further in the complements — but it tells us exactly that $u^{-1}(\{y\})$ is a connection for the singular set of u, taking the multiplicities of the singularities into account.

The proof is an elegant argument relying on the co-area formula.

Proof. Since u is smooth in $\mathbb{B}^{\ell+1}\setminus\mathcal{S}_u$, it follows from the Morse–Sard theorem that almost every point $y \in \mathbb{S}^\ell$ is a regular image for u. For such points, $u^{-1}(\{y\})$ is indeed a smooth 1-dimensional manifold by virtue of the submersion theorem.

We now invoke the co-area formula:

$$\int_{\mathbb{S}^\ell} \mathcal{H}^1(u^{-1}(\{y\}))\,\mathrm{d}y = \int_{\mathbb{B}^{\ell+1}} Ju,$$

where the Jacobian is computed as the determinant of the differential from $\mathbb{R}^{\ell+1}$ to the tangent bundle to $\mathbb{S}^\ell$. Since the Jacobian is the product of the singular values of $\mathrm{D}u$, while the norm of the differential is the square root of the sum of those singular values, the arithmetico-geometric inequality implies that

$$Ju \leq \frac{1}{\ell^{\ell/2}}|\mathrm{D}u|^\ell.$$

Hence,

$$\int_{\mathbb{S}^\ell} \mathcal{H}^1(u^{-1}(\{y\}))\,\mathrm{d}y \lesssim \int_{\mathbb{B}^{\ell+1}} |\mathrm{D}u|^\ell.$$

This ensures the existence of a point y, regular image of u, satisfying

$$\mathcal{H}^1(u^{-1}(\{y\}))\,\mathrm{d}y \lesssim \int_{\mathbb{B}^{\ell+1}} |\mathrm{D}u|^\ell. \qquad \square$$

Now, the proof of Theorem 7.4.1 is just a matter of combining the dipole construction with our connection result. Namely, one starts by invoking Theorem 1.2.6 to approximate any $u \in W^{1,\ell}(\mathbb{B}^{\ell+1};\mathbb{S}^\ell)$ map with mappings in the class $\mathscr{R}_0(\mathbb{B}^{\ell+1};\mathbb{S}^\ell)$.

For each map $v \in \mathscr{R}_0(\mathbb{B}^{\ell+1};\mathbb{S}^\ell)$, one can consider the connecting set $v^{-1}(\{y\})$ given by Proposition 7.4.2, and insert along it a dipole, following Proposition 7.3.2. The chosen topological charge for the dipoles should be a degree 1 map. After this insertion, one obtain a map $w \in \mathscr{R}_0(\mathbb{B}^{\ell+1};\mathbb{S}^\ell)$, which has the same singularities as w, whose restriction on every small sphere around one of these singularities is nullhomotopic, and satisfying the energy estimate

$$\int_{\mathbb{B}^{\ell+1}} |\mathrm{D}w|^\ell \leq \int_{\mathbb{B}^{\ell+1}} |\mathrm{D}v|^\ell + C\mathscr{H}^1(v^{-1}(\{y\})) \lesssim \int_{\mathbb{B}^{\ell+1}} |\mathrm{D}v|^\ell. \tag{7.4.1}$$

Moreover, the maps w and v coincide outside of a set of arbitrarily small measure. Now, as the topological singularities of the map w are all trivial, this map can in turn, but the topological extension and shrinking procedures, be approximated with smooth maps.

We apply this construction to the sequence in $\mathscr{R}_0(\mathbb{B}^{\ell+1};\mathbb{S}^\ell)$ approximating u. By (7.4.1), the Sobolev energy of the resulting approximating smooth maps remains bounded along the sequence, and we conclude via a diagonal argument.

To be completely rigorous, there are some technical details to be dealt with. Indeed, the dipole construction that we presented only allows to insert dipoles along segments, and the map has to be constant in a pencil-shaped neighbourhood of this segment to perform the insertion. The second issue is overcome by enlarging slightly the region where the map is constant, from the connecting curve itself to a neighbourhood of it. For smooth maps, this can be done with controlled energy, taking this region to be sufficiently small. For the first issue, several options are possible. A first one is to adapt the dipole construction to make it work along a curve. A second one is to instead connect singularities with segments. Indeed, as the shortest path between two points is a segment, there is a connection for the singularities, made of segments, whose total length is smaller than the one provided by Proposition 7.4.2. But then, these lines do not correspond to inverse images of points by u anymore, which causes extra technical difficulties to insert a dipole. An intermediate solution is, after having performed the enlargement of the constancy region, to approximate the connecting curves by broken lines staying inside this constancy region. This would be like a baby version of the *polyhedral approximation theorem* from geometric measure theory.

We insist on the importance of the bound (7.4.1). Indeed, as the previous argument essentially shows, any map in the class $\mathscr{R}$ can be weakly approximated with smooth maps: It suffices to connect the singularities together or to the boundary, and eliminate the topological singularities along this connection. However, without a bound on the energy of the approximating smooth maps obtained by this process, there is no guarantee that this energy would not blow up along an approximating sequence in the class $\mathscr{R}$, not allowing to extend the weak approximability to the whole Sobolev space by a density argument.

Actually, we are in presence of one more instance where the qualitative property of being weakly approximable can be associated with a quantitative problem, that is, what is the least Sobolev energy required to weakly approximate a given map. This, in turn, allows to hope relying on the nonlinear uniform boundedness to generate counterexamples in other cases. We shall come back to this in the complement section.

7.5 Comments and perspectives

7.5.1 The current landscape of the weak approximation problem

As we said at the beginning of the chapter, the landscape of the weak approximation problem is still widely open. Let us review the state of the art, with brief explanations and references. As we already said that $H^{s,p}_{\mathrm{W}} = H^{s,p}_{\mathrm{S}}$ whenever $sp \notin \mathbb{N}_*$, we restrict to $sp \in \mathbb{N}_*$.

- In [Haj94], P. Hajłasz proved that the weak approximation property holds in $W^{1,p}(\Omega; \mathscr{N})$ whenever $\mathscr{N}$ is $(p-1)$-connected, relying on the method of almost projection. Comparing with the strong approximation problem, here $\pi_p(\mathscr{N})$ can be arbitrary, but all the previous homotopy groups have to vanish. In particular, the weak approximation property always holds in $W^{1,1}$ — even though this is somehow one more hint at the fact that the weak convergence in L^1 is ill-behaved, see the work of Hang F. [Han02] when an equi-integrability condition is added, and the recent generalisation by J. Van Schaftingen [VS25a] — and holds for simply connected targets in $W^{1,2}$. This has been extended to $s \geq 1$ by P. Bousquet, A. Ponce, and J. Van Schaftingen [BPVS13], but the proof does not seem to adapt, at least not in a straightforward way, when $0 < s < 1$.

- In [PR03], M. R. Pakzad and T. Rivière divised another approach for weak approximation, relying on dipoles and connections, that yields weak approximation in $W^{1,2}(\Omega; \mathscr{N})$ for simply connected targets $\mathscr{N}$, reproving the result of Hajłasz with a different technique, but also covering some — but not all — non-simply connected targets.

- In [HR15], R. Hardt and T. Rivière proved that the weak approximation property holds in $W^{2,2}(\mathbb{B}^5; \mathbb{S}^3)$. Although the proof is restricted to this very specific setting for technical reasons, the key ingredient is that the relevant homotopy group for the strong approximation problem is finite: $\pi_4(\mathbb{S}^3) = \mathbb{Z}/2\mathbb{Z}$. At the core of the argument is still a construction with dipoles and connections.

- The first ever *local* obstruction to the weak approximation problem was provided by Bethuel's groundbreaking[3] counterexample [Bet20], showing that the weak approximation property fails in $W^{1,3}(\mathbb{B}^4; \mathbb{S}^2)$. Global obstruction were already known since the work of Hang and Lin [HL03b], but Bethuel's work features the first ever local obstruction, which arises when the domain is a ball for instance. As we already explained in Section 5.6, the proof relies on the nonlinear uniform boundedness principle. Ingredients to construct the sequence with superlinearly growing relaxed energy include the Hopf invariant (which is associated with the relevant group $\pi_3(\mathbb{S}^2)$) and the Pontryagin construction, and the estimate of the relaxed energy of the constructed sequence is carried out with the help of the theory of scans by Hardt and Rivière as well as branched optimal transportation.

  Two complete families of counterexamples were constructed in [DVS24], showing that on the one hand, the weak approximation property may fail in $W^{1,p}$ for every $p \geq 2$ for a well-chosen target, and on the other hand that every nontrivial Hopf invariant gives rise to an obstruction. This also relies on the nonlinear uniform boundedness principle, combined respectively with the isoperimetric inequality and estimates for the energy associated with the Hopf invariant. The counterexamples extend to $s \geq 1$, but do not seem to extend in a straightforward way to $0 < s < 1$.

7.5.2 Minimal connections and relaxed energies

As we mentioned in Section 5.6, the weak approximation problem can be quantified, defining the *relaxed energy* as

$$\mathcal{E}^{s,p}_{\mathrm{rel}}(u) = \inf\Big\{\liminf_{n\to+\infty} \mathcal{E}^{s,p}(u_n, \Omega) : u_n \in C^\infty(\Omega; \mathcal{N}),\, u_n \to u \text{ in measure}\Big\}.$$

This quantity was in turn intensively studied, and also led to new results, sometimes outside of the field of Sobolev mappings into manifolds.

A first notable result is that the relaxed energy of a map is connected to the minimal connection for its singular set. This was to be expected after the proof of the weak ap-

[3] To see how surprising this counterexample came, it is insightful to do some version study on the arXiv. Indeed, before Bethuel's work, it was a quite widespread belief in the community of experts of Sobolev mappings that the weak approximation property should always hold for integer p on a ball. This is more or less explicitly stated as a conjecture in the preprint version of [HR15]. Bethuel's work was then released as a preprint in 2014, and in the published version of [HR15], the corresponding explanation was rewritten and included notably a mention of Bethuel's recent work.

proximation property for sphere-valued maps that we presented, but the beautiful result is that there is an exact formula connecting both. In the model case $u \in W^{1,2}(\mathbb{B}^3; \mathbb{S}^2)$, we have

$$\mathcal{E}^{1,p}_{\mathrm{rel}}(u) = \mathcal{E}^{1,p}(u) + 8\pi L(u),$$

where $L(u)$ is the length of the minimal connection for the singular set of u, that is, the length of the minimal set whose boundary is Ju — using the language of geometric measure theory. This was proved by H. Brezis, J.-M. Coron, and E. Lieb [BCL86]. Generalisations to higher-dimensional domains and other targets where obtained by M. R. Pakzad [Pak02], and M. Giaquinta and D. Mucci [GM05]. A difficulty to extend the result to higher dimensions comes from subtle issues from geometric measure theory. Long story short, when connecting points, the minimal mass of a rectifiable current connecting the points — computed as the length of the associated rectifiable curves with multiplicities — coincides with the minimal mass of a general current connecting the points — computed by duality against Lipschitz functions. Actually, the minimal connection is given by straight lines. This is somehow connected to optimal transport, see [BM21] for a much complete discussion on this topic. On the contrary, in higher dimensions, minimising among rectifiable or general currents with a prescribed boundary might yield different results. This is a surprising result by L.C. Young, with other examples by F. Morgan and also B. White, that it might be more economic to use "half-surfaces" rather than surfaces of multiplicity 1. Hence, we have two quantities to work with, and the right one to extend the above formula for the relaxed energy is the largest one, where the minimisation is only over rectifiable currents. It is somehow interesting — or surprising — that such subtle issues from geometric measure theory have impact in the study of mappings to manifolds.

In turn, the relaxed energy has proved useful to study notably problems related to harmonic maps. Indeed, the extra term $L(u)$ in the formula for the relaxed energy is a null-Lagrangian, implying that minimisers of the relaxed energy are (weakly) harmonic maps. One can also replace $L(u)$ by $L(u,v)$, defined as the length of the minimal connection between the singularities of u and v, for some fixed Sobolev map v. Using either of these quantities allows intuitively to penalise the apparition of singularities for minimising maps, or to pin them according to the ones of v. As one example (among others) of such an application of the relaxed energy, we mention the result by T. Rivière [Riv95] according to which there exists *everywhere discontinuous* weakly harmonic maps into $\mathbb{S}^2$. This emphasises how different — and difficult — the regularity theory for harmonic maps is compared with corresponding linear problems. The key idea is to use a map v with a dense topological singular set, to "pin" singularities everywhere in the domain for the minimiser obtained using the relaxed energy.

7.5.3 Bubbling phenomena in geometric analysis

In Example 7.3.1, we saw that in critical dimension, the topology of Sobolev mappings can concentrate along a sequence, to form a “bubble” that disappears at the limit. Such phenomena are ubiquitous in critical problems, typically when the energy under consideration is invariant by scaling — which is a particular case of conformal invariance. They arise notably, but not only, in the study of topological properties of Sobolev mappings, for m-harmonic maps, for Yang–Mills in dimension 4, and so on.

To give (once again, in a highly non-exhaustive way) some more concrete examples, let us focus on harmonic maps. In this topic, a careful analysis of bubbling along a sequence was used by J. Sacks and K. Uhlenbeck [SU81] to establish an *energy identity*, showing that every homotopy class either admits a minimising harmonic map, or can be decomposed as a sum of homotopy classes with a minimising harmonic map. In other words, homotopy classes with a minimising harmonic map form a generating set of the homotopy group. In [Par96], T. Parker refined the bubbling analysis of Sacks and Uhlenbeck, showing that a weakly converging sequence of harmonic maps converges to a *bubble tree harmonic map*, which encodes the weak limit plus all the bubbles forming in the process, taking into account that small bubbles can form inside larger bubbles, and iteratively so. An application of the Sacks–Uhlenbeck theory is the proof by T. Rivière [Riv98] that infinitely many homotopy classes in $\pi_3(\mathbb{S}^2)$ admit a minimising harmonic map — even though the proof does not give an explicit such class, actually it is not even known if there exists a minimising harmonic map of Hopf degree 1.

Bibliography

[Ada75] R. A. Adams, *Sobolev spaces*, Pure Appl. Math., no. 65, Academic Press, 1975. 10, 22, 69

[ABO03] G. Alberti, S. Baldo, and G. Orlandi, *Functions with prescribed singularities*, J. Eur. Math. Soc. (JEMS) **5** (2003), no. 3, 275–311. 111

[ADF15] F. Alouges and G. Di Fratta, *Homogenization of composite ferromagnetic materials*, Proc. A. **471** (2015), no. 2182, 19, Id/No 20150365. 9

[Bal76] J. M. Ball, *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Ration. Mech. Anal. **63** (1976), 337–403. 35

[BS27] S. Banach and H. Steinhaus, *Sur le principe de la condensation de singularités*, Fundam. Math. **9** (1927), 50–61. 84

[Bet90] F. Bethuel, *A characterization of maps in $H^1(B^3, S^2)$ which can be approximated by smooth maps*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **7** (1990), no. 4, 269–286. 35, 112

[Bet91] F. Bethuel, *The approximation problem for Sobolev maps between two manifolds*, Acta Math. **167** (1991), 153–206. 12, 13, 61, 87, 108, 109

[Bet14] F. Bethuel, *A new obstruction to the extension problem for Sobolev maps between manifolds*, J. Fixed Point Theory Appl. **15** (2014), no. 1, 155–183. 65, 85

[Bet20] F. Bethuel, *A counterexample to the weak density of smooth maps between manifolds in Sobolev spaces*, Invent. Math. **219** (2020), no. 2, 507–651. 86, 116

[BC07] F. Bethuel and D. Chiron, *Some questions related to the lifting problem in Sobolev spaces*, Perspectives in nonlinear partial differential equations in honor of Haïm Brezis. (H. Berestycki, M. Bertsch, F. E. Browder, L. Nirenberg, L. A. Peletier, and L. Véron, eds.), Contemp. Math., vol. 446, American Mathematical Society (AMS), Providence, R.I, 2007, pp. 125–152. 8, 50

[BCDH91] F. Bethuel, J.-M. Coron, F. Demengel, and F. Hélein, *A cohomological criterion for density of smooth maps in Sobolev spaces between two manifolds*, no. 332, pp. 15–23, Springer, 1991. 35

[BD95] F. Bethuel and F. Demengel, *Extensions for Sobolev mappings between manifolds*, Calc. Var. Partial Differential Equations **3** (1995), no. 4, 475–491. 61

[BZ88] F. Bethuel and X. Zheng, *Density of smooth functions between two manifolds in Sobolev spaces*, J. Funct. Anal. **80** (1988), no. 1, 60–75. 11, 12, 13, 32, 51, 62

[BBM00] J. Bourgain, H. Brezis, and P. Mironescu, *Lifting in Sobolev spaces*, J. Anal. Math. **80** (2000), no. 1, 37–86. 50

[BBM05] J. Bourgain, H. Brezis, and P. Mironescu, *Lifting, degree, and distributional Jacobian revisited*, Comm. Pure Appl. Math. **58** (2005), no. 4, 529–551. 35

[Bou07] P. Bousquet, *Topological singularities in* $W^{S,P}(S^N, S^1)$, J. Anal. Math. **102** (2007), no. 1, 311–346. 35

[BM14] P. Bousquet and P. Mironescu, *Prescribing the Jacobian in critical spaces*, J. Anal. Math. **122** (2014), no. 1, 317–373. 35

[BPVS13] P. Bousquet, A. C. Ponce, and J. Van Schaftingen, *Density of smooth maps for fractional Sobolev spaces* $W^{s,p}$ *into* ℓ *simply connected manifolds when* $s \geq 1$, Confluentes Math. **5** (2013), no. 2, 3–22. 39, 46, 57, 115

[BPVS14] P. Bousquet, A. C. Ponce, and J. Van Schaftingen, *Strong approximation of fractional Sobolev maps*, J. Fixed Point Theory Appl. **15** (2014), no. 2, 133–153. 47, 62

[BPVS15] P. Bousquet, A. C. Ponce, and J. Van Schaftingen, *Strong density for higher order Sobolev spaces into compact manifolds*, J. Eur. Math. Soc. (JEMS) **17** (2015), no. 4, 763–817. 12, 13, 71, 87, 96, 97, 98

[BPVS25] P. Bousquet, A. C. Ponce, and J. Van Schaftingen, *Generic topological screening and approximation of Sobolev maps*, 2025, arXiv: `2501.18149 [math.FA]`. 27, 35, 71

[Bre11] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Universitext, Springer, 2011. 10, 22

[BCL86] H. Brezis, J.-M. Coron, and E. H. Lieb, *Harmonic maps with defects*, Comm. Math. Phys. **107** (1986), no. 4, 649–705. 111, 117

[BL01] H. Brezis and Y. Li, *Topology and Sobolev spaces*, J. Funct. Anal. **183** (2001), no. 2, 321–369. 71, 97

[BMM24] H. Brezis, J. Mawhin, and P. Mironescu, *A brief history of the Jacobian*, Commun. Contemp. Math. **26** (2024), no. 2, Id/No 2330001. 35

[BM01a] H. Brezis and P. Mironescu, *Gagliardo-Nirenberg, composition and products in fractional Sobolev spaces*, J. Evol. Equ. **1** (2001), no. 4, 387–404. 39

[BM01b] H. Brezis and P. Mironescu, *On some questions of topology for S^1-valued fractional Sobolev spaces*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **95** (2001), no. 1, 121–143. 50

[BM15] H. Brezis and P. Mironescu, *Density in $W^{s,p}(\Omega; N)$*, J. Funct. Anal. **269** (2015), no. 7, 2045–2109. 12, 13, 100

[BM18] H. Brezis and P. Mironescu, *Gagliardo-Nirenberg inequalities and non-inequalities: The full story*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **35** (2018), no. 5, 1355–1376. 24

[BM21] H. Brezis and P. Mironescu, *Sobolev maps to the circle*, Progr. Nonlinear Differential Equations Appl., no. 96, Birkhäuser, 2021. 16, 117

[BN95] H. Brezis and L. Nirenberg, *Degree theory and BMO; I: Compact manifolds without boundaries*, Selecta Math. (N.S.) **1** (1995), no. 2, 197–263. 30

[CO19] G. Canevari and G. Orlandi, *Topological singular set of vector-valued maps, I: applications to manifold-constrained Sobolev and BV spaces*, Calc. Var. Partial Differential Equations **58** (2019), no. 2, 40 p. 35, 61

[CO26] G. Canevari and G. Orlandi, *Topological singular set of manifold-valued maps weakly approximable by smooth maps*, 2026, arXiv: `2605.28622 [math.FA]`. 35

[CR24] R. Caniato and T. Rivière, *Coulomb gauges and regularity for stationary weak Yang–Mills connections in supercritical dimension*, 2024, arXiv: `2411.19910 [math.DG]`. 18, 61, 102

[DH24] E. Davoli and L. Happ, *Two-scale density of almost smooth functions in sphere-valued Sobolev spaces: A high-contrast extension of the Bethuel-Zheng theory*, 2024, arXiv: `2411.07838 [math.AP]`. 9

[Det25a] A. Detaille, *An improved dense class in Sobolev spaces to manifolds*, J. Funct. Anal. **289** (2025), no. 2, Article ID 110894, 43 p. 51

[Det25b] A. Detaille, *A surprising threshold for the validity of the method of singular projection*, 2025, arXiv: `2509.00920 [math.FA]`. 65

[Det26] A. Detaille, *A complete answer to the strong density problem in Sobolev spaces with values into compact manifolds*, J. Eur. Math. Soc. (JEMS) (2026), published online first. 12, 13, 87

[DMX25] A. Detaille, P. Mironescu, and K. Xiao, *Pullback of closed forms by low regularity maps to manifolds, and applications*, hal-05050996, 2025. 36

[DVS24] A. Detaille and J. Van Schaftingen, *Analytical obstructions to the weak approximation of Sobolev mappings into manifolds*, 2024, arXiv: `2412.12889 [math.FA]`. 86, 116

[DNPV12] E. Di Nezza, G. Palatucci, and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), no. 5, 521–573. 22

[ET58] J. L. Ericksen and C. Truesdell, *Exact theory of stress and strain in rods and shells*, Arch. Ration. Mech. Anal. **1** (1958), 295–323. 9

[FF60] H. Federer and W. H. Fleming, *Normal and integral currents*, Ann. of Math. **72** (1960), no. 3, 458–520. 14, 52, 60

[Foo84] R. L. Foote, *Regularity of the distance function*, Proc. Amer. Math. Soc. **92** (1984), no. 1, 153–155. 29

[Gag57] E. Gagliardo, *Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili*, Rend. Semin. Mat. Univ. Padova **27** (1957), 284–305. 16

[Gag59] E. Gagliardo, *Ulteriori proprietà di alcune classi di funzioni in più variabili*, Ric. Mat. **8** (1959), 24–51. 24

[Gas16] A. Gastel, *Partial regularity of polyharmonic maps to targets of sufficiently simple topology*, Z. Anal. Anwend. **35** (2016), no. 4, 397–410. 62

[GHP24] C. Gavioli, L. Happ, and V. Pagliari, *An extension operator for manifold-valued Sobolev maps on perforated domains*, 2024, arXiv: `2403.11690v2 [math.AP]`. 9

[GMS98a] M. Giaquinta, G. Modica, and J. Souček, *Cartesian currents in the calculus of variations I: Cartesian currents*, Ergeb. Math. Grenzgeb., no. 37, Springer, 1998. 35

[GMS98b] M. Giaquinta, G. Modica, and J. Souček, *Cartesian currents in the calculus of variations II: Variational integrals*, Ergeb. Math. Grenzgeb., no. 38, Springer, 1998. 35

[GM05] M. Giaquinta and D. Mucci, *The relaxed Dirichlet energy of mappings into a manifold*, Calc. Var. Partial Differential Equations **24** (2005), no. 2, 155–166. 117

[Haj94] P. Hajłasz, *Approximation of Sobolev mappings*, Nonlinear Anal. **22** (1994), no. 12, 1579–1591. 62, 115

[Han02] F. Hang, *Density problems for $W^{1,1}(M, N)$*, Comm. Pure Appl. Math. **55** (2002), no. 7, 937–947. 115

[HL03a] F. Hang and F. Lin, *Topology of Sobolev mappings II*, Acta Math. **191** (2003), no. 1, 55–107. 13, 16, 100

[HL03b] F. Hang and F. Lin, *Topology of Sobolev mappings III*, Comm. Pure Appl. Math. **56** (2003), no. 10, 1383–1415. 65, 86, 116

[HK88] R. Hardt and D. Kinderlehrer, *Mathematical questions of liquid crystal theory*, Nonlinear partial differential equations and their applications, Pitman Res. Notes Math. Ser., no. 181, Longman Sci. Tech., 1988, pp. 276–289. 8

[HKL86] R. Hardt, D. Kinderlehrer, and F. Lin, *Existence and partial regularity of static liquid crystal configurations*, Comm. Math. Phys. **105** (1986), no. 4, 547–570. 62

[HL87] R. Hardt and F. Lin, *Mappings minimizing the L^p norm of the gradient*, Comm. Pure Appl. Math. **40** (1987), no. 5, 555–588. 14, 52, 61, 62

[HR03] R. Hardt and T. Rivière, *Connecting topological Hopf singularities*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **2** (2003), no. 2, 287–344. 35

[HR08] R. Hardt and T. Rivière, *Connecting rational homotopy type singularities*, Acta Math. **200** (2008), no. 1, 15–83. 35

[HR15] R. Hardt and T. Rivière, *Sequential weak approximation for maps of finite Hessian energy*, Calc. Var. Partial Differential Equations **54** (2015), no. 3, 2713–2749. 115, 116

[Hat02] A. Hatcher, *Algebraic topology*, Cambridge University Press, 2002. 12

[Hed72] L. I. Hedberg, *On certain convolution inequalities*, Proc. Amer. Math. Soc. **36** (1972), no. 2, 505–510. 55

[HW08] F. Hélein and J. C. Wood, *Harmonic maps*, Handbook of global analysis, Elsevier, 2008, pp. 417–491. 17

[Hen26] S. Hencl, *Ball-Evans approximation problem: recent progress and open problems*, J. Math. Sci. (N.Y.) **298** (2026), no. 4, 524–531. 17, 101

[HP18] S. Hencl and A. Pratelli, *Diffeomorphic approximation of $W^{1,1}$ planar Sobolev homeomorphisms*, J. Eur. Math. Soc. (JEMS) **20** (2018), no. 3, 597–656. 17

[Hil23] T. H. Hildebrandt, *On uniform limitedness of sets of functional operations*, Bull. Amer. Math. Soc. **29** (1923), 309–315. 84

[HTWB11] J. Huang, Y. Tong, H. Wei, and H. Bao, *Boundary aligned smooth 3D cross-frame field*, ACM Transactions on Graphics **30** (2011), no. 6, 1–8. 9

[LMR26] T. Lan, D. Martino, and T. Rivière, *The Analysis of Willmore Surfaces and its Generalizations in Higher Dimensions*, 2026, arXiv: `2511.01777 [math.DG]`. 18

[Leo17] G. Leoni, *A first course in Sobolev spaces*, Grad. Stud. Math., no. 181, American Mathematical Society (AMS), 2017. 22

[Leo23] G. Leoni, *A first course in fractional Sobolev spaces*, Grad. Stud. Math., no. 229, American Mathematical Society (AMS), 2023. 10, 22

[MR] L. Martinazzi and T. Rivière, *Gluing instantons à la Brezis-Coron in dimension four and the dipole construction*. 15

[MMS18] K. Mazowiecka, M. Miśkiewicz, and A. Schikorra, *On the size of the singular set of minimizing harmonic maps*, 2018, arXiv: `1811.00515 [math.AP]`. 62

[Maz11] V. Maz'ya, *Sobolev spaces*, Grundlehren Math. Wiss., no. 342, Springer, 2011. 22

[MS02] V. Maz'ya and T. Shaposhnikova, *An elementary proof of the Brezis and Mironescu theorem on the composition operator in fractional Sobolev spaces*, J. Evol. Equ. **2** (2002), no. 1, 113–125. 39, 46, 57

[Mer79] N. D. Mermin, *The topological theory of defects in ordered media*, Rev. Modern Phys. **51** (1979), no. 3, 591–648. 8

[MS64] N. G. Meyers and J. Serrin, $H = W$, Proc. Natl. Acad. Sci. USA **51** (1964), no. 6, 1055–1056. 97

[Mil97] J. W. Milnor, *Topology from the differentiable viewpoint*, Princeton Landmarks in Mathematics, Princeton University Press, 1997. 53

[MVS21a] P. Mironescu and J. Van Schaftingen, *Lifting in compact covering spaces for fractional Sobolev mappings*, Anal. PDE **14** (2021), no. 6, 1851–1871. 16, 65, 86

[MVS21b] P. Mironescu and J. Van Schaftingen, *Trace theory for Sobolev mappings into a manifold*, Ann. Fac. Sci. Toulouse Math. (6) **30** (2021), no. 2, 281–299. 85

[MVS19] A. Monteil and J. Van Schaftingen, *Uniform boundedness principles for Sobolev maps into manifolds*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **36** (2019), no. 2, 417–449. 65, 71, 85

[Muc24] D. Mucci, *Strong density results for manifold valued fractional Sobolev maps*, Ann. Fac. Sci. Toulouse Math. (6) **33** (2024), no. 3, 581–610. 36

[Nas54] J. Nash, *C^1 isometric imbeddings*, Ann. of Math. (2) **60** (1954), no. 3, 383–396. 6

[Nas56] J. Nash, *The imbedding problem for Riemannian manifolds*, Ann. of Math. (2) **63** (1956), no. 1, 20–63. 6

[Nir59] L. Nirenberg, *On elliptic partial differential equations*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (3) **13** (1959), no. 2, 115–162. 24

[Pak02] M. R. Pakzad, *Relaxing the Dirichlet energy for maps into S^2 in high dimensions*, Commun. Contemp. Math. **04** (2002), no. 03, 513–528. 117

[PR03] M. R. Pakzad and T. Rivière, *Weak density of smooth maps for the Dirichlet energy between manifolds*, Geom. Funct. Anal. **13** (2003), no. 1, 223–257. 35, 61, 115

[Par96] T. H. Parker, *Bubble tree convergence for harmonic maps*, J. Differential Geom. **44** (1996), no. 3, 595–633. 118

[Riv95] T. Rivière, *Everywhere discontinuous harmonic maps into spheres*, Acta Math. **175** (1995), no. 2, 197–226. 15, 117

[Riv98] T. Rivière, *Minimizing fibrations and p-harmonic maps in homotopy classes from S^3 into S^2*, Comm. Anal. Geom. **6** (1998), no. 3, 427–483. 118

[Riv14] T. Rivière, *Conformally invariant variational problems*, 2014. 18

[RS96] T. Runst and W. Sickel, *Sobolev spaces of fractional order, Nemytskij operators and nonlinear partial differential equations*, De Gruyter Ser. Nonlinear Anal. Appl., no. 3, Walter de Gruyter, 1996. 22, 39, 41, 43, 50

[SU81] J. Sacks and K. Uhlenbeck, *The existence of minimal immersions of 2-spheres*, Ann. of Math. (2) **113** (1981), no. 1, 1–24. 118

[Sar75] D. Sarason, *Functions of vanishing mean oscillation*, Trans. Amer. Math. Soc. **207** (1975), 391–405. 24

[SU82] R. Schoen and K. Uhlenbeck, *A regularity theory for harmonic maps*, J. Differential Geom. **17** (1982), no. 2, 307–335. 98

[SU83] R. Schoen and K. Uhlenbeck, *Boundary regularity and the Dirichlet problem for harmonic maps*, J. Differential Geom. **18** (1983), no. 2, 253–268. 11, 32

[Tri83] H. Triebel, *Theory of function spaces*, Monogr. Math., no. 78, Birkhäuser, 1983. 22, 40, 41

[Tri92] H. Triebel, *Theory of function spaces II*, Monogr. Math., no. 84, Birkhäuser, 1992. 22

[Tri06] H. Triebel, *Theory of function spaces III*, Monogr. Math., no. 100, Birkhäuser, 2006. 22

[VS19] J. Van Schaftingen, *Sobolev mappings into manifolds: nonlinear methods for approximation, extension and lifting problems*, Lecture notes, Graduate course, Oxford, 2019. 59

[VS24] J. Van Schaftingen, *The extension of traces for Sobolev mappings between manifolds*, 2024, arXiv: `2403.18738 [math.AP]`. 16, 61, 85

[VS25a] J. Van Schaftingen, *Equi-integrable approximation of Sobolev mappings between manifolds*, 2025, arXiv: `2511.20064 [math.AP]`. 115

[VS25b] J. Van Schaftingen, *Lifting of fractional Sobolev mappings to noncompact covering space*, Ann. Inst. Henri Poincaré C Anal. Non Linéaire **42** (2025), no. 1, 41–84. 85

[Vin25] A. Vincent, *Hölder continuity of minimizing $W^{s,p}$-harmonic maps*, 2025, arXiv: `2506.16442 [math.AP]`. 60, 62

[Wil22] M. Willem, *Functional analysis. Fundamentals and applications*, 2nd ed., Cornerstones, Birkhäuser, 2022. 10, 22, 53