Nonlinear Tools for Geometric Analysis
Exam topics

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Contents

Foreword

The examination for this course will take the following format. You are expected to choose an article that involves a tool that has been discussed in the class (but not in the exact context discussed in class!), and should present it during the oral exam. We will discuss the paper together, with emphasis on the following points: What is the problem discussed in the paper? What are the main difficulties? How is the relevant tool addressing those difficulties, and how is it used? Of course, understanding of the material from the lecture related to the topic you chose is also expected. You can use the support of the blackboard, and also bring any personal notes or references — the goal is that you should not feel constrained to memorise anything, but rather focus on understanding the material. Final choices of topics should be communicated at least two weeks before the exam takes place.

In this document, you will find a list of recommended topics. For each of them, it is explicitly stated which tool it is about, and some guidance is provided, notably indicating parts of the paper that you can admit. This means that it is not expected that you understand how these things are proved, you can take them as "black boxes". Your understanding of the paper should focus on the points discussed above. All the suggested papers do not have the same difficulty. Some of them are notably longer or rely on more background. This unevenness is mitigated by the selection of topic inside the paper, which should make the proposed topics of more comparable difficulties. And of course, variations of difficulty between topics will be taken into account during the exam as well. In particular, the shorter the topic to study is, the more details you are expected to be able to provide during the exam.

You are free to choose any topic in the list. There is no limit in the number of people taking the same topic (in particular, even if the topics are not of equal difficulty, you are all equal in front of the choice of topic). In particular, you can also work together on a chosen topic. However, at the exam, you are expected to demonstrate personal understanding of the chosen topic, and not just repeat things that were explained to you by a friend. . . or an AI!

You are free to suggest a topic outside of this list, provided that you can justify convincingly its connection with a tool from the lecture. Beware that in this case, it will be up to you to choose what can be admitted and what are the important points to be understood, and there is no guarantee of balanced difficulty. The only restriction is that you cannot choose a topic (even from the list below) on which you have already worked (e.g., in a semester project, thesis, or anything related), I want this exam to also be the opportunity to learn something new!

1 Good and bad cubes for Sobolev homeomorphisms

The reference for this topic is the paper [HP18] by S. Hencl and A. Pratelli. The goal is to understand how the method of good and bad cubes can be used to prove approximation results of Sobolev homeomorphisms by smooth diffeomorphisms.

You will focus on Theorem 1.1. The goal is to understand the approximation theorem being proved and have an idea of the context in which it fits, and most importantly to understand the use of good and bad cubes in the proof. Typically, you are expected to understand how they are defined and handled, and what are the key differences with the context of Sobolev mappings we saw in the lecture.

You may take as granted the technical results used in the proof, the important being to understand how they fit into the big picture of the argument.

As a bonus question, you can think about why the result is restricted to dimension 2, and what would be the main difficulties to extend it to higher dimension.

2 Good and bad cubes for weak Yang–Mills connections

The reference for this topic is the paper [CR24] by R. Caniato and T. Rivière. The goal is to understand how the method of good and bad cubes can be used to prove approximation properties for weak Yang–Mills connections.

You will focus on Theorems IV.1 and V.1. The goal is to understand the approximation theorems being proved and have an idea of the context in which they fit, and most importantly to understand the use of good and bad cubes in the proof. Typically, you are expected to understand how they are defined and handled, and what are the key differences with the context of Sobolev mappings we saw in the lecture.

Beware that this topic is quite demanding in what concerns the geometry background required to understand fully the paper. However, you may take as granted all the auxiliary results used in the proof of both these theorems, notably all the analysis of Yang–Mills connections in 4 dimensions as well as Propositions II.1 and II.2 — but you should understand how they are used in the proof!

In the perspective of preparing the exam in groups, this topic would pair well with 11, and also 4.

3 Good and bad cubes for traces extensions

The reference for this topic is the paper [VS24] by J. Van Schaftingen. The goal is to understand how the method of good and bad cubes can be used to prove extension of traces results for Sobolev mappings into manifolds.

You will focus on Theorem 1.1, and can leave aside generalisations to more general domains as well as characterisations of the range of the trace and extensions with supercritical integrability. The goal is to understand the extension theorem being proved and the context in which it fits, and most importantly to understand the use of good and bad cubes in the proof. Typically, you are expected to understand how they are defined and handled, and what are the key differences with the context of Sobolev mappings we saw in the lecture. You are also expected to understand the role played by the topological assumptions in the proof, and to explain why other methods seen in the class would not apply to this situation.

4 The method of good and bad balls

The goal of this topic is to understand the method of good and bad balls, which is a variant of the method of good and bad cubes. You will focus on the application of this method to give an alternative proof of the strong approximation theorem in $W^{1,p}(\mathbb{B}^m; \mathscr{N})$ when $m - 1 < p < m$. A good reference to start with would be [VS19, Chapter 7], but you might also want to look at the papers quoted at the end of the chapter, notably the ones by A. Ponce and J. Van Schaftingen (2009), P. Bousquet, A. Ponce, and J. Van Schaftingen (2008), and A. Gastel and A. Nerf (2011).

You are expected to be able to provide a detailed sketch of the proof of the density theorem in this special case using the method of good and bad balls, especially highlighting the advantage of this method compared to good and bad cubes but also its limitations. Typically, two important questions would be: (i) how does this technique ease a generalisation to higher-order spaces; and (ii) what are the limitations to go to lower values of p, that is, $p < m - 1$.

To broaden your horizon, you may enjoy discussing with someone who chose topic 2, which features a variant of this method.

5 Composition operator for the lifting problem

The main reference for this topic is the paper [BC07] by F. Bethuel and D. Chiron, but you may want to start with the work of J. Bourgain, H. Brezis, and P. Mironescu [BBM00]. The goal is to understand the use of the composition (and product) results in

Sobolev spaces in the lifting problem.

You will focus on the case of lifting when $s > 1$ and $sp \geq 2$. In particular, you can take for granted the existence of the lifting when $s = 1$ and $p \geq 2$. The goal is to understand how composition and product theorems are used to deduce the existence of the lifting when $s > 1$ from the case $s = 1$ by improvement of regularity. You may be willing to start with the case where the target is $\mathbb{S}^1$ along with its universal covering $\mathbb{R}$ — which corresponds to [BBM00] — as the additional structure of the circle simplifies the argument. Beware that, in the general case featured in [BC07], there is a missing assumption that is used in the proof. You are expected a minima to find the issue and understand the proof under this assumption, but it is even better if you can discuss what happens without this assumption.

6 Nonlinear uniform boundedness for the extension of traces

The main reference for this topic is the paper [Bet14] by F. Bethuel, but you may also want to look at the work [MVS21b] by P. Mironescu and J. Van Schaftingen. The goal is to understand the use of the nonlinear uniform boundedness principle to construct counterexamples to extension of traces for mappings into a manifold.

You will focus on Theorem 1.1, and may ignore other discussions such as the one from Theorem 1.4. The goal is to understand the obstruction being featured and the context in which it fits, and most importantly to understand where and how the nonlinear uniform boundedness principle is applied — note that it is not invoked as such, but rather reproved in the special case of the extension of traces — and how the superlinear growth of the extension energy needed to apply it is obtained. You should also briefly discuss the topological assumptions involved in the proof, comparing with the discussion in [MVS21b]. In particular, you should be able to explain what exactly was the understanding of the extension of traces problem after the publication of both those papers.

7 Nonlinear uniform boundedness for the lifting problem

The main reference for this topic is the paper [MVS21a] by P. Mironescu and J. Van Schaftingen. The goal is to understand the use of the nonlinear uniform boundedness principle to construct analytical obstructions to the lifting property when $0 < s < 1$ and $sp = 1$.

You will focus on Theorem 1.4. The goal is to understand the obstruction being featured and the context in which it fits, and most importantly to understand where and how the nonlinear uniform boundedness principle is applied — note that it is not invoked as such, but rather reproved in the special case of the lifting problem — and how the superlinear growth of the lifting energy needed to apply it is obtained. You should also be able to compare this case with the case $1 < sp < 2$, and also $s = 1 = p$, and in particular discuss the difference between analytical obstruction — which are the ones featured here — with topological obstructions.

8 Nonlinear uniform boundedness for the weak approximation problem

The main reference for this topic is the paper [Bet20] by F. Bethuel. The goal is to understand the use of the nonlinear uniform boundedness principle to construct analytical obstructions to the weak approximation property of Sobolev mappings.

You will focus on Theorems 3 and 4, and can in particular completely ignore the whole discussion on liftings. The goal is to understand the obstruction being featured and the context in which it fits, and most importantly to understand where and how the nonlinear uniform boundedness principle is applied — note that it is not invoked as such, but rather reproved in the special case of the weak approximation problem — and how the superlinear growth of the weak approximation energy needed to apply it is obtained.

Especially given the length of the paper, you are not expected to study in full details the various constructions in the proof, but rather to explain how they fit together in the big picture of the argument. You can also take for granted the theory of scans by R. Hardt and T. Rivière used in the proof, as well as the final results of the discussion on branched optimal transport from the appendix.

9 Singular projection for harmonic maps — Compactness

The main reference for this topic is the paper [HL87] by R. Hardt and Lin F. The goal is to understand the use of the singular projection method to obtain compactness for minimising harmonic maps.

You will focus on Theorem 6.4. In particular, you make take for granted Corollary 2.8. The main goal is to understand the compactness result being proved and its importance in the study of harmonic maps, and especially where and how the method of singular projection is applied. In particular, you should understand what would be the counterpart of the proof in the real-valued setting, and how the singular projection helps dealing with the manifold constraint.

Given the short length of this topic, you are also expected to learn some basic notions concerning harmonic maps, especially the main challenges they raise compared to (real-valued) harmonic functions, in the spirit of what we did at the beginning of the course comparing Sobolev functions with Sobolev mappings.

10 Singular projection for harmonic maps — Energy bounds

The main reference for this topic is the paper [HKL86] by R. Hardt, D. Kinderlehrer, and Lin F. The goal is to understand the use of the singular projection method to obtain energy bounds for minimising harmonic maps.

You will focus on Lemma 2.3. The main goal is to understand the energy bound being proved and its importance in the study of harmonic maps, and especially where and how the method of singular projection is applied. In particular, you should understand what would be the counterpart of the proof in the real-valued setting, and how the singular projection helps dealing with the manifold constraint.

Given the short length of this topic, you are also expected to learn some basic notions concerning harmonic maps, especially the main challenges they raise compared to (real-valued) harmonic functions, in the spirit of what we did at the beginning of the course comparing Sobolev functions with Sobolev mappings.

11 Singular projection for weak Yang–Mills connections

The main reference for this topic is the paper [CR24] by R. Caniato and T. Rivière. The goal is to understand the use of the singular projection method to obtain extensions of gauge changes with suitable properties in the study of problems related to the Yang–Mills functional.

You will focus on Proposition B.1, and may essentially ignore the main results from the paper — even though some global understanding of the context in which this proposition fits would be appreciated. The main goal is to understanding the extension result being proved, and where and how the method of singular projection is used to obtain it. In particular, you should compare the conclusions obtained here with those that were covered in the lecture, and understand the difficulties raised by the extra conclusions required in the statement. You are also expected to be able to comment on the fact that the result is restricted to dimension 5, and explain whether or not it could be extended to other dimensions. A further question that you may want to consider is the need of the extra conclusions present in the statement for the purposes of the use that is made of Proposition B.1 in the main body of the paper. (You are not expected to work the full details of this question.)

In the perspective of preparing the exam in groups, this topic would pair well with 2.

12 Singular projection for the singular set of Sobolev mappings

The main reference for this topic is the paper [CO19] by G. Canevari and G. Orlandi. The goal is to understand the use of the method of singular projection to define the singular set of a Sobolev mapping.

You will mostly focus on Theorem 3.1. In particular, you may ignore the last section about applications, and also possibly the connection with the singular set defined by M.R. Pakzad and T. Rivière [PR03] (even though you may want to consider it, for instance, if you have someone working on topic 13 with who you can discuss). You may also take for granted all results from geometric measure theory used in the proofs, and just have some understanding of the objects involved. The main goal is to understand the construction of a singular set being featured and how it relates to approximation problems for Sobolev mappings, and especially the use of the method of singular projection in its construction. In particular, you should be able to comment on the fact that the singular set is defined more generally for mappings *with values into the ambient space*, and how it is crucial for the argument to work. If you are also having a look at [PR03], you might want to compare both approaches, compare the respective assumptions, and explain the contribution of Canevari and Orlandi in this respect.

13 Singular projection for the weak approximation problem

The main reference for this topic is the paper [PR03] by M.R. Pakzad and T. Rivière. The goal is to understand the use of the method of singular projection to prove weak approximation results for more general targets than the sphere.

You will focus on Theorem I in the special case where $\mathscr{N}$ is simply connected, that is, you may ignore the non-simply connected case starting from page 251. You will therefore mostly focus on Section 7, even though some tools and constructions from Sections 4 and 5 will also be needed. You may also, for simplicity, assume that the domain is a ball, to avoid issues related to global obstructions. Finally, you can take for granted all results from geometric measure theory used in the proofs, and just have some understanding of the objects involved. The main goal is to understand the weak approximation result being proved, and especially how the method of singular projection is used to bring the problem back to a simpler one that has been covered in the lecture. You should also be able to comment on the fact that the result is limit to $p = 2$, and on whether or not it could be extended to other values of p.

In the perspective of preparing the exam in groups, if you feel reading a bit more and especially material related to Theorem II, you may enjoy discussing with someone preparing topic 12.

14 The method of almost projection

The main reference for this topic is the paper [Haj94] by P. Hajłasz, but you may also want to have a look at the paper [BPVS13] by P. Bousquet, A. Ponce, and J. Van Schaftingen. The goal is to understand the method of almost projection, and to compare it with the method of singular projection seen in the lecture.

More precisely, you should understand what is the principle between the method of almost projection developed by Hajłasz, and what are the important differences with the method of singular projection. Typical questions to consider are notably how do the assumptions required to apply both these methods differ, and which results can be proved by one of these methods and not the other. To illustrate the implementation of this method, you are also expected to be able to give the key steps of the proof of the main results in [Haj94]. The paper [BPVS13] offers a different exposition that might help you complete your understanding, as well as a generalisation to $s \geq 1$. A possible additional question could be to discuss what happens if $0 < s < 1$.

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