About some approximation problems for Sobolev maps to manifolds

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Contents

Sobolev maps into manifolds

Let $\mathscr{N}$ be a smooth compact Riemannian manifold, $\mathscr{N} \subset \mathbb{R}^{\nu}$. Let $\mathscr{M}$ be a smooth compact Riemannian manifold of dimension m. Let $1 \leq p < +\infty$ and $0 < s < +\infty$.

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Definition

$$W^{s,p}(\mathscr{M};\mathscr{N}) = \{u \in W^{s,p}(\mathscr{M};\mathbb{R}^\nu) \colon u(x) \in \mathscr{N} \text{ for a.e. } x \in \mathscr{M}\}$$

Motivation

- Natural framework to study harmonic maps between manifolds (starting notably with Eells, Hildebrandt, Jost, Kaul, Lemaire, Sampson, and Widman).

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- More recently, used in the study of heterogeneous materials, notably ferromagnetism (see, e.g., recent contributions of Davoli, Gavioli, Happ, and Pagliari).

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- More recently, used in the study of heterogeneous materials, notably ferromagnetism (see, e.g., recent contributions of Davoli, Gavioli, Happ, and Pagliari).
- And many more (Ginzburg–Landau, Cosserat materials, etc.).

The strong approximation problem

Theorem

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Question

Do we have $W^{s,p}(\mathscr{M};\mathscr{N}) = H^{s,p}_S(\mathscr{M};\mathscr{N})$?

Schoen and Uhlenbeck's seminal counterexample

Proposition (Schoen and Uhlenbeck (1983))

The map $u_0 \colon \mathbb{B}^3 \to \mathbb{S}^2$ defined by $u_0(x) = x/|x|$ satisfies

$$u_0 \in W^{1,p}(\mathbb{B}^3; \mathbb{S}^2) \setminus H_S^{1,p}(\mathbb{B}^3; \mathbb{S}^2) \quad \text{whenever } 2 \leq p < 3.$$

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Two important features:

- the map u_0 is smooth *outside of a point singularity;*
- the obstruction comes from the fact that $\pi_2(\mathbb{S}^2) \neq \{0\}$.

Contents

Four natural questions

(Q1) Characterize those s, p, $\mathscr{M}$, and $\mathscr{N}$ for which strong density of smooth maps *does* occur.

(Q2) Find a suitable class of *almost smooth maps* which is always dense in $W^{s,p}(\mathscr{M};\mathscr{N})$.

(Q3) When strong density fails, characterize the space $H^{s,p}_S(\mathscr{M};\mathscr{N})$.

(Q4) What happens if strong convergence is replaced by a weaker notion?

A digression: the easy case $sp \geq m$

When $sp \geq m$, it *always* holds that $W^{s,p}(\mathscr{M};\mathscr{N}) = H^{s,p}_S(\mathscr{M};\mathscr{N})$ (Schoen and Uhlenbeck (1983)).

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The key ingredient is the Morrey–Sobolev embedding $W^{s,p} \hookrightarrow C^0$, or its limiting case $W^{s,p} \hookrightarrow \mathrm{VMO}$ when $sp = m$ (Brezis and Nirenberg (1995)).

Contents

A class of almost smooth maps

Definition

The class $\mathscr{R}_i(\mathscr{M};\mathscr{N})$ is the set of all maps $u\colon \mathscr{M} \to \mathscr{N}$ that are smooth outside of a singular set $\mathscr{S} = \mathscr{S}_u$ such that

- $\mathscr{S}$ is a finite union of i-dimensional submanifolds of $\mathscr{M}$;
- for every $j \in \mathbb{N}_*$, there exists a constant $C = C_j > 0$ such that

$$|\mathrm{D}^j u(x)| \leq \frac{C}{\operatorname{dist}(x, \mathscr{S})^j} \quad \text{for every } x \in \mathscr{M} \setminus \mathscr{S}.$$

The strong density theorem: the model case $s = 1$

Theorem (Bethuel (1991))

Assume that $p < m$. Then,

- $W^{1,p}(\mathbb{B}^m; \mathscr{N}) = H_S^{1,p}(\mathbb{B}^m; \mathscr{N})$ if and only if $\pi_{\lfloor p \rfloor}(\mathscr{N}) = \{0\}$;
- the class $\mathscr{R}_{m-\lfloor p \rfloor-1}(\mathbb{B}^m; \mathscr{N})$ is always dense in $W^{1,p}(\mathbb{B}^m; \mathscr{N})$.

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Partial cases were proved by Bethuel and Zheng (1988).

The case of an arbitrary domain $\mathscr{M}$ is known from Hang and Lin (2003). Global obstructions may arise as well.

Extensions to $0 < s < +\infty$

Case $0 < s < 1$: Brezis and Mironescu (2015), after partial contributions by Bethuel (1995), Rivière (2000), Bousquet (2007), Mucci (2009), and Bousquet, Ponce, and Van Schaftingen (2013).

Case $s \in \mathbb{N}_*$: Bousquet, Ponce, and Van Schaftingen (2015).

General case $0 < s < +\infty$: when $\mathscr{N}$ is a sphere, Escobedo (1988), and when $\mathscr{N}$ is $\lfloor sp \rfloor$-connected, Bousquet, Ponce, and Van Schaftingen (2014).

The complete answer to the strong approximation problem

Theorem (D. (2023))

Assume that $sp < m$. Then,

- $W^{s,p}(\mathbb{B}^m;\mathscr{N}) = H^{s,p}_S(\mathbb{B}^m;\mathscr{N})$ if and only if $\pi_{\lfloor sp \rfloor}(\mathscr{N}) = \{0\}$;
- the class $\mathscr{R}_{m-\lfloor sp \rfloor -1}(\mathbb{B}^m;\mathscr{N})$ is always dense in $W^{s,p}(\mathbb{B}^m;\mathscr{N})$.

The argument carries on to the case of a general domain $\mathscr{M}$, following Hang and Lin.

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A promising candidate: the Jacobian

Let $u \in W^{1,2}(\mathbb{B}^3; \mathbb{S}^2)$. Define $Ju = \mathrm{d}(u^\sharp \omega_{\mathbb{S}^2})$ in the sense of distributions:

$$\langle Ju, \alpha \rangle = -\int_{\mathbb{B}^3} \mathrm{d}\alpha \wedge u^\sharp \omega_{\mathbb{S}^2} \quad \text{for every } \alpha \in C^\infty_c(\mathbb{B}^3).$$

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Let us compute Ju_0, where $u_0(x) = x/|x|$.

Characterizing the closure of smooth maps with the Jacobian

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On the other hand, if $u \in H^{1,2}_S(\mathbb{B}^3; \mathbb{S}^2)$, then $Ju = 0$.

Theorem (Bethuel (1990))

$$H^{1,2}_S(\mathbb{B}^3; \mathbb{S}^2) = \{u \in W^{1,2}(\mathbb{B}^3; \mathbb{S}^2) \colon Ju = 0\}$$

Perspectives of generalization

This has been extended to manifolds whose *cohomology sees their homotopy* by Bethuel, Coron, Demengel, and Hélein (1991), and by Bousquet, Ponce, Van Schaftingen (2025) to higher order spaces.

See also

- the Cartesian currents of Giaquinta, Modica, and Souček;
- the flat chains into $\pi_{\lfloor p \rfloor}(\mathcal{N})$ by Pakzad and Rivière (2003), and Canevari and Orlandi (2019);
- the scans by Hardt and Rivière (2003, 2008).

An analytical characterization of the closure of smooth maps for fractional regularity

Theorem (D., Mironescu, Xiao (2025))

Assume that $0 < s < 1$ and that $sp < m$.

- For every closed $\lfloor sp \rfloor$-form ω on $\mathcal{N}$, there exists a well-defined Jacobian J_ω that extends by continuity the previous notion to $W^{s,p}(\mathbb{B}^m; \mathcal{N})$.
- If *the cohomology of* $\mathcal{N}$ *detects its homotopy*, then $u \in H_S^{s,p}(\mathbb{B}^m; \mathcal{N})$ if and only if $J_\omega u = 0$ for every ω.

This is in line with results by Bourgain, Brezis, and Mironescu (2005), and Bousquet and Mironescu (2014), to define the Jacobian and study its properties, and by Mucci (2024), for sphere-valued maps.

Contents

The weak approximation problem

We say that $(u_n)_{n\in\mathbb{N}}$ *weakly converges* to u in $W^{1,p}$, and we write $u_n \rightharpoonup u$, whenever $u_n \to u$ almost everywhere and

$$\sup_{n\in\mathbb{N}} \mathcal{E}^{1,p}(u_n, \mathscr{M}) = \sup_{n\in\mathbb{N}} \int_{\mathscr{M}} |\mathrm{D}u_n|^p < +\infty.$$

Define

$$H^{1,p}_W(\mathscr{M}; \mathscr{N}) = \{u \in W^{1,p}(\mathscr{M}; \mathscr{N}) \colon \text{there exists } (u_n)_{n\in\mathbb{N}} \text{ in } C^\infty(\mathscr{M}; \mathscr{N}) \text{ such that } u_n \rightharpoonup u\}.$$

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Question

Does it hold that $H^{1,p}_W(\mathscr{M};\mathscr{N}) = W^{1,p}(\mathscr{M};\mathscr{N})$?

A topological obstruction: here we go again?

If $p \notin \mathbb{N}$ and $\pi_{\lfloor p \rfloor}(\mathscr{N}) \neq \{0\}$, then $H^{1,p}_W(\mathscr{M};\mathscr{N}) \subsetneq W^{1,p}(\mathscr{M};\mathscr{N})$ whenever $\dim \mathscr{M} > p$.

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Theorem (Bethuel (1991))

If $p \notin \mathbb{N}$, then $H^{1,p}_W(\mathscr{M};\mathscr{N}) = H^{1,p}_S(\mathscr{M};\mathscr{N})$.

A new phenomenon: the case $p \in \mathbb{N}$

Unlike for $2 < p < 3$, we have $x/|x| \in H^{1,2}_W(\mathbb{B}^3; \mathbb{S}^2)$.

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- $H^{1,2}_W(\mathscr{M}; \mathscr{N}) = W^{1,2}(\mathscr{M}; \mathscr{N})$ for more general $\mathscr{N}$ (Pakzad and Rivière (2003));

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- $H^{1,2}_W(\mathscr{M}; \mathscr{N}) = W^{1,2}(\mathscr{M}; \mathscr{N})$ for more general $\mathscr{N}$ (Pakzad and Rivière (2003));
- $H^{2,2}_W(\mathbb{B}^5; \mathbb{S}^3) = W^{2,2}(\mathbb{B}^5; \mathbb{S}^3)$ (Hardt and Rivière (2015)).

Obstructions strike back: the analytical obstruction

Theorem (Bethuel (2020))

$$H^{1,3}_{W}(\mathbb{B}^4; \mathbb{S}^2) \subsetneq W^{1,3}(\mathbb{B}^4; \mathbb{S}^2)$$

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Theorem (Bethuel (2020))

$$H^{1,3}_W(\mathbb{B}^4;\mathbb{S}^2) \subsetneq W^{1,3}(\mathbb{B}^4;\mathbb{S}^2)$$

Global topological obstructions were already known (Hang and Lin (2003)).
Here, the obstruction is local.

Two families of analytical obstructions to the weak approximation property

Theorem (D. and Van Schaftingen (2024))

For every $p \in \mathbb{N} \setminus \{0, 1\}$, there exists a compact Riemannian manifold $\mathscr{N}$ such that, if $\dim \mathscr{M} > p$, then

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Theorem (D. and Van Schaftingen (2024))

For every $d \in \mathbb{N}_*$, if $\dim \mathscr{M} > 4d - 1$, then

$$H^{1,4d-1}_W(\mathscr{M}; \mathbb{S}^{2d}) \subsetneq W^{1,4d-1}(\mathscr{M}; \mathbb{S}^{2d}).$$

The key idea at the core of the construction

For $\mathscr{N} = \mathbb{S}^{2d}$, we construct u using periodic maps with singularities of Hopf invariant 2 on a grid, via a Whitehead product.

For the first family, we construct u using periodic quotient maps into the p-skeleton of $\mathbb{T}^{p+1}$.

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At the core of the proof for the first family lies the *isoperimetric inequality*, and for the second family, *Rivière's estimate on the Hopf invariant* (1998).

Thank you for your attention!