The weak approximation problem for manifold-valued Sobolev mappings

Antoine Detaille

ETH Zürich

The 14th of January 2026

Sobolev maps into manifolds

Let $\mathscr{N}$ be a smooth compact Riemannian manifold, $\mathscr{N} \subset \mathbb{R}^{\nu}$. Let $\mathscr{M}$ be a smooth compact Riemannian manifold of dimension m. Let $1 \leq p < +\infty$.

Sobolev maps into manifolds

Let $\mathscr{N}$ be a smooth compact Riemannian manifold, $\mathscr{N} \subset \mathbb{R}^\nu$. Let $\mathscr{M}$ be a smooth compact Riemannian manifold of dimension m. Let $1 \leq p < +\infty$.

Definition

$$W^{s,p}(\mathscr{M};\mathscr{N}) = \{u \in W^{s,p}(\mathscr{M};\mathbb{R}^\nu) \colon u(x) \in \mathscr{N} \text{ for a.e. } x \in \mathscr{M}\}$$

The strong approximation problem

Theorem

The space $C^\infty(\mathscr{M};\mathbb{R})$ is dense in $W^{1,p}(\mathscr{M};\mathbb{R})$.

The strong approximation problem

Theorem

The space $C^\infty(\mathscr{M}; \mathbb{R})$ is dense in $W^{1,p}(\mathscr{M}; \mathbb{R})$.

We define

$$H^{1,p}_S(\mathscr{M}; \mathscr{N}) = \overline{C^\infty(\mathscr{M}; \mathscr{N})}^{W^{1,p}}.$$

The strong approximation problem

Theorem

The space $C^\infty(\mathscr{M};\mathbb{R})$ is dense in $W^{1,p}(\mathscr{M};\mathbb{R})$.

We define

$$H^{1,p}_S(\mathscr{M};\mathscr{N}) = \overline{C^\infty(\mathscr{M};\mathscr{N})}^{W^{1,p}}.$$

Question

Do we have $W^{1,p}(\mathscr{M};\mathscr{N}) = H^{1,p}_S(\mathscr{M};\mathscr{N})$?

The strong density theorem

Theorem (Bethuel (1991))

Then, $W^{1,p}(\mathbb{B}^m; \mathscr{N}) = H_S^{1,p}(\mathbb{B}^m; \mathscr{N})$ if and only if $\pi_{\lfloor p \rfloor}(\mathscr{N}) = \{0\}$ or $p \geq m$.

The strong density theorem

Theorem (Bethuel (1991))

Then, $W^{1,p}(\mathbb{B}^m; \mathscr{N}) = H^{1,p}_S(\mathbb{B}^m; \mathscr{N})$ if and only if $\pi_{\lfloor p \rfloor}(\mathscr{N}) = \{0\}$ or $p \geq m$.

The case of an arbitrary domain $\mathscr{M}$ is known from Hang and Lin (2003). Global obstructions may arise as well.

The strong density theorem

Theorem (Bethuel (1991))

Then, $W^{1,p}(\mathbb{B}^m; \mathscr{N}) = H^{1,p}_S(\mathbb{B}^m; \mathscr{N})$ if and only if $\pi_{\lfloor p \rfloor}(\mathscr{N}) = \{0\}$ or $p \geq m$.

The case of an arbitrary domain $\mathscr{M}$ is known from Hang and Lin (2003). Global obstructions may arise as well.

This has been extended to $W^{s,p}$ for other values of s by Brezis and Mironescu ($0 < s < 1$, 2015), Bousquet, Ponce, and Van Schaftingen ($s = 2, 3, \dots$, 2015), and D. ($s > 1$ noninteger, 2023) .

The weak approximation problem

We say that $(u_n)_{n\in\mathbb{N}}$ *weakly converges* to u in $W^{1,p}$, and we write $u_n \rightharpoonup u$, whenever $u_n \to u$ almost everywhere and

$$\sup_{n\in\mathbb{N}} \mathcal{E}^{1,p}(u_n, \mathscr{M}) = \sup_{n\in\mathbb{N}} \int_{\mathscr{M}} |\mathrm{D}u_n|^p < +\infty.$$

Define

$$H^{1,p}_W(\mathscr{M};\mathscr{N}) = \{u \in W^{1,p}(\mathscr{M};\mathscr{N})\colon \text{there exists } (u_n)_{n\in\mathbb{N}} \text{ in } C^\infty(\mathscr{M};\mathscr{N}) \text{ such that } u_n \rightharpoonup u\}.$$

The weak approximation problem

We say that $(u_n)_{n\in\mathbb{N}}$ *weakly converges* to u in $W^{1,p}$, and we write $u_n \rightharpoonup u$, whenever $u_n \to u$ almost everywhere and

$$\sup_{n\in\mathbb{N}} \mathcal{E}^{1,p}(u_n, \mathscr{M}) = \sup_{n\in\mathbb{N}} \int_{\mathscr{M}} |\mathrm{D}u_n|^p < +\infty.$$

Define

$$H^{1,p}_W(\mathscr{M};\mathscr{N}) = \{u \in W^{1,p}(\mathscr{M};\mathscr{N}) \colon \text{there exists } (u_n)_{n\in\mathbb{N}} \text{ in } C^\infty(\mathscr{M};\mathscr{N}) \text{ such that } u_n \rightharpoonup u\}.$$

Question

Does it hold that $H^{1,p}_W(\mathscr{M};\mathscr{N}) = W^{1,p}(\mathscr{M};\mathscr{N})$?

A topological obstruction: here we go again?

If $2 < p < 3$, then $\frac{x}{|x|} \notin H^{1,p}_W(\mathbb{B}^3; \mathbb{S}^2)$.

A topological obstruction: here we go again?

If $2 < p < 3$, then $\frac{x}{|x|} \notin H^{1,p}_W(\mathbb{B}^3; \mathbb{S}^2)$.

Theorem (Bethuel (1991))

If $p \notin \mathbb{N}$, then $H^{1,p}_W(\mathscr{M}; \mathscr{N}) = H^{1,p}_S(\mathscr{M}; \mathscr{N})$.

A new phenomenon: the case $p \in \mathbb{N}$

Unlike for $2 < p < 3$, we have $x/|x| \in H^{1,2}_W(\mathbb{B}^3; \mathbb{S}^2)$.

A new phenomenon: the case $p \in \mathbb{N}$

Unlike for $2 < p < 3$, we have $x/|x| \in H^{1,2}_W(\mathbb{B}^3; \mathbb{S}^2)$.

More generally, we have:

- $H^{1,2}_S(\mathbb{B}^3; \mathbb{S}^2) \subsetneq H^{1,2}_W(\mathbb{B}^3; \mathbb{S}^2) = W^{1,2}(\mathbb{B}^3; \mathbb{S}^2)$ (Bethuel (1990));

A new phenomenon: the case $p \in \mathbb{N}$

Unlike for $2 < p < 3$, we have $x/|x| \in H^{1,2}_W(\mathbb{B}^3; \mathbb{S}^2)$.

More generally, we have:

- $H^{1,2}_S(\mathbb{B}^3; \mathbb{S}^2) \subsetneq H^{1,2}_W(\mathbb{B}^3; \mathbb{S}^2) = W^{1,2}(\mathbb{B}^3; \mathbb{S}^2)$ (Bethuel (1990));
- $H^{1,p}_W(\mathscr{M}; \mathscr{N}) = W^{1,p}(\mathscr{M}; \mathscr{N})$ whenever $\mathscr{N}$ is $(p-1)$-connected (Hajłasz (1994));

A new phenomenon: the case $p \in \mathbb{N}$

Unlike for $2 < p < 3$, we have $x/|x| \in H^{1,2}_W(\mathbb{B}^3; \mathbb{S}^2)$.

More generally, we have:

- $H^{1,2}_S(\mathbb{B}^3; \mathbb{S}^2) \subsetneq H^{1,2}_W(\mathbb{B}^3; \mathbb{S}^2) = W^{1,2}(\mathbb{B}^3; \mathbb{S}^2)$ (Bethuel (1990));
- $H^{1,p}_W(\mathscr{M}; \mathscr{N}) = W^{1,p}(\mathscr{M}; \mathscr{N})$ whenever $\mathscr{N}$ is $(p-1)$-connected (Hajłasz (1994));
- $H^{1,2}_W(\mathscr{M}; \mathscr{N}) = W^{1,2}(\mathscr{M}; \mathscr{N})$ for more general $\mathscr{N}$ (Pakzad and Rivière (2003));

A new phenomenon: the case $p \in \mathbb{N}$

Unlike for $2 < p < 3$, we have $x/|x| \in H^{1,2}_W(\mathbb{B}^3; \mathbb{S}^2)$.

More generally, we have:

- $H^{1,2}_S(\mathbb{B}^3; \mathbb{S}^2) \subsetneq H^{1,2}_W(\mathbb{B}^3; \mathbb{S}^2) = W^{1,2}(\mathbb{B}^3; \mathbb{S}^2)$ (Bethuel (1990));
- $H^{1,p}_W(\mathscr{M}; \mathscr{N}) = W^{1,p}(\mathscr{M}; \mathscr{N})$ whenever $\mathscr{N}$ is $(p-1)$-connected (Hajłasz (1994));
- $H^{1,2}_W(\mathscr{M}; \mathscr{N}) = W^{1,2}(\mathscr{M}; \mathscr{N})$ for more general $\mathscr{N}$ (Pakzad and Rivière (2003));
- $H^{2,2}_W(\mathbb{B}^5; \mathbb{S}^3) = W^{2,2}(\mathbb{B}^5; \mathbb{S}^3)$ (Hardt and Rivière (2015)).

Obstructions strike back: the analytical obstruction

Theorem (Bethuel (2020))

$$H^{1,3}_W(\mathbb{B}^4; \mathbb{S}^2) \subsetneq W^{1,3}(\mathbb{B}^4; \mathbb{S}^2)$$

Obstructions strike back: the analytical obstruction

Theorem (Bethuel (2020))

$$H^{1,3}_{W}(\mathbb{B}^4;\mathbb{S}^2) \subsetneq W^{1,3}(\mathbb{B}^4;\mathbb{S}^2)$$

Global topological obstructions were already known (Hang and Lin (2003)).
Here, the obstruction is local.

Two families of analytical obstructions to the weak approximation property

Theorem (D. and Van Schaftingen (2024))

For every $p \in \mathbb{N} \setminus \{0, 1\}$, there exists a compact Riemannian manifold $\mathscr{N}$ such that, if $\dim \mathscr{M} > p$, then

$$H^{1,p}_{W}(\mathscr{M}; \mathscr{N}) \subsetneq W^{1,p}(\mathscr{M}; \mathscr{N}).$$

Two families of analytical obstructions to the weak approximation property

Theorem (D. and Van Schaftingen (2024))

For every $p \in \mathbb{N} \setminus \{0, 1\}$, there exists a compact Riemannian manifold $\mathscr{N}$ such that, if $\dim \mathscr{M} > p$, then

$$H^{1,p}_W(\mathscr{M}; \mathscr{N}) \subsetneq W^{1,p}(\mathscr{M}; \mathscr{N}).$$

Theorem (D. and Van Schaftingen (2024))

For every $d \in \mathbb{N}_*$, if $\dim \mathscr{M} > 4d - 1$, then

$$H^{1,4d-1}_W(\mathscr{M}; \mathbb{S}^{2d}) \subsetneq W^{1,4d-1}(\mathscr{M}; \mathbb{S}^{2d}).$$

Weak approximation of mappings into manifold with a finite homotopy group

Theorem (D. and Van Schaftingen (in preparation))

For every $p \in \mathbb{N}_*$, if $\pi_p(\mathscr{N})$ is finite, then $H^{1,p}_W(\mathbb{B}^m; \mathscr{N}) = W^{1,p}(\mathbb{B}^m; \mathscr{N})$.

Weak approximation of mappings into manifold with a finite homotopy group

Theorem (D. and Van Schaftingen (in preparation))

For every $p \in \mathbb{N}_*$, if $\pi_p(\mathscr{N})$ is finite, then $H^{1,p}_W(\mathbb{B}^m; \mathscr{N}) = W^{1,p}(\mathbb{B}^m; \mathscr{N})$.

Combining this with a result of Serre, we obtain a complete answer for sphere-valued maps.

Corollary

For every $p \in \mathbb{N}_*$ and $\ell \in \mathbb{N}_*$, we have $H^{1,p}_W(\mathbb{B}^m; \mathbb{S}^\ell) = W^{1,p}(\mathbb{B}^m; \mathbb{S}^\ell)$ if and only if either ℓ is odd or $p \neq 2\ell - 1$.

Some remaining open problems

Some remaining open problems

- Give a full characterization of the situations where the weak approximation property holds.

Some remaining open problems

- Give a full characterization of the situations where the weak approximation property holds.
- What happens in $W^{s,p}(\mathscr{M};\mathscr{N})$ for $s \neq 1$?

Some remaining open problems

- Give a full characterization of the situations where the weak approximation property holds.
- What happens in $W^{s,p}(\mathscr{M};\mathscr{N})$ for $s \neq 1$?
- Is it possible to characterize those maps which are weakly approximable, in case the weak approximation property fails?

Some remaining open problems

- Give a full characterization of the situations where the weak approximation property holds.
- What happens in $W^{s,p}(\mathscr{M};\mathscr{N})$ for $s \neq 1$?
- Is it possible to characterize those maps which are weakly approximable, in case the weak approximation property fails?
- And many more!

Thank you for your attention!