A surprising threshold for the method of singular projection

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Context: nonlinear functional analysis

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Consider maps $u \in W^{s,p}(\mathscr{M}; \mathscr{N})$, where $\mathscr{M}$ and $\mathscr{N}$ are two compact Riemannian manifolds, with $\mathscr{N} \subset \mathbb{R}^\nu$ and $0 < s < +\infty$, $1 \leq p < +\infty$.

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- $\mathscr{N} = \mathbb{S}^2, \mathbb{RP}^2, SO(3)/H, \dots$: liquid crystals (connections with harmonic maps);
- $\mathscr{N} = \mathbb{S}^2$: ferromagnetism (connected to homogenization problems);
- $\mathscr{N} = G$ a Lie group: gauge theories.

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In general, usual linear construction for Sobolev functions are *not compatible* with the nonlinear constraint that $u(x) \in \mathscr{N}$ for a.e. $x \in \mathscr{M}$.

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- extend a trace $g \in W^{1-1/p,p}(\partial\mathscr{M};\mathscr{N})$ to a map defined on $\mathscr{M}$ (again convolution);
- construct competitors or localize in chart using a cut-off.

A workaround: the method of singular projection

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Typical statement

Let $p < \ell$ and let $g \in W^{1-1/p,p}(\partial \mathscr{M}; \mathbb{S}^{\ell-1})$. There exists $u \in W^{1,p}(\mathscr{M}; \mathbb{S}^{\ell-1})$ such that $\operatorname{tr} u = g$ and satisfying the estimate

$$\|Du\|_{L^p(\mathscr{M})} \leq C|g|_{W^{1-1/p,p}(\partial \mathscr{M})},$$

for some constant $C > 0$ depending only on $\mathscr{M}$, ℓ, and p.

Estimates for the method of singular projection

Theorem

Let $0 < s < +\infty$, $1 \leq p < +\infty$, $\ell \in \mathbb{N} \setminus \{0, 1\}$, and let $P \colon \mathbb{R}^\ell \setminus \{0\} \to \mathbb{S}^{\ell-1}$ be given by $P(x) = \frac{x}{|x|}$. Assume that either $s \geq 1$ and $sp < \ell$, or $0 < s < 1$ and $p < \ell$. For every $u \in W^{s,p}(\mathscr{M}; \mathbb{R}^\ell) \cap L^\infty(\mathscr{M}; \mathbb{R}^\ell)$ and every $\alpha > 0$,

$$\int_{B^\ell_\alpha} |P \circ (u - a)|^p_{W^{s,p}(\mathscr{M})} \, \mathrm{d}a \leq C \|u\|^p_{W^{s,p}(\mathscr{M})},$$

for some constant $C > 0$ depending only on s, p, α, $\mathscr{M}$, and ℓ.

The same estimate holds for *general singular projections*.

The case $s = 1$ is due to Hardt and Lin (1987). The case $0 < s < 1$ is due to Vincent (2025).

In full generality, the case $s > 1$ seems to be new (D. 2025).

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Theorem (D. 2025)

Assume that $0 < s < 1$ and $1 \leq p < +\infty$ are such that $sp < \ell$ but $p \geq \ell$, and let $P \colon \mathbb{R}^\ell \setminus \{0\} \to \mathbb{S}^{\ell-1}$ be given by $P(x) = \frac{x}{|x|}$.
There exists a map $u \in W^{s,p}(\mathbb{B}^\ell; \mathbb{R}^\ell) \cap L^\infty(\mathbb{B}^\ell; \mathbb{R}^\ell)$ such that $P \circ (u - a) \notin W^{s,p}$ for every $a \in \mathbb{B}^\ell$.

This shows that the threshold $p < \ell$ is sharp.

Thank you for your attention!